

SOME RESULTS ON AN ALGEBRO-GEOMETRIC CONDITION ON GRAPHS

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(Received 12 February 2016; accepted 24 January 2017; first published online 29 March 2017)

Communicated by B. Alspach

Abstract

Paolo Aluffi, inspired by an algebro-geometric problem, asked when the Kirchhoff polynomial of a graph is in the Jacobian ideal of the Kirchhoff polynomial of the same graph with one edge deleted. We give some results on which graph–edge pairs have this property. In particular, we show that multiple edges can be reduced to double edges, we characterize which edges of wheel graphs satisfy the property, we consider a stronger condition which guarantees the property for any parallel join, and we find a class of series–parallel graphs with the property.

2010 *Mathematics subject classification*: primary 05C31.

Keywords and phrases: graphs, Kirchhoff polynomial, Jacobian ideal.

1. Introduction

Over the last decade there has been an interest in taking an algebraic-geometry-inspired approach to understanding Feynman integrals [3, 4, 7–12, 15]. The key object of study is the *graph hypersurface*, which we can define as follows. Given a multigraph G (henceforth we will just say graph with the understanding that multiple edges and self-loops are permitted), to each edge e of G assign a variable t_e and define the *Kirchhoff polynomial*¹ of G by

$$\Psi_G = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \notin T} t_e.$$

The graph hypersurface is then simply the variety given by the zero set of this polynomial, viewed in projective space or in affine space depending on context. This

All three authors were at Simon Fraser University at the time of this research. AK was supported by an NSERC PGS D. GM was supported by an NSERC USRA during this project. KY is supported by an NSERC discovery grant.

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¹This polynomial is also known as the first Symanzik polynomial, and sometimes the Kirchhoff polynomial is instead defined dually with the condition $e \in T$ in place of $e \notin T$, see for example [5].

relates back to Feynman integrals because the Kirchhoff polynomial plays a key role in the integrand of the Feynman integral of G in parametric form, see, for example, [5]. In fact, viewing G as a massless scalar Feynman diagram, the first interesting piece of the Feynman integral is simply

$$\int_{t_i \geq 0} \frac{\Omega}{\Psi_G^2}$$

where $\Omega = \sum_{i=1}^{|E(G)|} (-1)^i dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_{|E(G)|}$. This has come to be known as the Feynman period, see [17], and is a very interesting object physically, number theoretically, and combinatorially.

The interplay between Kirchhoff polynomials for different graphs will be crucial to our argument. Therefore, to keep the notation light, we will abuse notation and simply write G for Ψ_G . We will take the convention that the Kirchhoff polynomial of a disconnected graph is 0.

At the level of the Kirchhoff polynomial, edge deletion (equivalently, partial derivative) will be indicated by superscripts and edge contraction (equivalently, setting variables to 0), by subscripts. That is, for Kirchhoff polynomials, in this notation,

$$G^e = G \setminus e = \partial_e G \quad \text{and} \quad G_e = G/e = G|_{e=0}.$$

These facts are elementary consequences of the definition of the Kirchhoff polynomial. A similar argument gives the classical contraction–deletion relation

$$G = t_e G^e + G_e \quad \text{for } e \text{ not a bridge or self-loop.}$$

Aluffi and Marcolli in [2] gave a definition of algebro-geometric Feynman rules that captured the most basic properties of Feynman rules in quantum field theory. Specifically, they require the multiplicative property for disjoint unions of graphs, which is a restatement of the multiplicative property of independent events in basic probability, and they require the formula for recasting Feynman diagrams as trees of one-particle-irreducible diagrams.

Other more advanced properties of physical Feynman rules can also be captured in algebraic language, see, for example, [16], but that is another story.

Aluffi and Marcolli then look at examples of their algebro-geometric Feynman rules which appear natively in the land of algebraic geometry. One example comes from classes in the Grothendieck ring; another comes from Chern class calculations and gives univariate polynomial output.

In the course of studying when this second example satisfies contraction–deletion relations, Aluffi in [1] needed to assume two technical conditions, which he calls Conditions 1 and 2, each of which is a condition on a pair of a graph G and an edge e of G .

Aluffi's Condition 1 is not difficult to state. Throughout the paper we follow [1] and work over \mathbb{Q} (see [1, Section 2.4]).

