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KRASNOSEL'SKII THEOREMS FOR NON-SEPARATING COMPACT SETS

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ABSTRACT. Let $S \subset R^d$, $d \geq 2$, be compact and let E denote the set $(d-2)-$extreme points of S. M. Breen has shown that if E is countable and $S \neq E$, then S is planar. A new proof of this result is given as well as a Krasnosl'skii theorem for $(d-2)$ extreme points which combines and generalizes previous results.

1. **Introduction.** If $S \subset R^d$, let E denote the set of $(d-2)$-extreme points of S. In [1], M. Breen proved that if S is compact, E countable and $S \neq E$, then S is planar. Section 2 of this paper gives a significantly shorter and more straightforward proof of her result. In [1], two Krasnosel'skii type theorems were proven. Section 3 of this paper gives a theorem which yields the latter two results as corollaries and its proof requires much less machinery than is used in [1]. Throughout, we employ the terminology of [1].

2. **The cardinality of** E. The following is Theorem 1 of [1] and we state it in the contrapositive form.

THEOREM 1. *Let $S \subset R^d$ be compact. If $S \not\subset R^2$ then $S = E$ or card $E = c$.*

Proof. Let H be a hyperplane with $H \cap S \neq \emptyset$. We claim $\text{ext}(H \cap S) \subset E$. Suppose not. Then there exists $x \in \text{ext}(H \cap S)$ and a $(d-1)$-simplex $D \subset S$ with $\mathbf{x} \in \text{rel int } D$. Then $\dim(D \cap H) \geq d-2 \geq 1$ and $x \in \text{rel int}(D \cap H)$, a contradiction.

We now prove the theorem in the case that S is connected. Without loss of generality, we suppose $S \not\subset R^{d-1}$. Then there exists a hyperplane H and an open half-space H^+ of H such that $S \cap H \neq \emptyset$ and $S \cap H^+ \neq \emptyset$. Let $x \in S \cap H^+$ and $y \in S \cap H$. Let $\mathscr{H}$ be the family of hyperplanes given by $\{H_z \mid z \in [x, y]$, with $z \in H_z$ and H_z parallel to $H\}$. Since S is connected and a hyperplane separates R^d, we must have $H_z \cap S \neq \emptyset$ for all $z \in [x, y]$. Since any two elements of $\mathscr{H}$ have empty intersection and card $\mathscr{H} = c$ we will be done if for any $z \in [x, y]$ we have that $H_z \cap E \neq \emptyset$. But the latter is true by the claim of the first paragraph, and this completes the proof in the case that S is connected.

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To prove the general case, not that if $S=E$ we are done, so suppose $S\neq E$. Let $x\in S\sim E$. Then there exists a $(d-1)$-simplex $D\subset S$ with $x\in\operatorname{rel\,int} D$. Let F be the flat generated by D and let C be the component of $F\cap S$ containing D. Now note rel int $C\neq\emptyset$ and so C is a compact, connected set of topological dimension $k=d-1\geq 2$. Hence card rel bd $C)=c$. If rel bd $C\subset E$, we are done. Thus, suppose there exists $y\in(\operatorname{rel\,bd} C)\sim E$. Let G be a $(d-1)$-simplex, with $G\subset S$ and $y\in\operatorname{rel\,int} G$. Note $G\not\subset F$, for otherwise we contradict that $y\in$ rel bd C. Then $G\cup C$ is a compact, connected subset of S with $G\cup C\not\subset R^2$. Let Q be the component of S containing $G\cup C$. Note that $Q\not\subset R^2$ and that any $(d-2)$-extreme point of Q is a $(d-2)$-extreme point of S. The proof is completed by applying the connected case of the theorem to Q.

3. **Helly-type results and** $(d-2)$-**extreme points.** The following two results are the main results of Section 3 of [1].

THEOREM 2. *Let $S\subset R^d$, $d\geq 2$, be a non-empty compact set having the half-ray property. Suppose for some $\varepsilon>0$ every $f(d,k)$ or fewer points of E see via S a common k-dimensional ε-neighborhood, where $f(d,0)=f(d,k)=d+1$ and $f(d,k)=2d$ for $1\leq k\leq d-1$. Then S is starshaped and* dim Ker $S\geq k$.

THEOREM 3. *Let $S\subset R^d$, $d\geq 2$ be a non-empty compact set with $\sim S$ connected. Suppose for some $\varepsilon>0$, every $d+1$ or fewer points of E see via S a common d-dimensional ε-neighborhood. Then* dim Ker $S=d$.

