AXIOMS FOR ELLIPTIC GEOMETRY

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Introduction. Until recently the literature contained little on the axiomatic foundations of elliptic geometry that was non-analytical and independent of projective geometry. During the past decade this subject has come in for further study, notably by Busemann [**2**] and Blumenthal [**1**], who supplied such foundations. This paper presents another and, it is believed, simpler effort in the same general direction, proceeding by the familiar synthetic methods of elementary geometry and using only elementary topological notions and ideas concerning metric spaces. Specifically, elliptic 2-space is obtained on the basis of six axioms, most notable of which is one assuming the existence of translations. The writer wishes to express his deep appreciation to Herbert Busemann for his invaluable help.

I. Some basic terms and notations

Small letters always denote points. The *distance* between two points a,b of any metric space is denoted by ab or ba. A point c is said to be *between* points a,b (denoted by acb or bca) if $c \neq a$ or b and $ac + cb = ab$, and is said to be a *midpoint* of a and b (denoted by $c = \text{mid}(a,b)$) if, moreover, $ac = cb$. An *arc*, a *simple arc*, and a *geodesic arc* mean, respectively, a continuous, a topological, and a congruent map of a closed Euclidean segment; a simple closed curve means the homeomorph of a Euclidean circle. An arc or geodesic arc with endpoints a,b is said to *lie between* a and b, and is denoted by (ab) or $[ab]$, respectively. When no confusion can arise "geodesic arc" is often shortened to "arc", as in the phrase "the arc $[ab]$". In the interest of clarity of presentation and ease of reference, as well as to offer brief proofs, the number of theorems used has been large. Each theorem, when first stated, is denoted merely by an Arabic numeral, the word "Theorem" being omitted. Some proofs are not given.

II. Geodesic arcs and straight lines

Axiom 1. Σ is a compact metric space with at least two points.

Σ is then bounded, and the function xy, where x and y are arbitrary points of Σ, has a maximum. We take this maximum as unit distance, calling points a and b *conjugate* if $ab = 1$.

Axiom 2. Any distinct points a,b have just one midpoint if non-conjugate, just two if conjugate.

1. *If abc, then a and b have a unique midpoint, and likewise for b and c.*

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Σ is *convex* by Axiom 2, i.e., there is a point between each two points. From Menger [**4**] we then infer Theorems 2 to 4; Theorem 5 is immediate; Theorem 6 follows from Theorem 1, Menger [**4**], and Axiom 1.

2. *There exists a geodesic arc between any two distinct points.*

3. *An arc (ab) is a geodesic arc if and only if its length equals ab, or if it is the shortest arc between a and b.*

4. *The geodesic arcs $[ab]$ are distinguished among all the arcs (ab) by the property that if p,q are inner points of $[ab]$, then apq or qpb or $p = q$.*

5. *If p is an inner point of $[ab]$, then apb.*

6. *There is just one geodesic arc between two non-conjugate points, just two between two conjugate points.*

7. *If $ab = 1$ the two geodesic arcs $[ab]$ have only a and b in common.*

8. *If p,q are inner points of $[ab]$, and apq, then pqb.*

9. *If abc, then there is only one arc $[ab]$ and one arc $[bc]$, $[ab] + [bc]$ is an arc $[ac]$, and b is an inner point of an arc $[ac]$.*

Axiom 3a. If abc and abd, then either $c = d$ or bcd or cbd.
b. If abc and bcd, where $ab + bc + cd \leqslant 1$, then abd.

Axiom 4. If c is a midpoint of a and b, then a point d exists such that $cd = 1$, cad, and cbd.

10. *If abc and abd, then either $c = d$; or acd, bcd; or adc, bdc.*

Proof. Menger showed that abc, abd, acd imply bcd [**4**, p. 107]. Similarly one can easily show that abc, abd, bcd imply acd. Now assume abc and abd. Then either $c = d$ or bcd or bdc by Axiom 3a. If bcd, then acd by the proposition stated two sentences back. If bdc, then adc, as can be seen by interchanging c and d in the proposition stated in the first sentence.