DEFINITION 1.1. Let G be a connected graph and e an edge of G . Then Condition 1 for the edge e of G , written $1(G, e)$, is the statement

$$G \in \langle \partial G^e \rangle$$

where $\langle \partial G^e \rangle$ is the ideal of partial derivatives, that is, the Jacobian ideal, of G^e , the Kirchhoff polynomial of G with e deleted.

There are a few immediate observations worth making. From the contraction–deletion relation and Euler’s homogeneous function theorem, see Theorem 2.1, Condition 1 for regular edges is equivalent to the statement

$$G_e \in \langle \partial G^e \rangle.$$

Certain special cases of edges are easy to understand. If e is a self-loop then $G = t_e G^e$ and so $1(G, e)$ is true. If e is a bridge then $G^e = 0$ since $G \setminus e$ is disconnected and so has no spanning trees; thus $1(G, e)$ is false. Finally, if $G \setminus e$ is a tree then $G^e = 1$ and so again $1(G, e)$ is false. Aluffi calls an edge which does not fall into one of the previous cases *regular*.

In this paper we investigate the graph-theoretic underpinning of Condition 1. We are not able to obtain a full characterization of graph–edge pairs which satisfy the condition, but we do obtain the following interesting results. Propositions 3.3 and 3.4 give that multiple edges of any multiplicity greater than 1 are equivalent to double edges from the perspective of Condition 1. Propositions 4.2 and 4.4 show that for wheels with at least 4 spokes, Condition 1 is false for all rim edges and true for all spoke edges. Then we move to focusing on series–parallel graphs. Definition 5.1 gives a stronger condition which, by Corollary 5.9, shows when Condition 1 is stable under parallel join. Finally, Corollary 6.14 describes a class of series–parallel graphs where Condition 1 holds for all edges, and Proposition 6.16 builds from this a much larger class of series–parallel graphs (and some other graphs) with specific edges for which Condition 1 holds.

2. Preliminaries

For our arguments we want to consider cases of identifying vertices. We will use the notation

$$\overset{12}{G}$$

for the Kirchhoff polynomial of the graph G with vertices 1 and 2 identified, and, more generally, if s_1, s_2, \dots are sets of vertices of G then

$$\overset{s_1, s_2, \dots}{G}$$

is the Kirchhoff polynomial of the graph G with the vertices of s_1 identified, the vertices of s_2 identified, and so on.

By considering the possible spanning trees, we can write down the 1- and 2-cut formulas for the Kirchhoff polynomial. Specifically, if G is formed from H and H' joined at a vertex then

$$G = HH'$$

and if G is formed from H and H' joined at two vertices, 1 and 2, then

$$G = \overset{12}{H}H' + H\overset{12}{H'}$$

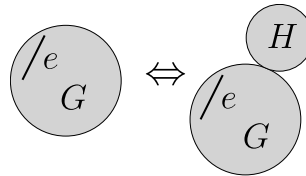
One of the main algebraic tools we will use is Euler's homogeneous function theorem.

THEOREM 2.1 (Euler's theorem). *Let R be a ring and let $f \in R[x_1, \dots, x_n]$ be a homogeneous function of degree m . Then*

$$m \cdot f = \sum_{j=1}^n x_j \frac{\partial f}{\partial x_j}$$

Next, we will give a few propositions explaining how small vertex and edge cuts affect Condition 1.

PROPOSITION 2.2. *If G and H are connected graphs that are joined at one vertex and e is a regular edge of G , then $1(G, e) \Leftrightarrow 1(G \cup H, e)$.*



PROOF. Let G and H be as in the statement.

(\Rightarrow) Suppose $1(G, e)$, so for some P_i in the edge variables a_i of G ,

$$G_e = \sum_i P_i G^{ea_i}$$

Then, since neither e nor the a_i are variables of H ,

$$\begin{aligned} (G \cup H)_e &= G_e \cup H = HG_e = H \sum_i P_i G^{ea_i} \\ &= \sum_i P_i HG^{ea_i} \\ &= \sum_i P_i (H \cup G)^{ea_i} \\ &= \sum_i P_i (G \cup H)^{ea_i} \end{aligned}$$

Hence $1(G \cup H, e)$.

