J. Austral. Math. Soc. (*Series A*) **56** (1994), 291–302

SOME GENERALIZATIONS IN H-MODULAR SPACES OF FAN'S BEST APPROXIMATION THEOREM

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(Received 17 May 1991; revised 15 November 1991)

Communicated by P. G. Dodds

Abstract

We state best approximation and fixed point theorems in modular spaces endowed with an H-space structure given by the modular topology. We consider both the cases of single valued functions and multifunctions. These theorems extend some previous results due to Ky Fan.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 47 H 10; secondary 54 H 25.

1. Introduction

The aim of this paper is to state non-convex versions of best approximation and fixed point theorems in modular spaces endowed with its modular topology. To do that we provide the modular space with an H-space structure in which the linear setting is replaced by merely topological assumptions. Precisely, in this structure the convex hulls are replaced by the contractible sets. The H-space theory based on Horvath's ideas [9, 10] has been developed in [1, 2, 3, 4]. Modular spaces with this structure are called H-modular spaces. This approach allows us to give a new application (Theorem 2) of our generalized KKM Theorem proved in [2] on which the main theorems of the present paper (Theorems 3, 4) are based. Moreover in Theorem 2 and so in Theorems 3 and 4 the classical compactness assumptions are relaxed as well, as in Lassonde's paper [13] and in our papers [1, 2, 3, 4]. Theorem 3 of Section 4 is a best

https://doi.org/10.1017/S1446788700035503 Published online by Cambridge University Press

approximation theorem for single valued functions and when the modular is a seminorm in a vector space it is a non-compact and non-convex version of the Ky Fan extension [8] of Browder's fixed point theorem [5]. In Section 5 the multivalued case is considered. We introduce the concept of $H\rho$-continuity for multifunctions in terms of modular convergence and in the spirit of the Hausdorff metric and then, a relaxed convexity assumption on the modular ρ compatible with the H-space structure (a related convexity definition can be found in [3]). These definitions, as well as other technical conditions, enable us to prove the best approximation theorem (Theorem 4) and, as a consequence, a fixed point property for multifunctions. Finally Theorem 4 is also compared with other generalizations to multifunctions of Ky Fan's theorem (see [15]).

2. H-spaces

We first recall some basic concepts in order to define the structure that we use in this paper. Further details can be found in [9, 10, 1, 2, 3, 4].

DEFINITIONS 1. An H-space is a pair $(X, \{\Gamma_A\})$ where X is a topological space and $\{\Gamma_A\}$ is a given family of non empty contractible subsets of X, indexed by the finite subsets of X, such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$. Let $(X, \{\Gamma_A\})$ be an H-space.

A subset $D \subset X$ is called *H-convex* if for every finite subset $A \subset D$ it follows $\Gamma_A \subset D$. A subset $D \subset X$ is called *weakly H-convex* if, for every finite subset $A \subset D$, the intersection $\Gamma_A \cap D$ is nonempty and contractible.

A subset $K \subset X$ is called *H-compact* if for every finite subset $A \subset X$ there is a compact, H-convex set $D \subset X$ such that $K \cup A \subset D$. A subset $X_1 \subset X$ is called compactly closed if X_1 is closed relative to every compact subset of X.

A multifunction $F : X \to X$ is called H-KKM if $\Gamma_A \subset \bigcup_{x\in A} F(x)$, for every finite subset $A \subset X$.

Theorem 2 below is proved as an application of the following generalization of the KKM Theorem given in [2].

THEOREM 1 [2]. *Let $(X, \{\Gamma_A\})$ be an H-space and $F : X \to X$ an H-KKM multifunction such that*

(a) *For every $x \in X$, $F(x)$ is compactly closed;*

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(b) *There is a compact set $L \subset X$ and an H-compact $K \subset X$ such that, for each weakly H-convex set D with $K \subset D \subset X$, we have $\bigcap_{x \in D}(F(x) \cap D) \subset L$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Note that hypothesis (b) is more general than the following one:

(I) There is an H-compact $K \subset X$ such that $\bigcap_{x \in K} F(x)$ is compact.