11. *The point d described in Axiom 4 is unique.*

Proof. Assume d' is another point having the same properties as d. Then $cd' = 1$, cad', and cbd'. Since cad and cad' we infer by Theorem 10 that either $d = d'$; or cdd', add'; or $cd'd$, $ad'd$. Now cdd' means that $cd + dd' = cd'$, where c,d,d' are all distinct, and this is impossible since $cd = cd' = 1$. Likewise $cd'd$ is impossible. Hence $d = d'$.

Definition 1. Let a,b be any distinct points, $c = \text{mid}(a,b)$, and d the unique point such that $cd = 1$, cad, and cbd. The point-set consisting of c, d, and all points between c and d is called a *straight line* (or *line*) determined by a and b.

12. *A unique straight line is determined by any two distinct points a and b, this straight line being denoted by -ab-. Every straight line is a simple closed curve of length 2.*

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Proof. If $ab < 1$, a and b have a unique midpoint c and hence determine a unique line. If d is the point such that $cd = 1$, cad, and cbd, it follows from Theorems 7 and 9 that this line is the simple closed curve of length 2 formed by the two arcs $[cd]$. If $ab = 1$, let $c' = \text{mid}(a,b)$ and d' be the point such that $c'd' = 1$, $c'ad'$, and $c'bd'$. As above, a and b determine a line consisting of the two arcs $[c'd']$. Now $ad' = \frac{1}{2}$ since $ac' = \frac{1}{2}$, $c'd' = 1$, and $c'ad'$. Likewise $bd' = \frac{1}{2}$. Hence

$$ad' + d'b = 1 = ab,$$

so that $d' = \text{mid}(a,b)$. By Theorem 9, $[ad'] + [d'b]$ is a geodesic arc between a and b, obviously not the geodesic arc between a and b which contains c'. Hence the line under discussion consists of the two arcs $[ab]$. If now we let c'' be the second midpoint of a,b and d'' the point such that $c''d'' = 1$, $c''ad''$, and $c''bd''$, then a and b will determine a line formed by the two arcs $[c''d'']$. But, as above, this line also consists of the two arcs $[ab]$, and it is clear that $c'' = d'$ and $c' = d''$. Hence a and b determine a unique line, which is again a simple closed curve of length 2.

13. *Every straight line is congruent to a Euclidean circle of length* 2.

Proof. Let $ab = 1$. Then, as shown in the proof of Theorem 12, -ab- consists of the two arcs $[cd]$, where $c = \text{mid}(a,b)$ and d is the unique point such that $cd = 1$, cad, and cbd. We take a Euclidean circle K of length 2, map a,b on any two antipodal points A,B of K, and c,d on C,D, the midpoints of A,B, then map geodesic arcs $[cad]$, $[cbd]$ congruently on semicircles CAD, CBD, respectively. Now if p,q be any distinct points of -ab-, and P,Q their corresponding points of K, we must show that $pq = PQ$, where PQ denotes the length of the shorter of the two arcs into which P and Q divide K. Since a,b,c,d divide -ab- into four equal quadrants, with a,b the midpoints of c,d, and vice versa, it is easy to see that $pq = PQ$ if p and q are in the same quadrant or in adjacent quadrants. But suppose p,q are interior points of opposite quadrants, e.g., let apc and bqd. Then

$$(pc + cb + bq) + (pa + ad + dq) = 2,$$

so that at least one expression in parentheses, say the first, does not exceed 1. Then $pc + cb + bq \leqslant 1$. Also pcb and cbq. Hence pcq by Axiom 3b, so that $pc + cq = pq \leqslant 1$. Since $PC = pc$, $CQ = cq$ we see that $PC + CQ \leqslant 1$. Hence $PQ = PC + CQ = pq$.

