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QUASICONFORMAL SOLUTIONS OF POISSON EQUATIONS

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Abstract

The main aim of this paper is to establish the Lipschitz continuity of the (K, K')-quasiconformal solutions of the Poisson equation $\Delta w = g$ in the unit disk $\mathbb{D}$.

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1. Introduction

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

We consider the matrix norm

$$|A| = \max\{|Az| : z \in \mathbb{C}, |z| = 1\}$$

and the matrix function

$$\ell(A) = \min\{|Az| : z \in \mathbb{C}, |z| = 1\}.$$

Let D and G be subdomains of the complex plane $\mathbb{C}$ and let $w = u + iv : D \to G$ be a function that has both partial derivatives at a point z in D. By ∇u, we denote the matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Obviously,

$$|\nabla w| = |w_z| + |w_{\bar{z}}| \quad \text{and} \quad \ell(\nabla w) = ||w_z| - |w_{\bar{z}}||.$$

We say that a function $u : D \to \mathbb{R}$ is *absolutely continuous on lines* in the region D if, for every closed rectangle $R \subset D$ with sides parallel to the axes x and y, u is absolutely continuous on almost every horizontal line and almost every vertical line in R. Such a function has, of course, partial derivatives u_x and u_y almost everywhere in D.

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A sense-preserving homeomorphism $w : D \to G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be:

(1) (K, K')-*quasiconformal* if

  (a) w is absolutely continuous on lines in D,
  (b) there are constants $K \geq 1$ and $K' \geq 0$ such that $|\nabla w|^2 \leq KJ_w + K'$,

  where J_w denotes the Jacobian of w, given by $J_w = |w_z|^2 - |w_{\bar{z}}|^2 = |\nabla w|\ell(\nabla w)$;

(2) K-*quasiconformal* if $K' = 0$;

(3) *Lipschitz continuous* if there exists a constant L such that, for all $z_1, z_2 \in D$,

$$|w(z_1) - w(z_2)| \leq L|z_1 - z_2|.$$

Let P be the Poisson kernel, that is, the function

$$P(z, e^{i\theta}) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2},$$

and let G denote the Green function of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, that is,

$$G(z, \omega) = \frac{1}{2\pi} \log\left|\frac{1 - z\overline{\omega}}{z - \omega}\right|,$$

where $z \in \mathbb{D} \setminus \{\omega\}$. Then, P is harmonic in $\mathbb{D}$ (see [2]) and G is harmonic in $\mathbb{D} \setminus \{\omega\}$. Let $f : \mathbb{S} \to \mathbb{C}$ be a bounded integrable function on the unit circle $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ and let $g : \mathbb{D} \to \mathbb{C}$ be continuous. The solutions to the Poisson equation $\Delta w = g$ in $\mathbb{D}$ satisfying the boundary condition $w|_{\mathbb{S}} = f \in L^1(\mathbb{S})$ have the representation

$$w = u - v, \tag{1.1}$$

where

$$u(z) = P[f](z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\varphi}) f(e^{i\varphi})\, d\varphi,$$
$$v(z) = G[g](z) = \int_{\mathbb{D}} G(z, \omega) g(\omega)\, dm(\omega)$$

and $dm(\omega)$ denotes the Lebesgue measure in $\mathbb{C}$. Further, if f (respectively, g) is continuous in $\mathbb{S}$ (respectively, in $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}$), then w has a continuous extension $\tilde{w}$ to $\mathbb{D}$, and $\tilde{w}|_{\mathbb{S}} = f$ (see [4]).

If u is a harmonic univalent function, then by Lewy's theorem (see [2]), u has a nonvanishing Jacobian and consequently, according to the inverse mapping theorem, u is a diffeomorphism.

Martio [13] was the first to consider harmonic quasiconformal mappings in $\mathbb{C}$. Recent papers [1, 5, 7, 8, 10, 15] shed much light on the topic of harmonic quasiconformal mappings in $\mathbb{C}$, and [6] extends the domain to the unit ball. In [12, 14, 16–19], the Lipschitz character of harmonic quasiconformal mappings is discussed. In particular, Kalaj and Mateljević [9] proved that a harmonic diffeomorphism between two Jordan domains with C^2 boundaries is a (K, K')-quasiconformal mapping for some constants $K \geq 1$ and $K' \geq 0$ if and only if it is Lipschitz continuous.