(\Leftarrow) Suppose $1(G \cup H, e)$, so for some sets of polynomials $\{P_i\}$ and $\{Q_j\}$, with all polynomials in the edge variables a_i of G and b_j of H ,

$$\begin{aligned} (G \cup H)_e &= HG_e = \sum_i P_i(G \cup H)^{ea_i} + \sum_j Q_j(G \cup H)^{eb_j} \\ &= \sum_i P_iHG^{ea_i} + \sum_j Q_jG^eH^{b_j} \\ &= H \sum_i P_iG^{ea_i} + G^e \sum_j Q_jH^{b_j}. \end{aligned}$$

Set all of the b_j equal to 1. Then

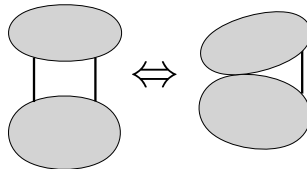
$$H \Big|_{\substack{b_j=1 \\ b_j \text{ edge} \\ \text{of } H}} G_e = H \Big|_{\substack{b_j=1 \\ b_j \text{ edge} \\ \text{of } H}} \sum_i P_i \Big|_{\substack{b_j=1 \\ b_j \text{ edge} \\ \text{of } H}} G^{ea_i} + G^e \left(\sum_u Q_u H^{b_u} \right) \Big|_{\substack{b_j=1 \\ b_j \text{ edge} \\ \text{of } H}}.$$

So

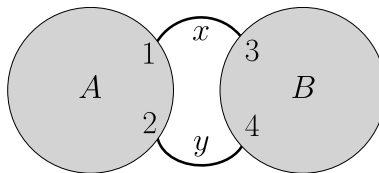
$$G_e = \sum_i R_i G^{ea_i} + S G^e,$$

where the R_i and S are polynomials in the variables a_i . Since Theorem 2.1 implies $G^e \in \langle \partial G^e \rangle$ and the G^{ea_i} are themselves partials, we have $1(G, e)$. \square

PROPOSITION 2.3. *Whenever there is a two-edge cut-set, contracting one of these edges has no effect on Condition 1 for the remaining edges. Formally, let G be a graph with a two-edge-cut set $\{x, y\}$. Then for all $e \in G \setminus \{x\}$ we have $1(G, e) \iff 1(G_x, e)$.*



PROOF. We can draw G as



With G as drawn, we see that

$$G = (x + y)AB + \overset{12}{AB} + \overset{34}{AB}$$

and

$$G_x = yAB + \overset{12}{AB} + \overset{34}{AB}.$$

To get G_x from G we contracted x by setting $x = 0$. Notice that in G , the variables x and y only appear in the term $(x + y)$. We also see that the occurrence of $(x + y)$ in G corresponds exactly to the occurrence of y in G_x . Therefore, we can recover G from G_x by making the replacement $y \rightarrow x + y$.

To delete y , we take the y derivatives of G and G_x , and we likewise delete x from G and G_y . Then we have the identity

$$G^x = G^y = G_x^y = G_y^x = AB.$$

We will also use the contraction–deletion relation

$$G_y = xG_y^x + G_{yx}.$$

There are two cases to be considered: (1) the edge that Condition 1 is being tested for belongs to the two-edge cut-set; (2) it does not belong to the cut-set.

PROOF OF CASE 1. Let y be the edge that we are testing for Condition 1.

(\Rightarrow) Suppose $1(G, y)$. Then

$$G_y \in \langle \partial G^y \rangle = \langle \partial G_x^y \rangle.$$

Hence,

$$G_{yx} = G_y - xG_y^x = G_y - xG_x^y \in \langle \partial G_x^y \rangle.$$

(\Leftarrow) Suppose $1(G_x, y)$. Then,

$$G_{yx} \in \langle \partial G_x^y \rangle.$$

Then

$$G_y = xG_y^x + G_{yx} = xG_x^y + G_{yx} \in \langle \partial G_x^y \rangle = \langle \partial G^y \rangle.$$

This concludes the proof of case 1. □

PROOF OF CASE 2. Let the edge e that we are testing for Condition 1 belong to either A or B , and wherever the a_i appear below, let them range over the edge variables not equal to x, y , or e .