Now let $ab < 1$. Again -ab- consists of the two arcs $[cd]$, as above, and we map geodesic arcs $[cad]$, $[cbd]$ congruently on semicircles CAD, CBD, respectively. Let e,f be the midpoints of c,d, chosen so that eac and fbc. Thus we have eac, acb, and $ea + ac + cb < 1$, from which we infer eab by Axiom 3b. Then eac, eab, acb permit us to infer ecb by Theorem 10. Now ecb, cbf, and $ec + cb + bf = 1$ imply ecf by Axiom 3b, and from this we infer ebf by Theorem 10. Hence $ef = ec + cf = 1$, and we infer by Theorem 3 that ecf is a geodesic arc between e and f. Also, since $ed + df = 1 = ef$ we infer that (edf) is a geodesic arc between e and f. Thus c and d are the midpoints of e and f, as well as vice versa, so that the proof of the congruence of $\text{-}ab\text{-}$ and K is just like that in the case $ab = 1$ except that now we use e and f instead of a and b.

14. *If -ab- is any straight line and p,q are any distinct points of -ab-, then each arc [pq] is contained in -ab-.*

Proof. It follows from previous discussions that each arc on $\text{-}ab\text{-}$ of length $\leqslant 1$ is a geodesic arc. There are two arcs (pq) on $\text{-}ab\text{-}$. If they are of equal length, each has length 1 and hence is a geodesic arc; in this case $pq = 1$ and $\text{-}ab\text{-}$ consists of the two arcs $[pq]$. If the two arcs (pq) are unequal in length, only the shorter is a geodesic arc since its length is less than 1; in this case $pq < 1$, there is just one arc $[pq]$, and $\text{-}ab\text{-}$ contains it.

15. *If -ab- is any straight line and p,q are any distinct points of -ab-, then -pq- = -ab-.*

Proof. Let $pq = 1$. Then, as shown in the proof of Theorem 12, $\text{-}pq\text{-}$ consists of the two arcs $[pq]$. In the proof of Theorem 14 we saw that $\text{-}ab\text{-}$ also consists of these two arcs. Hence $\text{-}ab\text{-} = \text{-}pq\text{-}$. Now let $pq < 1$. Then $[pq] \subset \text{-}ab\text{-}$ by Theorem 14, so that also $r \subset \text{-}ab\text{-}$, where $r = \text{mid}(p,q)$. Let d be the unique point such that $rd = 1$, rpd, and rqd. Then, as shown in the proof of Theorem 12, $\text{-}pq\text{-}$ consists of the two arcs $[rd]$. Now let s be the point of $\text{-}ab\text{-}$ antipodal to r. Then $rs = 1$, rps, and rqs. It follows from Theorem 11 that $d = s$. Hence $\text{-}pq\text{-}$ consists of the two arcs $[rs]$. But, by the proof of Theorem 14, $\text{-}ab\text{-}$ also consists of these two arcs. Hence $\text{-}ab\text{-} = \text{-}pq\text{-}$.

Combining Theorems 12 and 15 we get:

16. *Any two distinct points are on a unique straight line.*

III. Our space Σ and the S. L. spaces of Busemann

An S. L. space (straight line space) is defined as one satisfying the following five axioms [**2**]:

A. It is metric.

B. It is finitely compact.

C. It is convex.

D. Each point p has an N-neighborhood $xp < \rho$, $\rho < 0$, such that for any distinct points a,b of N and each $\epsilon > 0$ there is a positive $\delta(a,b,\epsilon) \leqslant \epsilon$ for which a unique point b_σ exists such that $bb_\sigma = \sigma$ and abb_σ.

E. Any two distinct points are on, at most, one geodesic (a geodesic is a locally congruent map of the real axis, and hence is not a geodesic arc).

Clearly Σ has properties A, B, C. To show it has property D, let p be any point and let $\rho < \frac{1}{2}$. If a,b are distinct points in this ρ-neighborhood of p, then $ab \leqslant ap + pb < 1$. Let a,c divide $\text{-}ab\text{-}$ into equal geodesic arcs. Then b divides one of these arcs into arcs $[ab]$ and $[bc]$, and also abc. Let x be a point such that