Hence, if X is compact, property (I) and so (b) is immediately fulfilled.

THEOREM 2. *Let $(X, \{\Gamma_A\})$ be an H-space, $D \subset X$ an H-convex subset. Let $G \subset D \times D$ be a subset such that*

(a) *For every $x \in D$, the set $\{y \in D : (x, y) \in G\}$ is compactly closed in D;*
(b) *For every $y \in D$, the set $\{x \in D : (x, y) \notin G\}$ is H-convex or empty;*
(c) *For every $x \in D$, $(x, x) \in G$;*
(d) *There is an H-compact $D_0 \subset D$ such that the set $T = \{y \in D : (x, y) \in G$ for every $x \in D_0\}$ is compact.*

Then there is $y_0 \in T$ such that $D \times \{y_0\} \subset G$.

Observe that in the case of a compact space X, condition (d) is automatically satisfied, so the statement of Theorem 2 becomes simpler.

PROOF. For every $x \in D$, let $F(x) = \{y \in D : (x, y) \in G\}$. By (a), $F(x)$ is compactly closed in D. By (b) and (c), if $A = \{x_1, \dots, x_n\} \subset D$, then $\Gamma_A \subset \bigcup_{i=1}^n F(x_i)$. Indeed if $z \in \Gamma_A$ and $z \notin \bigcup_{i=1}^n F(x_i)$, then $z \notin F(x_i)$ for every $i = 1, \dots, n$, that is $(x_i, z) \notin G$, for every $i = 1, \dots, n$ and so $\Gamma_A \subset \{x \in D : (x, z) \notin G\}$. In particular $(z, z) \notin G$, a contradiction. Finally, by (d), $\bigcap_{x \in D_0} F(x) = T$ is compact. Then, by Theorem 1 there is $y_0 \in \bigcap_{x \in D} F(x)$ that is $D \times \{y_0\} \subset G$.

We note that Theorem 2 was also proved in the meantime and independently by Chen [7, Theorem 1], using the same method based on our Theorem 1.

3. Some definitions concerning modular spaces

Let E be a real vector space. A functional $\rho : E \to [0, +\infty]$ is called a *modular* if the following conditions are verified

(1) $\rho(x) = 0$ if and only if $x = 0$;

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(2) $\rho(-x) = \rho(x)$, for every $x \in E$;

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for every $\alpha, \beta \geq 0, \alpha + \beta = 1$, and $x, y \in E$.

Let $E_\rho = \{x \in E : \rho(\lambda x) < +\infty$, for every $\lambda > 0\}$. It is well known that E_ρ is a vector subspace of E (see [14]).

A sequence $(x_n)_n \subset E_\rho$ is said to be *ρ-convergent* or *modular convergent* to x in E_ρ (we denote it by $x_n \xrightarrow{\rho} x$) if there is $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ for divergent n. A subset $A \subset E_\rho$ is called *ρ-closed* if for every sequence $(x_n)_n \subset A$, with $x_n \xrightarrow{\rho} x$, then $x \in A$. A subset $A \subset E_\rho$ is called *ρ-compact* if every sequence $(x_n)_n$ has a subsequence ρ-convergent in A. The topological space E_ρ provided with this topology will be denoted again by E_ρ.

A subset $D \subset E_\rho$ has the *property* (θ) if every sequence $(x_n)_n$ ρ-convergent to x_0 in D admits a subsequence $(x_{k_n})_n$ such that $\rho(x_{k_n}) \to \rho(x_0)$ as n diverges.

At first, we give an example of a subset of a modular space with property (θ). Let $(\Omega, \mathscr{F}, \mu)$ be a measurable space where μ is a non-negative, non-atomic and finite measure on Ω. Let X be the space of all μ-measurable real functions on Ω. Let us consider the Orlicz class E^ϕ with respect to the modular

$$\rho(x) = \int_\Omega \phi(|x(t)|) d\mu(t), \qquad x \in X,$$

where ϕ is a ϕ-function [14, page 4].