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Let g be a function from $\mathbb{D}$ to Ω with a continuous extension to the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$, let $f : \mathbb{S} \to \mathbb{C}$ be a bounded integrable function on $\mathbb{S}$ and let Ω be a Jordan domain with C^2 boundary. Further, let

(1) $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$ denote the family of solutions $w : \mathbb{D} \to \Omega$ of the Poisson equation $\Delta w = g$, where $w|_{\mathbb{S}} = f \in L^1(\mathbb{S})$ and each w is a sense-preserving diffeomorphism;
(2) $\mathcal{QC}_{\mathbb{D}\to\Omega}(K)$ denote the set of all K-quasiconformal mappings w from $\mathbb{D}$ to Ω;
(3) $\mathcal{QC}_{\mathbb{D}\to\Omega}(K, K')$ denote the set of all (K, K')-quasiconformal mappings w from $\mathbb{D}$ to Ω;
(4) $\mathcal{HQC}_{\mathbb{D}\to\Omega}(K, K')$ denote the set of all harmonic (K, K')-quasiconformal mappings w from $\mathbb{D}$ to Ω.

We remark that, if $w \in \mathcal{HQC}_{\mathbb{D}\to\Omega}(K, K')$, then $w = u$. By [9], we have the following result.

Proposition 1.1. *Suppose that $w \in \mathcal{HQC}_{\mathbb{D}\to\Omega}(K, K')$ and $z = re^{i\theta}$. Then $|\partial w/\partial r| = |\partial u/\partial r|$ is bounded.*

Assume that $\Omega = \mathbb{D}$ and $w(0) = 0$ for all elements w in $\mathcal{D}_{\mathbb{D}\to\Omega}(g)$. Under these assumptions, Kalaj and Pavlović proved that the family $\mathcal{D}_{\mathbb{D}\to\Omega}(g) \cap \mathcal{QC}_{\mathbb{D}\to\Omega}(K)$ is uniformly Lipschitz [11, Theorem 1.2]. Moreover, Kalaj and Mateljević discussed the Lipschitz continuity of elements in $\mathcal{HQC}_{\mathbb{D}\to\Omega}(K, K')$ [9, Theorem 1.1 and Corollary 1.3]. The aim of this paper is to generalise the main results in [9] to the setting of $\mathcal{D}_{\mathbb{D}\to\Omega}(g) \cap \mathcal{QC}_{\mathbb{D}\to\Omega}(K, K')$, that is, we consider the Lipschitz character of elements in $\mathcal{D}_{\mathbb{D}\to\Omega}(g) \cap \mathcal{QC}_{\mathbb{D}\to\Omega}(K, K')$ with the natural assumption that $|\partial u/\partial r|$ is bounded (see Proposition 1.1). Our main results are Theorems 3.2 and 3.3, which will be stated and proved in Section 3. In Section 2, we will construct examples to show the existence of the (K, K')-quasiconformal solutions of the Poisson equation $\Delta w = g$ with special functions g.

2. Examples

In this section we will construct two examples to show the existence of (K, K')-quasiconformal solutions of the Poisson equation $\Delta w = g$ with special functions g. Our first example shows that there is a (K, K')-quasiconformal solution of a Poisson equation, which is not M-quasiconformal for any $M \geq 1$. The second example shows that there is a (K, K')-quasiconformal solution of a Poisson equation with $K' \neq 0$, which is also M-quasiconformal for some $M > 1$.

Example 2.1. Let $w(z) = 3z - |z|^2 z$ in $\mathbb{D}$. Then

(1) w satisfies the equation $\Delta w = -8z$ and $w|_{\mathbb{S}} = 2e^{i\theta}$;
(2) w is a $(1, 9)$-quasiconformal mapping of $\mathbb{D}$ onto $\mathbb{D}_2 = \{z : |z| < 2\}$;
(3) w is not M-quasiconformal for any $M \geq 1$;
(4) w is Lipschitz continuous.