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abx. Then abc and abx, so that $c = x$; or acx, bcx; or axc, bxc by Theorem 10. Since $ac = 1$, acx is impossible, so that x is on the unique arc $[bc]$ by Theorem 9. For every positive δ less than bc and ϵ there is, of course, a unique point x on $[bc]$ such that $bx = \delta$. Thus Σ has property D. To show that Σ has property E we first note that any line $\text{-}ab\text{-}$ is a geodesic since each point of $\text{-}ab\text{-}$ has a neighborhood on $\text{-}ab\text{-}$ which is a geodesic arc. Conversely, if G is any geodesic in Σ it contains two distinct points a,b such that an arc $[ab]$ is contained in G. Now a,b determine $\text{-}ab\text{-}$, which also contains this arc $[ab]$. Thus $\text{-}ab\text{-}$, a geodesic, and G, also a geodesic, both contain $[ab]$. But in a space with properties A, B, C, D a unique geodesic contains a given geodesic arc [**2**, p. 21]. Hence $G = \text{-}ab\text{-}$. It then follows by Theorem 16 that Σ has property E. We have thus proved:

17. *Σ is an* S. L. *space, its straight lines and geodesics being identical.*

Wishing to confine ourselves to plane geometry we assume:

AXIOM 5. Σ is two-dimensional in the sense of Menger-Urysohn.

Since Σ is a two-dimensional S. L. space whose geodesics are all simple closed curves, we can infer the following [**2**, pp. 79, 81]:

18. *Σ is a projective plane and each two of its straight lines meet in a unique point.*

19. *If p and L are any point and line of Σ, respectively, where $p \not\subset L$, and S is the set of all points on the lines joining p to each point of L, then $S = \Sigma$.*

IV. MOTIONS AND TRANSLATIONS

DEFINITION 2. A *motion* M is a single-valued, distance-preserving transformation of Σ into itself. $M(\alpha,\beta) = \alpha', \beta'$ means that M sends subsets α,β into subsets α',β', respectively. We say α is *fixed* under M if $\alpha' = \alpha$. A sequence of motions M_n *converges* to a motion M if $M_n(x) \to M(x)$ as $n \to \infty$ for each point x of Σ. (The existence of motions other than the identity is assumed later.)

20. *Motions are topological transformations, the set of all motions forming a group.*

21. *Any infinite sequence of motions has a convergent subsequence.*

Proof. If, for an infinite sequence of motions M_n of a finitely compact metric space, a point b exists for which the set $\{M_n(b)\}$ is bounded, then M_n contains a convergent subsequence [**2**, p. 177]. Σ is finitely compact and bounded. Hence $\{M_n(b)\}$, b being arbitrary, is bounded. The theorem then follows.

22. *Each motion sends between-points into between-points, midpoints into midpoints, conjugate points into conjugate points, geodesic arcs into geodesic arcs, and straight lines into straight lines.*

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To arrive at our definition of a translation let us suppose that a motion M has a fixed line L (the existence of motions with fixed lines is formally assumed later). If $[ab] \subset L$ and $M(a,b) = a',b'$, then $M([ab]) = [a'b'] \subset L$. Since L is congruent to a Euclidean circle and M preserves distance on L, it follows that if the oriented geodesic arcs $[ab]$, $[a'b']$ have the same sense so will each oriented geodesic arc $[xy]$ of L and its transform $[x'y'] = M([xy])$ have the same sense, whereas if $[ab]$, $[a'b']$ have unlike senses so will $[xy]$, $[x'y']$ have unlike senses. Thus M is either *sense-preserving* or *sense-reversing* on L.

DEFINITION 3. A *translation* (of Σ) along a line is a motion of Σ leaving that line fixed and preserving sense on it. (The existence of translations is assumed later.)

23. *The set of all translations along the same line forms a group.*

24. *Each infinite sequence of translations along the same line has a subsequence converging to a translation along that line.*

Proof. If L is the line, each infinite sequence of translations along L has a subsequence T_n converging to a motion T by Theorem 21. For any point p of L let $T(p) = p'$, $T_n(p) = p_n$. Then $p_n \subset L$, and $p_n \to p'$ when $n \to \infty$. Line L being a closed set, $p' \subset L$, that is, $T(L) = L$. If $q \subset L$, where $0 < pq < 1$, let $T(q) = q'$, $T_n(q) = q_n$. Then $q_n \to q'$. Since $[pq]$, $[p_nq_n]$ have the same sense, $[p'q']$ has this same sense. T must then preserve sense for all geodesic arcs of L, and hence be a translation along L.