(\Rightarrow) Suppose $1(G, e)$, so that for some polynomials P_i, Q, R we have

$$G_e = \sum_i P_i G^{ea_i} + QG^{ex} + RG^{ey}.$$

Now we use $G^{ex} = G^{ey} = G_x^{ey}$ and let $S = Q + R$ so that

$$G_e = \sum_i P_i G^{ea_i} + S G_x^{ey}.$$

Then we set $x = 0$ to get

$$G_{ex} = \sum_i P_i(x=0) G_x^{ea_i} + S(x=0) G_x^{ey}.$$

(\Leftarrow) Suppose $1(G_x, e)$ so that for some polynomials P_i, R we have

$$G_{ex} = \sum_i P_i G_x^{ea_i} + R G_x^{ey}.$$

Then, by the replacement $y \rightarrow x + y$, we recover G_e from G_{ex} on the left-hand side and we recover G^{ea_i} and G^{ey} on the right-hand side. Therefore,

$$G_e = \sum_i P_i(y \rightarrow x + y) G^{ea_i} + R(y \rightarrow x + y) G^{ey}.$$

This concludes the proof of case 2. □

Next we will define some special classes of graphs that we will use.

DEFINITION 2.4. The *wheel with n spokes* is the graph with $n + 1$ vertices consisting of a cycle of length n along with an additional vertex that is adjacent to all the vertices of the cycle. The edges of the cycle are called *rim edges* while the other edges are called *spoke edges*.

DEFINITION 2.5. A *source-terminal graph* is a graph G with two distinct marked vertices $s, t \in V(G)$.

If G and H are two source-terminal graphs then we can define their *parallel join* as being the source-terminal graph $G \star H$, which is the disjoint union of G and H with the sources and terminals identified and with these two vertices forming the source and terminal of $G \star H$.

If G and H are two source-terminal graphs then we can define their *series join* as being the source-terminal graph $G \sharp H$, which is the disjoint union of G and H with the source of H identified with the terminal of G , with the source of G becoming the source of $G \sharp H$ and the terminal of H becoming the terminal of $G \sharp H$.

DEFINITION 2.6. We take a *series-parallel graph* to be a source-terminal graph G such that G is either:

- (1) $G = K_2$.
- (2) G is the parallel join of two series-parallel graphs H, H' , that is, $G := H \star H'$.
- (3) G is the series join of two series-parallel graphs H, H' , that is, $G := H \sharp H'$.

In the case of series-parallel graphs, the Kirchhoff polynomial for the graph with the two terminals identified is particularly important and so we will use the notation

$$\overline{G} = G^{st}$$

for any source-terminal graph G .

If we interpret $s, t \in V(G)$ as the only vertices with external edges then we recover \overline{G} as the second Symanzik polynomial. That is,

$$\overline{G} = \sum_{T_1, T_2} \prod_{e \notin T_1 \cup T_2} x_e$$

where T_1, T_2 are trees, $s \in T_1, t \in T_2, T_1 \cap T_2 = \emptyset$, and $V(G) \subseteq T_1 \cup T_2$. We call spanning forests of two trees with these properties *spanning-st-forests*.

The following are all reformulations of the 1- and 2- vertex cut formulas for the Kirchhoff polynomial as applied to the series and parallel operations.

LEMMA 2.7. *Let H, H' be source–terminal graphs. Then:*

- (a) $H \star H' = \overline{HH'} + \overline{H'H}$;
- (b) $\overline{H \star H'} = \overline{H} \overline{H'}$;
- (c) $\overline{H \not\star H'} = \overline{HH'}$;
- (d) $\overline{H \not\star H'} = \overline{HH'} + \overline{H'H}$;
- (e) $\deg(\overline{H}) = \deg(H) + 1$ and $\deg(\overline{H}) > 0$.

Any series–parallel graph has a natural recursive structure, which we can capture in a tree.

DEFINITION 2.8. For any series–parallel graph G we may associate to it a (not necessarily unique) decomposition tree Υ , which is the rooted tree whose leaves represent the edges of G and whose interior vertices represent the operations $\not\star, \star$ used in the construction of G ; the root vertex corresponds to the last operation used in the construction. Conversely, any such tree uniquely defines a series–parallel graph.

The Υ -dual is the series–parallel graph associated to the decomposition tree Υ^\vee obtained by exchanging every $\not\star$ with a \star and *vice versa*. Finally, $\text{ht}(\Upsilon)$ is the height of Υ as a rooted tree.

