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A characterization of boolean spaces

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A boolean space is a compact Hausdorff space which is zero-dimensional. In this paper, a boolean space X is characterized in terms of its ring of real-valued functions $C(X)$. The result is sharpened for the case when X is an F-space (every finitely generated ideal of $C(X)$ is principal).

1. Introduction

A boolean space is a compact Hausdorff space which is zero-dimensional. The purpose of this paper is to characterize a boolean space X in terms of its ring of real-valued continuous functions $C(X)$. The result will be sharpened for the case when X is an F-space (every finitely generated ideal of $C(X)$ is principal).

2. B-rings

Let S be a commutative ring with identity 1, and let $\{M_\alpha \mid \alpha \in A\}$ be the set of all maximal ideals of S. The Jacobson radical of S is the set $J(S) = \cap\{M_\alpha \mid \alpha \in A\}$. S is called a B-ring if for each integer $n \geq 3$ and each $s_1, \ldots, s_n \in S$ such that $(s_1, \ldots, s_{n-2}) \not\subseteq J(S)$ and $1 \in (s_1, \ldots, s_n)$, there exists $t \in S$ such that $1 \in (s_1, \ldots, s_{n-2}, s_{n-1}+ts_n)$; see [4] for details. Here, the notation $(s_1, \ldots, s_n)$ means the ideal of S generated by $s_1, \ldots, s_n$.

Since every set of the form $M_x = \{f \in C(X) \mid f(x) = 0\}$ is a maximal ideal of $C(X)$, it follows that if $g \in J(C(X))$ then $g \in M_x$ for each

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$x \in X$ so that $g(x) = 0$ for each $x \in X$, or equivalently, $g = 0$. Hence, $J(C(X)) = (0)$. We can now simplify the definition of B-rings in the special case of $C(X)$.

PROPOSITION 2.1. *$C(X)$ is a B-ring if and only if $f, g, h \in C(X)$ with $f \neq 0$ and $1 \in (f, g, h)$ implies there exists $t \in C(X)$ such that $1 \in (f, g+th)$.*

Proof. The direct implication is obvious. To see the converse let $n \geq 3$ with $(f_1, \ldots, f_{n-2}) \nsubseteq (0)$ and $1 \in (f_1, \ldots, f_n)$; then $f_1^2 + \ldots + f_{n-2}^2 \neq 0$ and $Z\left(f_1^2 + \ldots + f_n^2\right) = \emptyset$, where $Z\left(f_1^2 + \ldots + f_n^2\right)$ denotes the zero set of the function $f_1^2 + \ldots + f_n^2$. Note that $Z\left(\left[f_1^2 + \ldots + f_{n-2}^2\right]^2 + f_{n-1}^2 + f_n^2\right) = \emptyset$ must also hold. Consequently, $1 \in \left(f_1^2 + \ldots + f_{n-2}^2, f_{n-1}, f_n\right)$. By hypothesis, there exists $t \in C(X)$ such that $1 \in \left(f_1^2 + \ldots + f_{n-2}^2, f_{n-1}+tf_n\right)$. From this we see that $Z\left(f_1^2 + \ldots + f_{n-2}^2 + \left[f_{n-1}+tf_n\right]^2\right) = Z\left(\left[f_1^2 + \ldots + f_{n-2}^2\right]^2 + \left[f_{n-1}+tf_n\right]^2\right) = \emptyset$. Therefore, $1 \in \left(f_1, \ldots, f_{n-2}, f_{n-1}+tf_n\right)$.

3. B-rings and boolean spaces

In this section we shall assume that X is a compact Hausdorff space. We begin by proving a lemma similar to Lemma 4.3 of [1].

LEMMA 3.1. *Let $f, g, h \in C(X)$ and denote $g^{-1}(0, \infty)$ as $P(g)$ and $g^{-1}(-\infty, 0)$ as $N(g)$. If there is a connected subset Z of $Z(f)$ such that $(Z \cap Z(h)) \cap P(g) \neq \emptyset$ and $(Z \cap Z(h)) \cap N(g) \neq \emptyset$, then for each $t \in C(X)$, $1 \notin (f, g+th)$.*

Proof. Note that there must be $x, y \in Z$ such that $(g+th)(x) > 0$ and $(g+th)(y) < 0$. Since Z is connected, the continuity of $g + th$ implies the existence of some $z \in Z$ such that $(g+th)(z) = 0$. This shows that $Z(f) \cap Z(g+th) \neq \emptyset$, or equivalently, $1 \notin (f, g+th)$.

LEMMA 3.2. *If $C(X)$ is a B-ring, then for each closed connected set Z and each closed set S, $Z \cap S$ must be connected.*

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Proof. The proof follows Lemma 4.5 of [1]. Suppose that Z is a closed connected set and that S is a closed set such that $Z \cap S$ is not connected. Write $Z \cap S = F_1 \cup F_2$ where F_1, F_2 are disjoint non-empty closed subsets of $Z \cap S$, hence closed subsets of X. Since X is assumed to be a compact Hausdorff space, and therefore normal, there are open sets $U_1 \supseteq F_1$ and $U_2 \supseteq F_2$ whose closures are disjoint. Put $U = U_1 \cup U_2$. The closed sets $Z - U$ and $S - U$ are disjoint, hence contained in disjoint open sets V_1, V_2 respectively. By Urysohn's Lemma, choose $f, g, h \in C(X)$ such that $f(Z) = 0$ and $f(X-V_1-U) = 1$, $g(\overline{U}_1) = 1$ and $g(\overline{U}_2) = -1$, $h(S) = 0$ and $h(X-V_2-U) = 1$. Then f, g, h satisfy the hypothesis of the previous lemma and $1 \in (f, g, h)$. It follows that $C(X)$ is not a B-ring.