25. *A translation along a line leaving a point of that line fixed leaves each point of the line fixed.*

26. *A translation leaving fixed each of two non-conjugate points leaves fixed each point of their line.*

DEFINITION 4. Distinct translations S, T along the same line are called *equivalent* along the line if $S(x) = T(x)$ for each point x of the line.

27. *Distinct translations S, T along the same line are equivalent along the line if a point p exists on the line so that $S(p) = T(p)$.*

Proof. Let L be the line, x any point of it, $S(p,x) = q,x'$, and $T(x) = x''$. Then $TS^{-1}(q,x') = q,x''$, the translation T being applied second. TS^{-1} leaves each point of L fixed by Theorems 23, 25. Hence $x' = x''$.

28. *Every motion (and hence every translation) has at least one fixed point.*

Proof. A motion being a continuous mapping and Σ being a projective plane, the assertion follows from the fact that a continuous mapping of a projective plane into itself has a fixed point [**3**, p. 80].

29. *A translation along a line having no fixed point on that line has one fixed point all told.*

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Proof. Let T and L be the translation and line, respectively. A point a exists so that $T(a) = a$. Let $b \neq a$, $T(b) = b$. Then $T(\text{-}ab\text{-}, L) = \text{-}ab\text{-}, L$ by Theorem 22. Let $\text{-}ab\text{-}, L$ meet in c. Then $T(c) = c$, which contradicts the hypothesis.

AXIOM 6. Distinct lines G, H exist, each with the property that if a, b are any points on it (not necessarily distinct), there are exactly two distinct translations along it sending a into b.

30. *There is just one translation along G other than the identity leaving each point of G fixed.*

This translation is denoted by R, the identity by I. (A corresponding assertion, of course, holds for H. For brevity we shall usually state things only in terms of G.)

31. *If S, T are equivalent translations along G, then $S^2 = T^2$ and $ST = TS$. Furthermore, $TS^{-1} = S^{-1}T = R$.*

Proof. Let $S(p) = T(p) = q$, where $p \subset G$. Then $T^{-1}S(p) = p$. Hence $T^{-1}S$, a translation along G, leaves each point of G fixed. Suppose $T^{-1}S = I$. Then

$$T(T^{-1}S) = TI = T,$$

so that $(TT^{-1})S = T$, or $IS = T$, and finally $S = T$, which is a contradiction. Hence $T^{-1}S = R$. Likewise

$$TS^{-1} = S^{-1}T = ST^{-1} = R.$$

From $ST^{-1} = S^{-1}T$ and $ST^{-1} = T^{-1}S$, respectively, we get $S^2 = T^2$ and $ST = TS$.

32. *If S, T are equivalent translations along G with no fixed point on G, they have a common fixed point, but no other fixed point.*

Proof. S and T have unique fixed points by Theorem 29, which we denote by f and g, respectively. Suppose $f \neq g$. Let $S(g) = g'$, $T(f) = f'$. Then $g \neq g'$, $f \neq f'$. Also

$$S^2(f) = T^2(f),$$

or $T(f') = f$. Likewise $S(g') = g$. By Theorem 31, $ST(f) = TS(f)$, or $S(f') = f'$, which contradicts the fact that S has f as its only fixed point. Hence $f = g$.