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PROOF. Since $w(z) = 3z - |z|^2 z$, we have

$$w_z = 3 - 2|z|^2 \quad \text{and} \quad w_{\bar{z}} = -z^2.$$

Obviously, $\Delta w = -8z$. It follows from

$$J_w = |w_z|^2 - |w_{\bar{z}}|^2 = 9 - 12|z|^2 + 3|z|^4 > 0,$$

together with the fact that $w(z)|_{\mathbb{S}} = 2e^{i\theta}$ and the degree principle, that w is a sense-preserving homeomorphism from $\mathbb{D}$ onto $\mathbb{D}_2$. Further, we know that w is a $(1, 9)$-quasiconformal mapping since

$$|\nabla w|^2 \le J_w + (|w_z| + |w_{\bar{z}}|)^2 \le J_w + 9.$$

The limit

$$\lim_{|z|\to 1} \frac{|w_{\bar{z}}|}{|w_z|} = \lim_{|z|\to 1} \frac{|z|^2}{3 - 2|z|^2} = 1$$

tells us that w is not M-quasiconformal for any $M \ge 1$.

For $z_1, z_2 \in \mathbb{D}$, here and henceforth, we let $[z_1, z_2]$ denote the segment in $\mathbb{D}$ with the endpoints z_1 and z_2. Then

$$|w(z_1) - w(z_2)| = \left| \int_{[z_1,z_2]} w_z(z)\, dz + w_{\bar{z}}(z)\, d\bar{z} \right| \le \int_{[z_1,z_2]} |\nabla w|\, |dz|.$$

Thus, the Lipschitz continuity of w follows easily from the estimate:

$$|\nabla w| = |w_z| + |w_{\bar{z}}| = 3 - |z|^2 \le 3.$$

Hence the proof is complete. □

EXAMPLE 2.2. Let $w(z) = \frac{1}{3}z + |z|^2 z$ in $\mathbb{D}$. Then

(1) w satisfies the equation $\Delta w = 8z$ and $w|_{\mathbb{S}} = \frac{4}{3}e^{i\theta}$;
(2) w is a $(1, \frac{20}{3})$-quasiconformal mapping of $\mathbb{D}$ onto $\mathbb{D}_{4/3} = \{z : |z| < \frac{4}{3}\}$;
(3) w is a $\frac{5}{2}$-quasiconformal mapping;
(4) w is Lipschitz continuous.

PROOF. Since $w(z) = \frac{1}{3}z + |z|^2 z$, we have

$$w_z = \frac{1}{3} + 2|z|^2 \quad \text{and} \quad w_{\bar{z}} = z^2,$$

which implies $\Delta w = 8z$. It follows from

$$J_w = |w_z|^2 - |w_{\bar{z}}|^2 = \frac{1}{9} + \frac{4}{3}|z|^2 + 3|z|^4 > 0,$$

together with the fact that $w|_{\mathbb{S}} = \frac{4}{3}e^{i\theta}$ and the degree principle, that w is a sense-preserving homeomorphism from $\mathbb{D}$ onto $\mathbb{D}_{4/3}$. Since

$$|\nabla w|^2 = (|w_z| + |w_{\bar{z}}|)^2 = \frac{1}{9} + 2|z|^2 + 9|z|^4 = J_w + \frac{2}{3}|z|^2 + 6|z|^4 \le J_w + \frac{20}{3},$$

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we see that w is a $(1, \frac{20}{3})$-quasiconformal mapping. Moreover, we infer from

$$\frac{|w_{\bar{z}}|}{|w_z|} = \frac{|z|^2}{\frac{1}{3} + 2|z|^2} \leq \frac{3}{7}$$

that w is a $\frac{5}{2}$-quasiconformal mapping. The Lipschitz continuity of w follows easily from the estimate:

$$|\nabla w| = \tfrac{1}{3} + 3|z|^2 \leq \tfrac{10}{3}.$$

Hence the proof is complete. □

3. Statements and proofs of the main results

We start with a lemma which will be useful for the proofs of the main results.

