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DECOMPOSING LINEAR TRANSFORMATIONS

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Abstract

Let R be the ring of linear transformations of a right vector space over a division ring D. Three results are proved: (1) if $|D| > 4$, then for any $a \in R$ there exists a unit u of R such that $a + u$, $a - u$ and $a - u^{-1}$ are units of R; (2) if $|D| > 3$, then for any $a \in R$ there exists a unit u of R such that both $a + u$ and $a - u^{-1}$ are units of R; (3) if $|D| > 2$, then for any $a \in R$ there exists a unit u of R such that both $a - u$ and $a - u^{-1}$ are units of R. The second result extends the main result in H. Chen, ['Decompositions of countable linear transformations', *Glasg. Math. J.* (2010), doi:10.1017/S0017089510000121] and the third gives an affirmative answer to the question raised in the same paper.

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Let R be a ring with identity and let $U(R)$ be the group of units of R. In this note, we are concerned with the following three conditions on R:

$$\forall a \in R,\ \exists u \in U(R) \text{ such that } a + u,\ a - u,\ a - u^{-1} \in U(R). \tag{O}$$

$$\forall a \in R,\ \exists u \in U(R) \text{ such that } a + u,\ a - u^{-1} \in U(R). \tag{P}$$

$$\forall a \in R,\ \exists u \in U(R) \text{ such that } a - u,\ a - u^{-1} \in U(R). \tag{Q}$$

Connections of these conditions with some well-known conditions in ring theory will be briefly explained later. In 1954 Zelinsky [9] proved that every element in the ring of linear transformations of a right vector space over a division ring D is a sum of two units unless $D = \mathbb{Z}_2$ and $\dim(V) = 1$. This is the motivation for the work of Chen [4] where it is proved that the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| \neq 2, 3$ satisfies (P). Chen [4] is also motivated to raise the question whether the ring of linear transformations of a countably generated right vector space over a division ring D with $|D| \neq 2$ satisfies (Q). The main result of this note is the following theorem.

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THEOREM 1. *Let* $\mathrm{End}(V_D)$ *be the ring of linear transformations of a right vector space* V *over a division ring* D.

(1) *If* $|D| > 4$, *then* $\mathrm{End}(V_D)$ *satisfies* (O).
(2) *If* $|D| > 3$, *then* $\mathrm{End}(V_D)$ *satisfies* (P).
(3) *If* $|D| > 2$, *then* $\mathrm{End}(V_D)$ *satisfies* (Q).

Part (2) of the theorem is an improvement of the main result of [4, Theorem 5] where (2) is proved for any countably generated vector space V. Part (3) of the theorem is an affirmative answer to Chen's question [4, p. 6] whether the ring of linear transformations of a countably generated right vector space over a division ring of more than two elements satisfies (Q).

Three lemmas are needed for the proof of the theorem. For a countably infinite-dimensional right vector space V_D, a linear transformation $f \in \mathrm{End}(V_D)$ is called a *shift operator* if there exists a basis $\{v_1, v_2, \ldots, v_n, \ldots\}$ of V such that $f(v_i) = v_{i+1}$ for all i.

LEMMA 2. *Let* V *be a countably infinite-dimensional right vector space over a division ring* D *and let* $f \in \mathrm{End}(V_D)$ *be a shift operator. Then there exists* $g \in U(\mathrm{End}(V_D))$ *such that* $f + g,\ f - g,\ f - g^{-1} \in U(\mathrm{End}(V_D))$.

PROOF. By fixing a basis of V_D, we can identify f with a matrix

$$A = \begin{pmatrix} X & 0 & 0 & \cdots \\ Y & X & 0 & \cdots \\ 0 & Y & X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{where } X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} X & 0 & 0 & \cdots \\ 0 & X & 0 & \cdots \\ 0 & 0 & X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ Y & 0 & 0 & \cdots \\ 0 & Y & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $B^2 = C^2 = 0$ and $A = B + C$. Thus, $1 + B$ is invertible with inverse $1 - B$. We see that $A - (1 + B) = C - 1$ is invertible, and

$$A - (1 - B) = \begin{pmatrix} 2X - 1 & 0 & 0 & \cdots \\ Y & 2X - 1 & 0 & \cdots \\ 0 & Y & 2X - 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is invertible with inverse

$$\begin{pmatrix} -(2X+1) & 0 & 0 & \cdots \\ -(2X+1)Y(2X+1) & -(2X+1) & 0 & \cdots \\ 0 & -(2X+1)Y(2X+1) & -(2X+1) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