A set $D \subset E^\phi$ is *sequentially equi-absolutely ϕ-integrable* if for every $\epsilon > 0$ and every sequence $(x_n)_n$ in D there is $\delta > 0$, such that $\int_A \phi(|x_n(t)|) d\mu(t) < \epsilon$ for every $A \in \mathscr{F}$ with $\mu(A) < \delta$.

It is easy to show that every sequentially equi-absolutely ϕ-integrable subset D of E^ϕ has the property (θ). Indeed, if $x_n \xrightarrow{\rho} x_0$ in D, then there is a subsequence $x_{k_n} \to x_0$ almost everywhere in Ω. Since ϕ is continuous, $\phi(|x_{k_n}|) \to \phi(|x_0|)$ almost everywhere in Ω, too. By the Vitali Theorem $\rho(x_{k_n}) \to \rho(x_0)$ and so property (θ) holds.

Remark that the subspace ρ-topology on a subset $D \subset E_\rho$ with the property (θ) is weaker than the strong topology induced by the F-norm

$$\|x\|_\rho = \inf\left\{u > 0 : \rho\left(\frac{x}{u}\right) \leq 1\right\}.$$

As an example we consider the Orlicz class E^ϕ induced by $\phi(u) = e^u - 1$, and the sequence $x_n : [0, 1] \to \mathbb{R}$ defined by $x_n = n\chi_{[0,1/ne^n]}$. Let D be the ρ-closure of the set $\{x_n : n \in \mathbb{N}\}$. Since $x_n \xrightarrow{\rho} x_0 = 0$ in D, this holds for every its subsequence and so D satisfies the property (θ). But $(\rho(2x_n))_n$ diverges.

The aim of this paper is to state non-convex best approximation theorems in modular spaces. For this purpose we introduce in E_ρ an H-space structure. For every finite subset $A \subset E_\rho$ let us denote by Γ_A the corresponding contractible set. The space E_ρ with this H-space structure will be called an H-modular space.

4. A best approximation theorem for single valued functions

In this section we state a best approximation result as well as a fixed point theorem for functions defined in H-convex subset of E_ρ. In order to do this, we introduce the following definition.

DEFINITION 2. Let D be any subset in E. A function $f : D \to E_\rho$ is called *ρ-continuous* if for every sequence $(x_n)_n \subset D$ with $x_n \xrightarrow{\rho} x$ in D we have $f(x_n) \xrightarrow{\rho} f(x)$.

Now we are ready to prove the main theorem of this section.

THEOREM 3. *Let $C \subset E_\rho$ be an H-convex set and $f : C \to E_\rho$ a ρ-continuous function such that $(C - f(C))$ has the property (θ). Assume that there is an H-ρ compact $C_0 \subset C$ such that the set $T = \{y \in C : \rho(x - f(y)) \geq \rho(y - f(y))$, for every $x \in C_0\}$ is ρ-compact. Finally suppose that*

(+) *for every $w \in E_\rho$, $\lambda \in \mathbb{R}$ the set $\{x \in E_\rho : \rho(x - w) < \lambda\}$ is H-convex.*

Then there is $y_0 \in T$ such that

$$\rho(y_0 - f(y_0)) = \min_{x \in C} \rho(x - f(y_0)).$$

PROOF. Let $G = \{(x, y) \in C \times C : \rho(x - f(y)) \geq \rho(y - f(y))\}$. By definition, condition (c) and (d) of Theorem 2 of Section 2 are fulfilled. Now, we prove that the section $G(x) = \{y \in C : (x, y) \in G\}$ is ρ-closed for every $x \in C$. Let x be fixed in C.

Let $(y_n)_n \subset G(x)$ a sequence ρ-convergent to y_0. By ρ-continuity of f, $(f(y_n))_n$ ρ-converges to $f(y_0)$ and so $(x - f(y_n))_n$ ρ-converges to $(x - f(y_0))$ and $(y_n - f(y_n))_n$ ρ-converges to $(y_0 - f(y_0))$. Since $\rho(x - f(y_n)) \geq \rho(y_n - f(y_n))$, by the property (θ) we have:

$$\rho(x - f(y_0)) \geq \rho(y_0 - f(y_0))$$

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and so $y_0 \in G(x)$, that is, $G(x)$ is ρ-closed.