Lemma 3.1. *Suppose that $w \in \mathcal{D}_{\mathbb{D}\to\Omega}(g) \cap \mathcal{QC}_{\mathbb{D}\to\Omega}(K, K')$ with the representation (1.1) and $|\partial u/\partial r| \leq L$ in $\mathbb{D}$ for some constant L. Then, for $z_1, z_2 \in \mathbb{D}$,*

$$|w(z_1) - w(z_2)| \leq (KL + \tfrac{2}{3}K|g|_\infty + \sqrt{K'})|z_1 - z_2|, \tag{3.1}$$

where $|g|_\infty = \sup_{|z|<1} |g(z)|$.

Proof. For $z_1, z_2 \in \mathbb{D}$, it is easy to see that

$$|w(z_1) - w(z_2)| = \left| \int_{[z_1,z_2]} w_z(z)\, dz + w_{\bar{z}}(z)\, d\bar{z} \right| \leq \int_{[z_1,z_2]} |\nabla w|\, |dz|.$$

Thus, in order to prove inequality (3.1), we only need to show that

$$|\nabla w| \leq KL + \tfrac{2}{3}K|g|_\infty + \sqrt{K'}. \tag{3.2}$$

Obviously,

$$|\nabla w|^2 \leq KJ_w + K' = K|\nabla w|\ell(\nabla w) + K',$$

which implies

$$|\nabla w| \leq \frac{K\ell(\nabla w) + \sqrt{K^2\ell(\nabla w)^2 + 4K'}}{2} \leq K\ell(\nabla w) + \sqrt{K'}.$$

For $z \in \mathbb{D}$, set $z = re^{i\theta}$. Then

$$\frac{\partial w(re^{i\theta})}{\partial r} = e^{i\theta} w_z + e^{-i\theta} w_{\bar{z}}.$$

Hence

$$\ell(\nabla w) \leq \left|\frac{\partial w}{\partial r}\right|,$$

and we deduce that

$$|\nabla w| \leq K\left|\frac{\partial w}{\partial r}\right| + \sqrt{K'}.$$

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To prove (3.2), it suffices to derive the estimate

$$\left|\frac{\partial w}{\partial r}\right| \le L + \frac{2}{3}|g|_\infty.$$

For all $z \ne \omega$,

$$G_z(z,\omega) = \frac{1}{4\pi}\left(\frac{1}{\omega - z} - \frac{\overline{\omega}}{1 - z\overline{\omega}}\right) = \frac{1}{4\pi}\frac{1-|\omega|^2}{(z-\omega)(z\overline{\omega}-1)}$$

and

$$G_{\overline{z}}(z,\omega) = \frac{1}{4\pi}\frac{1-|\omega|^2}{(\overline{z}-\overline{\omega})(\overline{z}\omega - 1)},$$

so we see that

$$\left|\frac{\partial G(re^{i\theta})}{\partial r}\right| = |e^{i\theta}G_z + e^{-i\theta}G_{\overline{z}}| \le 2|G_z|$$

and [11, Lemma 2.7] implies

$$\left|\frac{\partial v}{\partial r}\right| \le \frac{2}{3}|g|_\infty.$$

It follows that

$$\left|\frac{\partial w}{\partial r}\right| \le \left|\frac{\partial u}{\partial r}\right| + \left|\frac{\partial v}{\partial r}\right| \le L + \frac{2}{3}|g|_\infty,$$

as required, and the lemma is proved. □

The Hilbert transform of a function f in $L^1(\mathbb{S})$ is defined by the formula

$$H[f](\varphi) = -\frac{1}{\pi}\int_{0^+}^{\pi}\frac{f(\varphi+t) - f(\varphi - t)}{2\tan(t/2)}\,dt.$$

The integral is improper and converges for almost all $\varphi \in [0, 2\pi]$. See [20, Ch. VII] for more properties of this transform. Our first main result is the following theorem.

Theorem 3.2. *Suppose that $w \in \mathcal{D}_{\mathbb{D}\to\Omega}(g)$ with the representation (1.1) and that Ω is convex. Then the following conditions are equivalent:*

(1) *w is a (K,K')-quasiconformal mapping and $|\partial u/\partial r| \le L$ in $\mathbb{D}$;*
(2) *w is Lipschitz with respect to the Euclidean metric;*
(3) *u is Lipschitz with respect to the Euclidean metric;*
(4) *u is a (K,K')-quasiconformal mapping;*
(5) *f is absolutely continuous on $\mathbb{S}$, $f' \in L^\infty(\mathbb{S})$ and $H[f'] \in L^\infty(\mathbb{S})$.*

Proof. Obviously, the assumptions on w and on Ω show that f is sense-preserving. Thus it follows from the Radó–Kneser–Choquet theorem (see [2]) that u is a harmonic univalent function from $\mathbb{D}$ to Ω and also that u is sense-preserving.