33. *All translations along G have a common fixed point, to be denoted by g.*

Proof. Let a, a_1 be points of G with $aa_1 = \frac{1}{2}$, and T_1 a translation along G such that $T_1(a,g) = a_1,g$, where g is the fixed point of T_1. Let $a_2 = \text{mid}(a,a_1)$ and $T_2(a) = a_2$; $a_3 = \text{mid}(a,a_2)$ and $T_3(a) = a_3$; and in general $a_n = \text{mid}(a,a_{n-1})$ and $T_n(a) = a_n$, where $n > 1$. Then $T_n(g) = g$ for all positive integers n. For any point x, where axa_1, we can construct a translation S from the translations T_n and their limiting translations such that $S(a,g) = x,g$. The powers of all

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such translations S send a into all the points of G. Now the totality of translations sending a into all the points of G is identical with the set of all translations along G. Hence if y, z are any points of G, at least one of the two translations along G sending y into z leaves g fixed. Call this translation U, and let V be the equivalent translation along G. If $y \neq z$ then, by Theorem 25, U and V have no fixed point on G; from Theorem 32 and the fact that $U(g) = g$ we then infer that $V(g) = g$. If $y = z$ we note that R and I are the only translations along G sending y into z, and that $I(g) = g$. To show that $R(g) = g$ let T_0, T_+ be any pair of translations equivalent along G, but with no fixed point on G. Then $T_+^{-1}T_0 = R$ by Theorem 31, and $T_0(g) = T_+(g) = g$, as just shown above. From this we see that $R(g) = g$.

34. *Each point of G is conjugate to g.*

Proof. gx is constant for any point x on G by Theorem 33 and Axiom 6. Assume $gx < 1$. Now $R(x,g) = x,g$. It follows from Theorem 26 that R leaves fixed each point of -gx-, and hence, by Theorem 19, each point of Σ, so that $R = I$. From this contradiction we infer $gx = 1$.

35. *The translation R has no fixed points other than g and each point of G.*

We let h denote the common fixed point for all translations along H.

36. *The points g and h are distinct, and g is on H if, and only if, h is on G.*

V. Rotations, poles and polars, and reflections

Definition 5. A motion leaving a point c fixed is called a *rotation about c*. If for all points x, y such that $xc = yc$ a rotation about c exists sending x into y, we say that *all rotations about c exist*.

37. *All rotations about g and h exist.*

Proof. Considering only g, let a, b be any points such that $ga = gb$. If a is on G, so is b, in which case a translation along G, that is, a rotation about g, exists sending a into b. Suppose a and b are not on G. Let -ga- $\neq$ -gb-, let -ga-, -gb- meet G in a', b', respectively, and let S, T be the distinct translations along G such that $S(a') = T(a') = b'$. Now each of these translations sends a into a point of -gb'- whose distance from g equals gb. Let b, b'' be the two points of -gb'- at distance gb from g. S and T cannot both send a into b'', for suppose $S(a) = T(a) = b''$. Since $S(a') = T(a') = b'$, we have $ST^{-1}(b', b'') = b', b''$. Hence ST^{-1}, a translation along G, has b', b'', as well as g, as fixed points. By Theorem 25, then, each point of G is fixed under ST^{-1}, and from Theorem 35 we see that $ST^{-1} = I$, and hence that $S = T$. Since this contradicts the fact that $S \neq T$, we infer that S, T cannot both send a into b''. Similarly they cannot both send a into b. Hence either $S(a) = b$ or $T(a) = b$. Finally, let -ga- = -gb-. Then $R(a) = b$, for R leaves -ga- fixed, but not point a, by Theorem 35.

38. *If all rotations about a point p exist, and a motion exists sending p into a point $q(\neq p)$, then all rotations about q exist.*

Proof. Let c, d be any points such that $qc = qd$, and M a motion such that $M(p) = q$, $M^{-1}(c,d) = a,b$. Then $pa = pb$. If N be a rotation about p such that $N(a) = b$, then $MNM^{-1}(q,c) = q,d$.

39. *All rotations exist about some point of G.*

Proof. If $g \subset H$ the assertion is a consequence of Theorems 36 and 37. If $g \not\subset H$, take point $c(\neq g)$ on $-gh-$ so that $hc = hg$. A rotation exists about h sending g into c; hence all rotations about c exist. Take $d(\neq h)$ on $-gh-$, so that $cd = ch$. There is a rotation about c sending h into d; hence all rotations about d exist. Thus we obtain a sequence of points $c,d,e, \ldots$ on $-gh-$ about each of which all rotations exist. The function hx, where x ranges over G, attains its maximum and minimum at points of G since G is a closed set and Σ is compact. Let G and H meet at p; then hp is such a maximum, with the value 1. Let a be a point of G such that ha is a minimum of hx.