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and

$$A+(1+B)=\begin{pmatrix} 1+2X & 0 & 0 & \cdots \\ Y & 1+2X & 0 & \cdots \\ 0 & Y & 1+2X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is invertible with inverse

$$\begin{pmatrix} 1-2X & 0 & 0 & \cdots \\ -(1-2X)Y(1-2X) & 1-2X & 0 & \cdots \\ 0 & -(1-2X)Y(1-2X) & 1-2X & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This completes the proof. □

The $n \times n$ matrix ring over a ring R is denoted by $\mathbb{M}_n(R)$. Part (3) of Lemma 3 comes from Chen [3, Theorem 4.1]. But the proof given here is shorter.

LEMMA 3. *Let R be a ring and $n \geq 1$.*

(1) *If for any $a, b, c \in R$ there exists $u \in U(R)$ such that $a+u$, $b-u$, $c-u^{-1}$ are units of R, then the same is true of $\mathbb{M}_n(R)$.*

(2) *If for any $a, b \in R$ there exists $u \in U(R)$ such that $a+u$, $b-u^{-1}$ are units of R, then the same is true of $\mathbb{M}_n(R)$.*

(3) *If for any $a, b \in R$ there exists $u \in U(R)$ such that $a-u$, $b-u^{-1}$ are units of R, then the same is true of $\mathbb{M}_n(R)$.*

PROOF. (1) If $n = 1$, there is nothing to prove. Suppose that $n > 1$ and let

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}$$

be matrices in $\mathbb{M}_n(R)$, where the upper left-hand blocks are elements of R, the upper right-hand blocks are $1 \times (n-1)$ matrices, the lower left-hand blocks are $(n-1) \times 1$ matrices, and the lower right-hand blocks are matrices in $\mathbb{M}_{n-1}(R)$. By our assumption, there exists $u \in U(R)$ such that $x := \alpha_{11} + u$, $y := \beta_{11} - u$, $z := \gamma_{11} - u^{-1}$ are all units of R. Now $\alpha_{22} - \alpha_{21}x^{-1}\alpha_{12}$, $\beta_{22} - \beta_{21}y^{-1}\beta_{12}$, $\gamma_{22} - \gamma_{21}z^{-1}\gamma_{12}$ are matrices in $\mathbb{M}_{n-1}(R)$. By the induction hypothesis, there exists a unit μ of $\mathbb{M}_{n-1}(R)$ such that

$$X := (\alpha_{22} + \alpha_{21}x^{-1}\alpha_{12}) + \mu,$$
$$Y := (\beta_{22} - \beta_{21}y^{-1}\beta_{12}) - \mu,$$
$$Z := (\gamma_{22} - \gamma_{21}z^{-1}\gamma_{12}) - \mu^{-1}$$

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are units of $\mathbb{M}_{n-1}(R)$. Then $\lambda := \left(\begin{smallmatrix} u & 0 \\ 0 & \mu \end{smallmatrix}\right)$ is a unit of $\mathbb{M}_n(R)$ such that

$$\alpha + \lambda = \begin{pmatrix} x & \alpha_{12} \\ \alpha_{21} & \alpha_{21}x^{-1}\alpha_{12} + X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha_{21}x^{-1} & 1 \end{pmatrix} \begin{pmatrix} x & \alpha_{12} \\ 0 & X \end{pmatrix},$$

$$\beta - \lambda = \begin{pmatrix} y & \beta_{12} \\ \beta_{21} & \beta_{21}y^{-1}\beta_{12} + Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta_{21}y^{-1} & 1 \end{pmatrix} \begin{pmatrix} y & \beta_{12} \\ 0 & Y \end{pmatrix},$$

$$\gamma - \lambda^{-1} = \begin{pmatrix} z & \gamma_{12} \\ \gamma_{21} & \gamma_{21}z^{-1}\gamma_{12} + Z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \gamma_{21}z^{-1} & 1 \end{pmatrix} \begin{pmatrix} z & \gamma_{12} \\ 0 & Z \end{pmatrix}$$

are all units of $\mathbb{M}_n(R)$. This completes the proof.

The proofs of (2) and (3) are similar to the proof of (1). □

Part (2) of Lemma 4 below comes from Chen [4, Lemma 2].