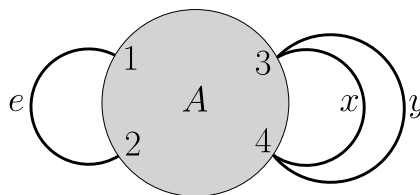
3. Multiple edges

There are several interesting results concerning parallel edges and Condition 1. Aluffi in [1] showed what we give as Proposition 3.1 to prove that the Chern class obeys a multiple-edge formula. Extending beyond his work, Propositions 3.3 and 3.4 imply that where there is a pair of parallel edges, adding a third parallel edge or more has no effect on condition 1, and if there are three or more edges, deleting all of them except two has no effect. In the context of Condition 1, one can look at a multigraph as a simple graph with two types of edges: the single edge and the multiple edge.

PROPOSITION 3.1 (Aluffi, 2011). *If e is a regular edge and e has at least one other edge parallel to it in G then $1(G, e)$ is true.*

The main idea for the proof of Proposition 3.1 is Euler’s homogeneous function theorem, see [1].

The next three propositions have a common set-up.



For a graph with edge e connected to A at vertices 1, 2 and parallel edges x, y connected to A at vertices 3, 4, the condition $1(G, e)$ reads as

$$G_e = xyA^{12} + (x + y)A^{12,34} = \sum_i P_i(xyA^{a_i} + (x + y)A^{34a_i}) + Q(yA + A) + R(xA + A). \tag{3.1}$$

As the ideal generated by G_e is homogeneous, we may assume that P_i, Q, R are homogeneous degree 2 polynomials in the variables x, y, a_i (cf. [13, Section 4.2]). Let us write these polynomials as the solution set

$$P_i, \quad Q, \quad R.$$

Equation (3.1) has a symmetry which is quite useful. Let τ_{xy} be the operation where x and y are swapped. Applying τ_{xy} to both sides gives

$$xyA^{12} + (x + y)A^{12,34} = \sum_i \tau_{xy}P_i(xyA^{a_i} + (x + y)A^{34a_i}) + \tau_{xy}Q(xA + A) + \tau_{xy}R(yA + A). \tag{3.2}$$

The left-hand side is invariant under this operation. Thus we can sum equations (3.1) and (3.2) and divide by 2, yielding the polynomials

$$\tilde{P}_i = \frac{P_i + \tau_{xy}P_i}{2}, \quad \tilde{Q} = \frac{Q + \tau_{xy}Q}{2}, \quad \tilde{R} = \frac{R + \tau_{xy}R}{2}$$

which satisfy Equation (3.1). Hence, from any given solution set P_i, Q, R , one can construct $\tilde{P}_i, \tilde{Q}, \tilde{R}$ such that

$$\tau_{xy}\tilde{P}_i = \tilde{P}_i, \quad \tau_{xy}\tilde{Q} = \tilde{R}, \quad \tau_{xy}\tilde{R} = \tilde{Q}.$$

Focusing only on the x, y dependence, in order to satisfy these relations and Equation (3.1), the $\tilde{P}_i, \tilde{Q}, \tilde{R}$ must have the forms

$$\begin{aligned} \tilde{P}_i &= p_{0i} + (x + y)p_{1i} + (x^2 + y^2)p_{2i} + xyp_{11i} \\ \tilde{Q} &= q_0 + xq_{10} + yq_{01} + x^2q_{20} + y^2q_{02} + xyp_{11} \\ \tilde{R} &= q_0 + xq_{01} + yq_{10} + x^2q_{02} + y^2q_{20} + xyp_{11}. \end{aligned}$$

Substituting these expressions into Equation (3.1) and collecting terms, we get the equation:

$$\begin{aligned}
 &xyA + (x + y)A^{12,34} = \\
 &xy \sum_i p_{0i} A^{a_i} + (x + y)q_0 A + 2q_0 A^{34} \\
 &+ (x^2y + xy^2) \sum_i p_{1i} A^{a_i} + 2xyq_{10} A + (x + y)q_{10} A^{34} \\
 &+ (x^3y + xy^3) \sum_i p_{2i} A^{a_i} + (x^2 + y^2)q_{01} A + (x + y)q_{01} A^{34} \\
 &+ x^2y^2 \sum_i p_{11i} A^{a_i} + (x^2y + xy^2)q_{20} A + (x^2 + y^2)q_{20} A^{34} \\
 &+ (x + y) \sum_i p_{0i} A^{34a_i} + (x^3 + y^3)q_{02} A + (x^2 + y^2)q_{02} A^{34} \\
 &+ (x^2 + y^2 + 2xy) \sum_i p_{1i} A^{34a_i} + (x^2y + xy^2)q_{11} A + 2xyq_{11} A^{34} \\
 &+ (x^3 + y^3 + x^2y + xy^2) \sum_i p_{2i} A^{34a_i} \\
 &+ (x^2y + xy^2) \sum_i p_{11i} A^{34a_i}.
 \end{aligned}$$