Case 1. $gh \geqslant ha$. Then $hp > gh \geqslant ha$ since $1 > gh$. Since G is connected and closed, hx takes on all values between its maximum hp and minimum ha. Hence for some point $x = x'$ we have $hx' = gh$. A rotation about h exists sending g into x', so that all rotations exist about x', a point of G.

Case 2. $gh < ha$. Let $-gh-$ meet G in r, whence $gh + hr = gr$. Then $ha \leqslant hr$, so that $gh < hr$. The latter relation may be written

$$hr = K\cdot gh + F\cdot gh, \tag{1}$$

where K is a positive integer and $0 \leqslant F < 1$. We then have

$$hr = K\cdot gh + gh - (1 - F)gh,$$

or

$$hr + (1 - F)gh = (K + 1)gh. \tag{2}$$

Since $1 - F > 0$ we infer from (2) that

$$hr < (K + 1)gh. \tag{3}$$

Add gh to each side of (1), obtaining

$$hr + gh = (K + 1)gh + F\cdot gh$$

or

$$gr = (K + 1)gh + F\cdot gh.$$

Since $F\cdot gh \geqslant 0$, we get

$$(K + 1)gh \leqslant gr. \tag{4}$$

From (3) and (4) we have

$$hr < (K + 1)gh \leqslant gr.$$

Since $ha \leqslant hr$ we get

$$ha < (K + 1)gh \leqslant gr.$$

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Taking $N = K + 1$, and noting that $gr = hp = 1$, we obtain

$$ha < N \cdot gh \leqslant hp.$$

Now we take that one of the points $c,d,e, \ldots$ mentioned previously whose distance from h equals $N \cdot gh$, and denote the point by y. As shown in Case 1 there exists a point on G, which we denote by z, such that $hy = hz$. Hence a rotation about h exists which sends y into z. Since all rotations exist about y they likewise exist about z.

40. *All rotations exist about each point of G and H.*

DEFINITION 6. The locus of points conjugate to any point p is called the *polar* of p, and p is called the *pole* of the locus.

41. *G and H are the polars of g and h, respectively.*

42. *If a motion sends a point p into a point q, it sends the polar of p into the polar of q.*

43. *The polar of each point of G or H is a straight line.*

Proof. Let x be any point of G, and r its conjugate on G. Since $xg = xr = 1$, a rotation exists about x sending g into r. Some translation along G sends r into x. Hence a motion exists sending g into x, and hence G into X, the polar of x. Then X is a straight line by Theorem 22.

DEFINITION 7. A group of motions of a metric space into itself is *transitive* if for each pair of points a,b of the space there exists a motion of the group sending a into b.

44. *The group of all motions of Σ is transitive.*

Proof. Let x,y be any distinct points, p any point of G, P the polar of p, and q the intersection of G and P. Let $x' \subset P$ so that $xg = x'g$. A rotation exists about g sending x into x' by Theorem 37. Since $px' = pq = 1$, a rotation exists about p sending x' into q by Theorem 40. Let $y' \subset G$ so that $py = py'$. Some translation along G sends q into y', and a rotation exists about p sending y' into y. The resultant of these four motions is a motion sending x into y.

45. *All rotations exist about each point of Σ.*

Proof. This follows directly from Theorems 37, 38, 44.

46. *The polar of each point of Σ is a straight line.*

47. *Distinct points have distinct polars.*

Proof. Let a,b be any distinct points, with polars A,B, respectively. If $b \subset A$, clearly $A \neq B$. If $b \not\subset A$, let $\text{-}ab\text{-}$ meet A in c. Then $ac = 1$ and abc. Thus $ab + bc = ac$, so that $bc < ac$. Since bc, the distance from b to a point of A, is not 1 we infer that $A \neq B$.