LEMMA 4. *Let D be a division ring and $n \geq 1$.*

(1) *If $|D| > 4$, then $\mathbb{M}_n(D)$ satisfies (O).*
(2) *If $|D| > 3$, then $\mathbb{M}_n(D)$ satisfies (P).*
(3) *If $|D| > 2$, then $\mathbb{M}_n(D)$ satisfies (Q).*

PROOF. (1) It is easily seen that if $|D| > 4$ then for any $a, b, c \in D$ there exists $u \in U(D)$ such that $a + u$, $b - u$, $c - u^{-1}$ are units of D. Thus (1) follows from Lemma 3(1).

The proofs of (2) and (3) are similar to the proof of (1). □

PROOF OF THEOREM 1. (1) Let $f \in \mathrm{End}(V_D)$. Let $\mathcal{S}$ be the set of all ordered pairs (W, g), where W is an f-invariant subspace of V and g, $f|_W + g$, $f|_W - g$, and $f|_W - g^{-1}$ are units of $\mathrm{End}(W_D)$ (where $f|_W$ is the restriction of f to W). Clearly, $((0), 1) \in \mathcal{S}$.

Define a partial ordering on $\mathcal{S}$ by setting $(W', g') \leq (W, g)$ whenever both are in $\mathcal{S}$, $W' \subseteq W$ and $g' = g|_{W'}$.

Suppose that $\{(W_\alpha, g_\alpha) : \alpha \in \Lambda\}$ is a totally ordered subset of $\mathcal{S}$. We define $g \in \mathrm{End}((\cup W_\alpha)_D)$ by setting $g(x) = g_\alpha(x)$ ($\alpha \in \Lambda$, $x \in W_\alpha$), and it is easy to see that $(\cup W_\alpha, g) \in \mathcal{S}$ and $(W_\alpha, g_\alpha) \leq (\cup W_\alpha, g)$ for all $\alpha \in \Lambda$. It follows from Zorn's lemma that there exists a maximal element (U, h) in $\mathcal{S}$; we prove (1) by showing that $U = V$. Hence we assume that $U \neq V$, and show that this leads to a contradiction.

Let us fix $x \in V \setminus U$. Let $V_0 := U + K$ where K is the subspace of V spanned by $\{x, f(x), f^2(x), \ldots\}$, and write $V_0 = U \oplus N$ where N is a nonzero subspace of V_0. Since U is f-invariant, there is a linear transformation $\bar{f} : V_0/U \to V_0/U$ given by $\bar{f}(\bar{v}) = \overline{f(v)}$ (for $v \in V_0$). Let $\pi : V_0 \to N$ be the projection on N along U. There is a natural isomorphism $\varphi : V_0/U \to N$ such that $\varphi(\bar{v}) = \pi(v)$ (for $v \in V_0$). Thus $\theta := \varphi \bar{f} \varphi^{-1} \in \mathrm{End}(N_D)$, and so $\theta\varphi = \varphi\bar{f}$. Since V_0/U is spanned by $\{\bar{x}, \bar{f}(\bar{x}), \bar{f}^2(\bar{x}), \ldots\}$, N is spanned by $\{\varphi(\bar{x}), \varphi(\bar{f}(\bar{x})), \varphi(\bar{f}^2(\bar{x})), \ldots\} = \{\varphi(\bar{x}), \theta\varphi(\bar{x}), \theta^2\varphi(\bar{x}), \ldots\}$. Thus, either $\theta \in \mathrm{End}(N_D)$ is a shift operator or N_D is finite-dimensional. So, by Lemmas 2 and 4(1), there exists $\alpha \in U(\mathrm{End}(N_D))$ such

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that $\theta+\alpha$, $\theta-\alpha$ and $\theta-\alpha^{-1}$ are all units of $\mathrm{End}(N_D)$. Let $g : V_0 \to V_0$ be given by $g(u+v) = h(u) + \alpha(v)$ $(u \in U,\ v \in N)$. Then g is a unit of $\mathrm{End}((V_0)_D)$.

We next show that $f+g$, $f-g$ and $f-g^{-1}$ are units of $\mathrm{End}((V_0)_D)$. For $u \in U$ and $v \in N$,

$$(f-g)(u+v) = (f-h)(u) + [f(v) - \alpha(v)]. \qquad (*)$$

Applying π to both sides of $(*)$ gives

$$\begin{aligned}\pi(f-g)(u+v) &= \pi f(v) - \alpha(v) = \varphi\overline{f(v)} - \alpha(v) = \varphi\bar{f}(\bar{v}) - \alpha(v)\\ &= \theta\varphi(\bar{v}) - \alpha(v) = \theta\pi(v) - \alpha(v) = \theta(v) - \alpha(v)\\ &= (\theta-\alpha)(v).\end{aligned}$$

If $(f-g)(u+v) = 0$, then $(\theta-\alpha)(v) = 0$ and so $v = 0$. It follows from $(*)$ that $(f-h)(u) = 0$, and hence $u = 0$. Thus, $f-g : V_0 \to V_0$ is one-to-one.