Let now $y \in C$, be fixed. By property (+), the set $C \setminus G(y) = \{x \in C : (x, y) \notin G\}$ is H-convex, if non-empty, on putting $w = f(y)$ and $\lambda = \rho(y - f(y))$. Therefore by Theorem 2 there is $y_0 \in T$ such that $C \times \{y_0\} \subset G$ or $\rho(x - f(y_0)) \geq \rho(y_0 - f(y_0))$, for every $x \in C$ and so the assertion follows.

COROLLARY 1. (Fixed point theorem) *Under the assumptions of Theorem* 3, *if $f(C) \subset C$, there is $y_0 \in T$ such that $f(y_0) = y_0$.*

PROOF. It is sufficient to observe that $\min_{x \in C} \rho(x - f(y_0)) = 0$.

REMARKS. (1) In the convex case, that is, $\Gamma_A = \mathrm{co}A$ for every finite set $A \subset E_\rho$, the property (+) of Theorem 3 is equivalent to quasi-convexity of the modular ρ, namely the set $\{x \in E_\rho : \rho(x) < \lambda\}$ is convex for every $\lambda \in \mathbb{R}$.

(2) If the H-space structure in E_ρ is *translation invariant*, that is, for every finite subset $A \subset E_\rho$ and $y \in E_\rho$, we have $\Gamma_A - y = \Gamma_{A-y}$, then the property (+) is equivalent to saying that the set $\{x \in E_\rho : \rho(x) < \lambda\}$ is H-convex, for every $\lambda \in \mathbb{R}$.

(3) When ρ is an F-norm in a vector space E, with $\rho(\alpha x) \leq \rho(x)$ for every $x \in E$, $\alpha \in [0, 1]$, every subset $D \subset E$ has the property (θ) with respect to strong convergence. Thus, in this particular case, Theorem 3 is a non-compact and non-convex version for the homogeneous subadditive functionals of the Ky Fan Extension of Browder's fixed point theorem [8].

(4) In Theorem 3 the property (θ) assumed on $C - f(C)$ can be relaxed to the following one:

(θ^*) for all sequences $(x_n)_n$, $(y_n)_n$ from $C - f(C)$ ρ-convergent to x_0, y_0 respectively with $\rho(x_n) \geq \rho(y_n)$, we have $\rho(x_0) \geq \rho(y_0)$.

It is easy to show that the property (θ) implies the property (θ^*).

(5) All the results of Section 4 can be extended to the whole modular space L_ρ, namely $L_\rho = \{x \in E : \rho(\lambda x) \to 0\}$. In this case it is sufficient to require in Theorem 3 the following further condition: for every $y \in C$, there is $x \in C$ such that $\rho(x - f(y)) < +\infty$.

5. A best approximation theorem for multifunctions

In this section we state a best approximation result as well as a fixed point theorem for multifunctions defined in H-convex subsets of E_ρ. In order to do

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this, we will introduce some background concepts.

Given a subset $K \subset E_\rho$ we set $\rho(K, x) = \inf_{t \in K} \rho(t - x)$.

Let 2^{E_ρ} be the family of non-empty subsets of E_ρ and for $A, B \in 2^{E_\rho}$,

$$e_\lambda(A, B) = \sup_{a \in A} \inf_{b \in B} \rho(\lambda(a - b)).$$

DEFINITION 3. A sequence $(A_n)_{n=0,1,2,\dots}$, $A_n \in 2^{E_\rho}$, is said to be *ρ-convergent* to A_0 *in the sense of Hausdorff* (*$H\rho$-convergent*) if there is $\lambda > 0$ such that $\lim_{n\to+\infty} \max(e_\lambda(A_n, A_0), e_\lambda(A_0, A_n)) = 0$. We will write briefly $A_n \overset{H\rho}{\to} A_0$.

DEFINITION 4. A multifunction $L : E_\rho \to 2^{E_\rho}$ is said to be *continuous in the sense of Hausdorff* (*$H\rho$-continuous*) if $L(y_n) \overset{H\rho}{\to} L(y_0)$ whenever $y_n \overset{\rho}{\to} y_0$.