For the equivalence between (2) and (3), we remark that, by [11, Lemma 2.6],

$$\nabla v = \int_{\mathbb{D}} \nabla_z G(z,\omega) g(\omega)\, dm(\omega),$$

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and [11, Lemma 2.3] leads to

$$|\nabla v| \le \tfrac{2}{3}|g|_\infty.$$

Thus the equivalence between (2) and (3) easily follows from the estimates

$$|\nabla u| \le |\nabla v| + |\nabla w| \quad \text{and} \quad |\nabla w| \le |\nabla u| + |\nabla v|.$$

The implication from (1) to (2) is obvious from Lemma 3.1. For the converse implication, the assumption that 'w is Lipschitz' and the equivalence of (2) and (3) show that u is Lipschitz and so there are constants P and L such that

$$|\nabla w| \le P \quad \text{and} \quad |\nabla u| \le L.$$

Thus

$$|\nabla w|^2 \le J_w + |\nabla w|^2 \le J_w + P^2,$$

from which we see that w is $(1, P^2)$-quasiconformal. Further,

$$\left|\frac{\partial u}{\partial r}\right| \le |\nabla u| \le L.$$

Hence (1) and (2) are equivalent.

The equivalence between (3), (4) and (5) easily follows from [9, Corollary 1.3] and the proof of the theorem is complete. □

The next theorem is our second main result.

Theorem 3.3. *Suppose that $h : \Omega \to \mathbb{D}$ is a sense-preserving diffeomorphism and $\phi : \mathbb{D} \to \Omega$ is a conformal transformation. If $w = h \circ \phi$ and $w \in \mathcal{D}_{\mathbb{D}\to\mathbb{D}}(g)$ with the representation (1.1), then the following two statements are equivalent:*

(1) *h is (K, K')-quasiconformal and $|\partial u/\partial r| \le L$ in $\mathbb{D}$;*

(2) *h is Lipschitz with respect to the Euclidean metric.*

Proof. Assume that h is (K, K')-quasiconformal and $|\partial u/\partial r| \le L$. Then

$$|\nabla w|^2 = |\nabla h|^2|\phi'|^2 \le K(J_h|\phi'|^2) + K'|\phi'|^2 = KJ_w + K|\phi'|^2.$$

Thus w is (K, K_1')-quasiconformal with $K_1' = K\|\phi'\|_\infty^2$, where $\|\phi'\|_\infty = \sup\{|\phi'| : z \in \mathbb{D}\}$ is finite by Kellogg's theorem (see [7, Proposition 2.1] or [3]), and then it follows from Lemma 3.1 that w is Lipschitz. Also, Kellogg's theorem guarantees that ϕ^{-1} is Lipschitz, and therefore, $h = w \circ \phi^{-1}$ is Lipschitz. This completes the proof of the implication from (1) to (2). To prove the converse implication, we assume that h is Lipschitz. Then there is a constant M such that $|\nabla h| \le M$, and thus,

$$|\nabla h|^2 \le J_h + M^2.$$

This implies that h is $(1, M^2)$-quasiconformal. Again, Kellogg's theorem shows that ϕ is Lipschitz and so $w = h \circ \phi$ is Lipschitz. Hence $|\nabla w| \le M'$, where M' is a constant. By [11, Lemma 2.6],

$$\nabla v = \int_{\mathbb{D}} \nabla_z G(z, \omega) g(\omega)\, dm(\omega)$$

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and so [11, Lemma 2.3] gives

$$|\nabla v| \le \tfrac{2}{3}|g|_\infty.$$

Consequently,

$$\left|\frac{\partial u}{\partial r}\right| \le |\nabla u| \le |\nabla w| + |\nabla v| \le M' + \frac{2}{3}|g|_\infty.$$

Hence the proof of the theorem is complete. □

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