THEOREM 3.3. *Let X be a compact Hausdorff space. If $C(X)$ is a B-ring, then X is a boolean space.*

Proof. Let $x \in X$. If C is the connected component of X containing x, then C is a closed connected set. If $C \neq \{x\}$ then it would follow that the discrete set $C \cap \{x, y\} = \{x, y\}$ must be connected, where $y \in C - \{x\}$. We conclude that $C = \{x\}$ and, hence, X is totally disconnected. By compactness, X is zero-dimensional.

Next we prove the converse of Theorem 3.3. We begin by defining $A(X)$ to be all those functions $f \in C(X)$ whose range is a finite set. In particular, $A(X)$ contains the constant functions. It is well known that for compact spaces X, we may apply the Stone-Weierstrass Theorem to conclude that $A(X)$ is dense in $C(X)$, under the topology of uniform convergence, if X is zero-dimensional.

THEOREM 3.4. *If X is a boolean space, then $C(X)$ is a B-ring.*

Proof. If $f, g, h \in C(X)$ with $1 \in (f, g, h)$, then a straightforward computation shows that there exist $\delta, \varepsilon > 0$ such that if $f', g', h' \in C(X)$ with $|f-f'| < \varepsilon$, $|g-g'| < \varepsilon$, and $|h-h'| < \varepsilon$ then $|f'| + |g'| + |h'| > \delta$. Let $\xi = \min(\varepsilon, \delta/3)$ and choose $f', g', h' \in A(X)$ within ξ of f, g, h respectively. Note then that $|f'| + |g'| + |h'| > \delta$.

Since functions in $A(X)$ have finite range, it follows that there exist functions $u, v, w \in A(X)$ satisfying $uf' = |f'|$, $vg' = |g'|$, $wh' = |h'|$, and $|u| = |v| = |w| = 1$. Define $c \in A(X)$ by $c = 1/(|f'|+|g'|+|h'|) < 1/\delta$. Choosing $p = uc$, $q = vc$, and $t = w/v$ gives $1 = pf' + q(g'+th')$. Thus, we have appropriately written the identity in the subring $A(X)$.

Now set $d_1 = f' - f$, $d_2 = g' - g$, and $d_3 = h' - h$; then $|d_i| < \xi$ for each i and

$$\begin{aligned} 1 &= p(f+d_1) + q[(g+d_2)+t(h+d_3)] \\ &\le |pf+q(g+th)| + |pd_1+qd_2+qtd_3| . \end{aligned}$$

Letting $s = |pd_1+qd_2+qtd_3|$ it follows that $1 - s \le |pf+q(g+th)|$. By direct calculation,

$$\begin{aligned} s &\le |p|\cdot|d_1| + |q|\cdot|d_2| + |qt|\cdot|d_3| \\ &< (1/\delta)\cdot\xi + (1/\delta)\cdot\xi + (1/\delta)\cdot\xi \le 1 . \end{aligned}$$

This gives that $0 < 1 - s \le |pf+q(g+th)|$ so that $pf + q(g+th)$ is a unit in $C(X)$. Since $pf + q(g+th) \in (f, g+th)$, it follows that $1 \in (f, g+th)$.

Thus, we have shown that a compact space X is a boolean space if and only if $C(X)$ is a B-ring. It is interesting to note that we did not need $f \ne 0$.

By assuming X is Lindelöf and using the Stone-Čech compactification of X , one can easily show that X is zero-dimensional if and only if $C^*(X)$ is a B-ring.

4. SB-rings and boolean F-spaces

Let S be a commutative ring with identity. S is called an SB-ring if for each $s, c, d, e \in S$ with $s \in (c, d, e)$ and $c \notin J(S)$, it follows that $s \in (c, d+te)$ for some $t \in S$; see [4] for details.

A topological space X is called an F-space if every finitely generated ideal of $C(X)$ is principal. X is called a T-space if $C(X)$ is an Hermite ring; and X is called a U-space if for each $f \in C(X)$ there exists a unit $u \in C(X)$ such that $f = u|f|$. In [1] it is shown that every U-space is a T-space.

THEOREM 4.1. *Suppose X is a compact F-space. Then X is a boolean space if and only if $C(X)$ is an SB-ring.*

Proof. Since every SB-ring is a B-ring [4, p. 457], it suffices to show that if X is a boolean F-space then $C(X)$ is an SB-ring. Now, every boolean F-space is a U-space [1, Theorem 5.5]. Hence, X is a T-space and $C(X)$ is a Hermite ring. Since Hermite B-rings are SB-rings [4, Theorem 3.3], it follows that $C(X)$ is an SB-ring.

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