The following result gives some properties of $H\rho$-continuous multifunctions.

PROPOSITION 1. *Let $L : E_\rho \to 2^{E_\rho}$ be an $H\rho$-continuous multifunction with ρ-closed values. If $y_n \overset{\rho}{\to} y_0$ in E_ρ, then the following properties hold:*

(i) *for every $z_0 \in L(y_0)$ there is $z_n \in L(y_n)$ such that $z_n \overset{\rho}{\to} z_0$;*

(ii) *for every sequence $(z_n)_{n=0,1,2,\dots}$ with $z_n \in L(y_n)$ for $n \geq 1$ and $z_n \overset{\rho}{\to} z_0$ then $z_0 \in L(y_0)$.*

PROOF. (i) Let $z_0 \in L(y_0)$ be fixed. By $H\rho$-continuity of L, from $y_n \overset{\rho}{\to} y_0$ it follows that $L(y_n) \overset{H\rho}{\to} L(y_0)$; thus there is $\lambda > 0$ such that $\lim_{n\to+\infty} e_\lambda(L(y_0), L(y_n)) = 0$. So, for every $\epsilon > 0$ there is $\bar{n}_\epsilon$ such that $\sup_{z\in L(y_0)} \inf_{t\in L(y_n)} \rho(\lambda(z - t)) < \epsilon$, for every $n > \bar{n}_\epsilon$, from which $\inf_{t\in L(y_n)} \rho(\lambda(z_0 - t)) < \epsilon$, for every $n > \bar{n}_\epsilon$. Hence for every $n > \bar{n}_\epsilon$ there is $t_n \in L(y_n)$ such that $\rho(\lambda(z_0 - t_n)) < \epsilon$ that is $t_n \overset{\rho}{\to} z_0$.

(ii) Let $(z_n)_{n=0,1,2,\dots}$ with $z_n \in L(y_n)$, $n \geq 1$ and $z_n \overset{\rho}{\to} z_0$. By $H\rho$-continuity of L there is $\lambda > 0$ such that $\lim_{n\to+\infty} e_\lambda(L(y_n), L(y_0)) = 0$. So for every $\epsilon > 0$ there is $\bar{n}_\epsilon$ such that for every $n > \bar{n}_\epsilon$ and $t \in L(y_n)$ we have $\inf_{z\in L(y_0)} \rho(\lambda(t - z)) < \epsilon$. In particular $\inf_{z\in L(y_0)} \rho(\lambda(z_n - z)) < \epsilon$ where $z_n \in L(y_n)$ and $z_n \overset{\rho}{\to} z_0$. Hence for every $z_n \in L(y_n)$, $n > \bar{n}_\epsilon$, there is $\bar{z}_n \in L(y_0)$ such that $\rho(\lambda(z_n - \bar{z}_n)) < \epsilon/2$. From $z_n \overset{\rho}{\to} z_0$ it follows $\lim_{n\to+\infty} \rho(\tilde{\lambda}(z_n - z_0)) = 0$ for some $\tilde{\lambda} > 0$ and so we can assume $\rho(\tilde{\lambda}(z_n - z_0)) < \epsilon/2$, for every $n > \bar{n}_\epsilon$. Putting $\mu = \min\{\lambda, \tilde{\lambda}\}$, by properties of ρ we have for $n > \bar{n}_\epsilon$,

$$\rho\Big(\frac{\mu}{2}(\bar{z}_n - z_0)\Big) = \rho\Big(\frac{\mu}{2}(\bar{z}_n - z_n + z_n - z_0)\Big) \leq \rho(\mu(\bar{z}_n - z_n)) + \rho(\mu(z_n - z_0)) < \epsilon$$

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and so $\bar{z}_n \xrightarrow{\rho} z_0$. The assertion follows by ρ-closedness of $L(y_0)$.