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48. *Each straight line of Σ is the polar of some point.*

Proof. Let C be any line, a,b distinct points of C, and A,B the polars of a,b, respectively. A,B meet in a point c. Then $ca = cb = 1$ since c is on A and B. Hence the polar of c must contain a and b, and must therefore be $\text{-}ab\text{-}$. Thus C is the polar of c.

DEFINITION 8. A *k-dimensional linear subspace* of an S. L. space is any closed k-dimensional set (of the space) which, if it contains any two distinct points, also contains the geodesic through them. An *involutory motion* is a motion, not the identity, whose square is the identity. An involutory motion M of an S. L. space is called a *reflection* in the linear subspace S when all points of S are fixed under M, and S is maximal, i.e., it is not a proper subset of any other linear subspace whose points are fixed under M [**2**, pp. 113, 179].

49. *Each straight line is a 1-dimensional linear subspace of Σ.*

50. *If p is any point, and P its polar, one of the rotations about p is a reflection in P.*

Proof. Let $p = g$. Then $R^2 = I^2$ by Theorem 31, or $R^2 = I$. By Theorem 35 only g, apart from each point of G, is fixed under R. Thus R is a reflection in G by Theorem 49. Now let $p \neq g$, and $M(p) = g$, M being a motion. Then $M(P) = G$. Let x be any point of P, y any point between x and p, and $z(\neq y)$ the point on $\text{-}xp\text{-}$ such that $py = pz$. Let

$$M(x,y,z) = x',y',z',$$

in which case $M(\text{-}xp\text{-}) = \text{-}x'g\text{-}$ and y',z' are on $\text{-}x'g\text{-}$, with $gy' = gz'$. Then

$$M^{-1}RM(p,x) = p,x.$$

Thus the motion $M^{-1}RM$ leaves p, and also each point of P, fixed. But it leaves no other point of Σ fixed. For, suppose $M^{-1}RM(y) = y$, where y is any point of Σ not on P and distinct from p; then $RM(y) = M(y)$, or $R(y') = y'$, where, as above, $M(y) = y'$. But this contradicts Theorem 35. Now $M^{-1}RM \neq I$, otherwise $R = I$. Also $R(y') = z'$ since $gy' = gz'$. Hence

$$M^{-1}RM(p,x,y,z) = p,x,z,y,$$

so that $(M^{-1}RM)^2 = I$. Also $M^{-1}RM$ leaves fixed each point of the linear subspace P, and P is maximal. Hence $M^{-1}RM$ is a reflection in P, and since it leaves p fixed it is also a rotation about p.

From Theorems 48 and 50 we then obtain:

51. *A reflection exists in each straight line of Σ.*

DEFINITION 9. A metric space is called *homogeneous* if and only if it is congruent to a Euclidean, hyperbolic, or elliptic space of finite dimension.

52. *Σ is congruent to a two-dimensional elliptic space.*

Proof. An S. L. space is homogeneous if a reflection exists in each geodesic [**2**, p. 181]. Hence Σ is homogeneous by Theorems 17 and 51 and, being a projective plane, must be congruent to a two-dimensional elliptic space.

VI. A FINAL REMARK ON Σ AND S. L. SPACES

Busemann states that if all translations exist along two geodesics of a closed S. L. plane, the metric of the latter is elliptic [**2**, p. 219]. The proof of this, which was left to the writer, can now be supplied. For brevity we merely outline its main features. First, we can use results in [**2**] to show that Axioms 1 to 5 for Σ are valid propositions in any closed S. L. plane S. Thus S is a compact metric space, two points at maximum distance have exactly two midpoints, etc. Then, noting that Busemann's translations are defined somewhat differently than are translations in Σ, we can show, nevertheless, that the assumption that all of Busemann's translations exist along a geodesic of S implies that exactly two translations as defined in Σ exist along it, sending an arbitrary one of its points into an arbitrary one of its points. Then, whenever all Busemann's translations exist along two geodesics of S, so do all translations exist along them in the sense of Axiom 6. It follows that Axioms 1 to 6 are valid propositions in S if all Busemann's translations exist along two geodesics of S. The metric of S would then be elliptic by Theorem 52.

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