The main tool in the proofs of Theorems 2 and 3 is the Lemma of [2] but both proofs required additional non-trivial lemmas. We will prove Theorem 4, from which Theorems 2 and 3 follow as corollaries. The proof will use the Lemma of [2] but will require no additional results.

THEOREM 4. *Let $S\subset R^d$, $d\geq 2$, be a non-empty compact set with $\sim S$ connected. Suppose for some $\varepsilon>0$, every $f(d,k)$ or fewer points of E see via S a common k-dimensional ε-neighborhood where $f(d,0)=f(d,d)=d+1$ and $f(d,k)=2d$ for $1\leq k\leq d-1$. Then S is starshaped and* dim Ker $S\geq k$.

Proof. Let $\mathcal{K}=\{\operatorname{conv} S_x \mid x\in E\}$. The hypotheses imply that every $f(n,k)$ members of $\mathcal{H}$ have a non-empty k-dimensional intersection. Note that $\mathcal{H}$ is a uniformly bounded family of compact convex sets. Depending on the value of k, Helly's theorem or the Lemma of [2] gives that $\dim\bigcap_{R\in H} R\geq k$. Let $z\in\bigcap_{R\in H} R$. To show $z\in\operatorname{Ker} S$ it suffices to prove that given any $y\in\sim S$, that $L(y,z)\subset\sim S$ where $L(y,z)$ si the closed half-line with vertex y not containing z determined by the line containing y and z. Suppose the latter is false. Without loss of generality we take z as 0_v, the origin, and suppose that $y\in\sim S$ with $L(y,0_v)\cap S\neq\emptyset$. Choose w with $w\notin\operatorname{conv} S$. Since $\sim S$ is an open connected set, it is polygonally connected. Since $w\in\sim S$, we may choose a polygonal arc $l\subset\sim S$ joining y and w. Let the vertices of l be $x_1, x_2, \ldots x_n$ with $x_1=w$ and

$x_n = y$. Since $l \subset \sim S$ there exists $\varepsilon > 0$ with $l_\varepsilon \subset \sim S$ where l_ε is the ball about l of radius ε in the Hausdorff metric. Since $w \in \operatorname{conv} S$, we have $L(w, 0_v) \subset \sim S$. Let l be the homeomorphic image of f on the interval $[1, n]$ with $f(i) = x_i$, $1 \leq i \leq n$. Let $j = \max\{i \mid L(x_i, 0_v) \cap S = \emptyset\}$. Let $C(L(x_j, 0_v), \delta)$ denote the closed half cylinder centered about $L(x_j, 0_v)$ of radius δ. Choose δ so that $\delta < \varepsilon/2$ and $C(L(x_j, 0_v), \delta) \cap S = \emptyset$. Let $\gamma = \sup\{\alpha \mid \alpha \in [j, j+1]$ and $C(L(f(\alpha), 0_v), \delta) \cap S = \emptyset\}$. Note $j < \gamma < j+1 \leq n$ and $B \cap S \neq \emptyset$, where $B = C(L(f(\gamma), o_v), \delta)$. Since $B \cap S$ is compact we may choose $q \in B \cap S$ with $\|q\| = \sup\{\|r\| \mid r \in B \cap S\}$ where $\| \ \|$ is the Euclidean norm. Since $\delta < \varepsilon/2$, q is not an element of $d-2$ dimensional sphere centered about $f(\gamma)$ at the "beginning" of B. Then there exists a unique hyperplane G of support to B containing q. The definition of B implies $S \cap \operatorname{int} B = \emptyset$. Thus we have $S_q \subset G^+$ where G^+ is the closed half-space of G not containing 0_v. Thus $\operatorname{conv} S_q \subset G^+$. We will be done if we can show $q \in E$ because this will contradict the fact that we have $z \in \bigcap_{R \in H} R$. Now suppose that $q \notin E$. Then there exists a $(d-1)$-simplex $D \subset S$ with $q \in \operatorname{rel\,int} D$. Note $D \subset G$, lest we contradict the definition of B. We then can produce $q_1 \in D \cap B$, with $\|q_1\| > \|q\|$, contradicting the definition of q.

In conclusion, we remark that the latter proof is an adaptation of an argument of Goodey used in [3] to generalize a result in [4].

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