Now, we introduce the following generalization of the convexity of the modular ρ. Let

$$\Sigma^{(n-1)} = \left\{(t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i=1}^{n} t_i = 1;\ t_i \geq 0,\ i = 1, \dots, n\right\}$$

be the $(n-1)$-dimensional standard simplex. We say that ρ has the property (+) if given $A = \{x_1, \dots, x_n\} \subset E_\rho$, $B = \{y_1, \dots, y_n\} \subset E_\rho$, for every $z \in \Gamma_A$ there are $\xi \in \Gamma_B$ and $(t_1(z,\xi), \dots, t_n(z,\xi)) \in \Sigma^{(n-1)}$ such that $\rho(z-\xi) \leq \sum_{i=1}^{n} t_i(z,\xi)\rho(x_i - y_i)$.

Note that in the convex case, that is, $\Gamma_A = \text{co}A$ for every finite subset $A \subset E_\rho$, the property (+) generalizes the convexity of ρ.

In [3] we introduced the following concept of H-convexity: for every $\Delta = \{w_1, \dots, w_n\} \subset E_\rho$, and for every $\eta \in \Gamma_\Delta$ there is an n-tuple $(t_1(\eta), \dots, t_n(\eta)) \in \Sigma^{(n-1)}$ such that $\rho(\eta) \leq \sum_{i=1}^{n} t_i(\eta)\rho(w_i)$.

In order to compare the property (+) with H-convexity we remark the following.

Given $A = \{x_1, \dots, x_n\}$, $B = \{y_1, \dots, y_n\}$, let $\Delta = \{x_1 - y_1, \dots, x_n - y_n\}$. If for every $z \in \Gamma_A$ there is $\xi \in \Gamma_B$ such that $z - \xi \in \Gamma_\Delta$, then H-convexity implies the property (+).

However the following proposition holds.

PROPOSITION 2. *If ρ satisfies the property (+), then the function $\rho(K, \cdot)$ is H-convex in E_ρ for every H-convex $K \subset E_\rho$.*

PROOF. Let $A = \{x_1, \dots, x_n\} \subset E_\rho$ be fixed. For every $m \in \mathbb{N}^+$ there is $T_m = \{\tau_1^m, \dots, \tau_n^m\} \subset K$ such that

$$\rho(K, x_i) > \rho(\tau_i^m - x_i) - \frac{1}{m}, \qquad i = 1, \dots, n. \tag{1}$$

For a fixed $\xi \in \Gamma_A$ and $m \in \mathbb{N}^+$, by property (+) there is $\tau_m^* \in \Gamma_{T_m} \subset K$ and $\{t_1(\tau_m^*, \xi), \dots, t_n(\tau_m^*, \xi)\}$ in Σ^{n-1} such that

$$\rho(K, \xi) \leq \rho(\tau_m^* - \xi) \leq \sum_{i=1}^{n} t_i(\tau_m^*, \xi)\rho(\tau_i^m - x_i). \tag{2}$$

For every $m \in \mathbb{N}^+$ we put $t^m = (t_1(\tau_m^*, \xi), \dots, t_n(\tau_m^*, \xi))$. By the compactness of Σ^{n-1}, the sequence $(t^m)_{m\in\mathbb{N}^+}$ admits a subsequence still denoted by $(t^m)_{m\in\mathbb{N}}$ convergent to $\bar{t} = (\bar{t}_1(\xi), \dots, \bar{t}_n(\xi)) \in \Sigma^{n-1}$. By (1) and (2) $\rho(K,\xi) < \sum_{i=1}^n t_i(\tau_m^*, \xi)\rho(K, x_i) + 1/m$ for every $m \in \mathbb{N}^+$ and so $\rho(K,\xi) \leq \sum_{i=1}^n \bar{t}_i(\xi)\rho(K, x_i)$, as asserted.

Finally we will use the following elementary result.

PROPOSITION 3. *Let X be any topological space, $\phi : X \to \mathbb{R}^+$ be a given function such that for every sequence $(x_n)_n$ convergent to x_0 in X there is a subsequence $(x_{k_n})_n$ such that $\phi(x_{k_n}) \to \phi(x_0)$. Then, if X is compact, ϕ attains its minimum value.*

Let $C \subset E_\rho$ be an H-convex set and L be a multifunction defined on C. From now on we suppose that

(i) the set $L(C) = \bigcup_{x\in C} L(x)$ is ρ-compact;
(ii) the set $C - L(C)$ has the property (θ);
(iii) the multifunction L is $H\rho$-continuous with non-empty ρ-closed and H-convex values.