Clearly, $U \subseteq \mathrm{Im}(f-g)$. For any $w \in N$, there exists an element $v \in N$ such that $(\theta-\alpha)(v) = w$. Thus, $w = (\theta-\alpha)(v) = \pi(f-g)(u+v) \in \mathrm{Im}(f-g)$ because $U \subseteq \mathrm{Im}(f-g)$. So $f-g : V_0 \to V_0$ is onto. Hence $f-g$ is a unit of $\mathrm{End}((V_0)_D)$.

Similarly, one can show that $f+g$, $f-g^{-1}$ are units of $\mathrm{End}((V_0)_D)$.

Thus, $(V_0, g) \in \mathcal{S}$ and $(U, h) \le (V_0, g)$, contradicting the maximality of (U, h). So $U = V$ and the proof is complete.

The proofs of (2) and (3) are similar to the proof of (1). □

Following Menal and Moncasi [6], a ring R is said to satisfy *unit* 1*-stable range* if, whenever $aR + bR = R$, there exists $u \in U(R)$ such that $a + bu \in U(R)$. This condition has been discussed by several authors. For example, Menal and Moncasi [6] proved that if R satisfies the unit 1-stable range condition, then $K_1(R) = U(R)/V(R)$, where $V(R)$ is the subgroup of $U(R)$ generated by $\{(ab+1)(ba+1)^{-1} : ab+1 \in U(R)\}$. The unit 1-stable range is always satisfied by a ring R such that, for any $x, y \in R$, there exists $u \in U(R)$ such that $x-u$ and $y-u^{-1}$ are both units of R (see Goodearl and Menal [5]). The latter condition is called the *Goodearl–Menal condition* by Chen [4]. Proposition 9 in [4] and the remarks on page 6 in [4] indicate that, for a semilocal ring R, R satisfies (P) if and only if R satisfies the Goodearl–Menal condition if and only if no homomorphic image of R is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_3$. On the other hand, by [8, Corollary 4] and the remarks on page 6 in [4], one has that, for a semilocal ring R, R satisfies (Q) if and only if R satisfies unit 1-stable range if and only if no homomorphic image of R is isomorphic to $\mathbb{Z}_2$.

It is easy to verify that the ring $\mathbb{Z}_3$ satisfies (Q), but not (P); and any field of four elements satisfies (P), but not (O). Condition (O) certainly implies both (P) and (Q), but it is unknown whether (P) implies (Q). We close with a sufficient condition for (P) to imply (Q). A ring R is called *right continuous* if every right ideal is essential in a direct summand of R_R and every right ideal isomorphic to a direct summand of R_R is itself a direct summand. The Jacobson radical of a ring R is denoted by $J(R)$.

PROPOSITION 5. *Let $R/J(R)$ be a right continuous ring. If R satisfies (P), then it satisfies (Q).*

PROOF. Because every unit of $R/J(R)$ can be lifted to a unit of R, R satisfies (P) (respectively (Q)) if and only if $R/J(R)$ satisfies (P) (respectively (Q)). Thus, we can assume that R is semiprimitive, right continuous. By Utumi [7], R is von Neumann regular; so 2 is a regular element of R. By [10, Lemma 7], $R = S \times T$ where 2 is a unit of S and 2 is a nilpotent element of T. Thus $2 \in J(T) \subseteq J(R)$. Since $J(R) = 0$, $2 = 0$ in T. Since R satisfies (P), T satisfies (P). This, together with the fact that $2 = 0$ in T, implies that T satisfies (Q). It remains to show that S satisfies (Q). Because R is right continuous, S is right continuous. So S is a clean ring by [1, Theorem 3.9], and $2 \in U(S)$. Thus, by [2, Theorem 11], for any $a \in S$, $a = u + v$ where $u \in U(S)$ and $v^2 = 1$. This shows $a - v = a - v^{-1} = u \in U(S)$. So S satisfies (Q). Hence $R = S \times T$ satisfies (Q). □

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