We refer to these assumptions as hypotheses (I).

The following lemma gives a key result for the main theorem of this section (Theorem 4).

LEMMA 1. *Suppose hypotheses* (I) *hold. If $(y_n)_n \subset C$ and $y_n \xrightarrow{\rho} y_0$ in C; then there is a subsequence $(y_{k_n})_n$ such that $\rho(L(y_{k_n}), x) \to \rho(L(y_0), x)$ for every $x \in C$.*

PROOF. For a fixed $x \in C$, and $n \in \mathbb{N}$ we define $\phi(t) = \rho(x - t)$, $t \in L(y_n)$. The function ϕ satisfies the hypotheses of Proposition 3. Indeed, let $(t_k)_k \subset L(y_n)$ be a sequence ρ-convergent to $t_0 \in L(y_n)$. As $(x - t_k) \xrightarrow{\rho} (x - t_0)$ in $C - L(y_n)$, then by (ii) there is a subsequence $(t_{h_k})_k$ such that $\rho(x - t_{h_k}) \to \rho(x - t_0)$, that is, $\phi(t_{h_k}) \to \phi(t_0)$. So by Proposition 3 and the ρ-compactness of $L(y_n)$ there is $\xi_n \in L(y_n)$ such that $\inf_{t\in L(y_n)} \rho(x - t) = \rho(x - \xi_n)$.

Let us consider the sequence of minimizing elements $(\xi_n)_n$. By ρ-compactness of $L(C)$ there is a subsequence $(\xi_{k_n})_n$ and $\xi_0 \in L(C)$ such that $\xi_{k_n} \xrightarrow{\rho} \xi_0$ and $\rho(\xi_{k_n} - x) \to \rho(\xi_0 - x)$. By $H\rho$-continuity of L, $\xi_0 \in L(y_0)$. So we have

$$\lim_{n\to+\infty} \rho(L(y_{k_n}), x) = \rho(\xi_0 - x) \geq \rho(L(y_0), x).$$

Next, we prove that equality occurs. If it is false, there is a $\delta > 0$ and $\bar{n} \in \mathbb{N}$ such that for every $n > \bar{n}$ we have $\rho(L(y_{k_n}), x) - \rho(L(y_0), x) > \delta$ and so

$$\rho(t - x) - \rho(L(y_0), x) > \delta, \qquad \text{for every } t \in L(y_{k_n}). \tag{3}$$

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Let $z_0 \in L(y_0)$ be fixed. By $H\rho$-continuity of L there is a sequence $(z_{k_n})_n$ with $z_{k_n} \in L(y_{k_n})$ such that $z_{k_n} \xrightarrow{\rho} z_0$ and by (3)

$$\rho(z_{k_n} - x) - \rho(L(y_0), x) > \delta, \qquad \text{for every } n \in \mathbb{N}.$$

Finally, by (ii) using a further subsequence, we have $\rho(z_0 - x) - \rho(L(y_0), x) \geq \delta$, for every $z_0 \in L(y_0)$, a contradiction.

THEOREM 4. *Suppose that property* (+) *on* ρ *and hypotheses* (I) *hold. If there is an* H-ρ *compact* $C_0 \subset C$ *such that the set* $T = \{y \in C :$ *there is* $z \in L(y)$ *with* $\rho(L(y), x) \geq \rho(z - y)$, *for every* $x \in C_0\}$ *is* ρ-*compact, then there is* $y_0 \in T$ *such that*

$$\min_{z \in L(y_0)} \rho(y_0 - z) = \inf_{x \in C} \rho(L(y_0), x). \tag{4}$$

Observe that if C is ρ-compact the existence of such a subset C_0 is automatically satisfied, so the statement of Theorem 4 becomes simpler.

PROOF. Let $G = \{(x, y) \in C \times C :$ there is $z \in L(y)$ with $\rho(L(y), x) \geq \rho(z - y)\}$. At first, we prove that $(x, x) \in G$, for every $x \in C$. Fixed $x \in C$, let $\phi : L(x) \to \mathbb{R}^+$ defined by $\phi(t) = \rho(x - t)$. As proved in Lemma 1, the function ϕ satisfies the hypotheses of Proposition 3. So, there is $z \in L(x)$ such that $\rho(x - z) = \min_{t \in L(x)} \rho(x - t) \equiv \rho(L(x), x)$, that is, $(x, x) \in G$.

Next, we prove that for every fixed $x \in C$ the set $G(x) = \{y \in C :$ there is $z \in L(y)$ with $\rho(L(y), x) \geq \rho(z - y)\}$ is ρ-closed.

Let $(y_n)_n \subset G(x)$ with $y_n \xrightarrow{\rho} y_0$. For every $n \in N$, let $z_n \in L(y_n)$ such that

$$\rho(L(y_n), x) \geq \rho(z_n - y_n) \tag{5}$$

By (i) and (iii) there is a subsequence $(z_{k_n})_n$ ρ-convergent to $z_0 \in L(y_0)$. By (ii) and Lemma 1 we can suppose, up to subsequences, that $\rho(z_{k_n} - y_{k_n}) \to \rho(z_0 - y_0)$ and $\rho(L(y_{k_n}), x) \to \rho(L(y_0), x)$. Hence by (5)

$$\rho(L(y_0), x) = \lim_n \rho(L(y_{k_n}), x) \geq \lim_n \rho(z_{k_n} - y_{k_n}) = \rho(z_0 - y_0),$$

that is, $G(x)$ is ρ-closed.

Finally, we prove that for every $y \in C$ the set

$$C \setminus G(y) = \{x \in C : \text{ for every } z \in L(y),\ \rho(L(y), x) < \rho(z - y)\}$$

is H-convex.

Let $y \in C$ and $z \in L(y)$ be fixed. Let $A = \{x_1, \dots, x_n\} \subset C \setminus G(y)$, then

$$\rho(L(y), x_i) < \rho(z - y), \qquad \text{for every } i = 1, \dots, n. \tag{6}$$

For a fixed $\xi \in \Gamma_A$, by Proposition 2, H-convexity of $L(y)$ and (6), there is $(\bar{t}_1(\xi), \dots, \bar{t}_n(\xi)) \in \Sigma^{n-1}$ such that

$$\rho(L(y), \xi) \leq \sum_{i=1}^{n} \bar{t}_i(\xi)\rho(L(y), x_i) < \rho(z - y),$$

which is the H-convexity of $C \setminus G(y)$.

Finally, hypothesis (d) of Theorem 1 is automatically fulfilled. By Theorem 1, the assertion follows.

COROLLARY 2. (Fixed point theorem) *Under the assumptions of Theorem* 4, *if* $L(C) \subset C$, *there is* $y_0 \in T$, *such that* $y_0 \in L(y_0)$.

PROOF. Let $y_0 \in T$ satisfy (4). If $x \in L(y_0) \subset C$ then

$$\inf_{z \in L(y_0)} \rho(x - z) = 0, \quad \text{and so} \quad \inf_{x \in C} \inf_{z \in L(y_0)} \rho(x - z) = 0.$$

Therefore the assertion follows again by (4).

REMARKS. Theorem 4 is an extension in various directions (non-convex and non-compact case, multifunctions, non-homogeneous functionals) of Ky Fan's extension of Browder's fixed point theorem [8].

Among the other generalizations to multifunctions of the Ky Fan theorem [8], we limit ourselves to quote Theorem 1 of Sehgal-Singh in [15]. There, the authors work with convex subsets in a topological vector space and use a relaxed compactness hypothesis, different with respect to our one.

In the compact case, our theorem is an extension to H-modular spaces of the Sehgal-Singh theorem.

Finally we note that all the results of Section 5 can be extended to the whole modular space L_ρ, namely, $L_\rho = \{x \in E : \lim_{\lambda \to 0} \rho(\lambda x) = 0\}$. In this case it is sufficient to require in Theorem 4 the following further condition: for every $x, y \in C$, there is $z \in L(y)$ such that $\rho(z - x) < +\infty$.

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