This is an equation as polynomials in x and y , so we can get a list of equations by equating coefficients. This gives

$$\begin{aligned}
 [x^0y^0]: & 0 = 2q_0A^{34} \\
 [x^1y^0]: & A^{12,34} = \sum_i p_{0i} A^{34a_i} + q_0A + (q_{10} + q_{01})A^{34} \\
 [x^1y^1]: & A^{12} = \sum_i p_{0i} A^{a_i} + 2 \sum_i p_{1i} A^{34a_i} + 2q_{10}A + 2q_{11}A^{34} \\
 [x^2y^0]: & 0 = \sum_i p_{1i} A^{34a_i} + q_{01}A + (q_{20} + q_{02})A^{34} \\
 [x^3y^0]: & 0 = \sum_i p_{2i} A^{34a_i} + q_{02}A \\
 [x^2y^1]: & 0 = \sum_i p_{1i} A^{a_i} + \sum_i p_{2i} A^{34a_i} + \sum_i p_{11i} A^{34a_i} + (q_{20} + q_{11})A^{34} \\
 [x^3y^1]: & 0 = \sum_i p_{2i} A^{a_i} \\
 [x^2y^2]: & 0 = \sum_i p_{11i} A^{a_i}.
 \end{aligned}$$

By following the steps in this derivation, we can determine a set of polynomials satisfying these eight equations when Equation (3.1) is satisfied. Conversely, given a solution set for these equations, we can obtain a solution set for Equation (3.1). In this sense, the eight equations above are equivalent to Equation (3.1).

We take the equation $[x^1y^1]$, and use $[x^2y^0]$ to substitute out $\sum_i p_{1i} A^{34a_i}$, and obtain the equation

$$[x^1y^1]^*: \quad A^{12} = \sum_i p_{0i} A^{a_i} + 2(q_{10} - q_{01})A + 2(q_{11} - q_{20} - q_{02})A^{34}.$$

As a consequence of the $[x^0y^0]$ equation, we have $q_0 = 0$. (We see A is not zero, so q_0 must be.) Hence we can remove the q_0A term from the $[x^1y^0]$ equation, yielding

$$[x^1y^0]^*: \quad A^{12,34} = \sum_i p_{0i} A^{34a_i} + (q_{10} + q_{01})A^{34}.$$

These two expressions will be used in the next three proofs.

PROPOSITION 3.2. *If x, y are parallel edges then $1(G, e) \Rightarrow 1(G_x^y, e)$.*

PROOF. Suppose $1(G, e)$. Then Equation (3.1) has a solution set. We look at the $[x^1y^0]^*$ equation:

$$A^{12,34} = \sum_i p_{0i} A^{34a_i} + (q_{10} + q_{01})A^{34}.$$

The left-hand side is $A^{12,34} = G_{xe}^y$. We want to show that it is in the ideal $\langle \partial G_x^{ye} \rangle$. Here $G_x^{ye} = A^{34}$. By Theorem 2.1 we have $A \in \langle \partial A \rangle$. The A^{34a_i} are partials of A themselves. Thus $A^{12,34} \in \langle \partial A \rangle$, so $1(G_x^y, e)$ is satisfied. □

PROPOSITION 3.3. *If x, y are parallel edges then $1(G, e) \Rightarrow 1(G^y, e)$.*

PROOF. First, note that $G^{ye} = xA + A^{34}$. Suppose $1(G, e)$ and create the equation $x[x^1y^1]^* + [x^1y^0]^*$.

$$\begin{aligned} xA^{12} + A^{12,34} &= \sum_i p_{0i}(xA^{a_i} + A^{34a_i}) + 2x(q_{10} - q_{01})A^{34} \\ &\quad + [2x(q_{11} - q_{20} - q_{02}) + (q_{10} + q_{01})]A^{34}. \end{aligned} \tag{3.3}$$

The left-hand side of Equation (3.3) is G_e^y . Note that A and A^{34} are both in the ideal of partial derivatives of $(xA + A^{34})$ because $A = (\partial/\partial x)(xA + A^{34})$ and $A^{34} = (xA + A^{34}) - xA$.

Hence G_e^y is in the ideal $\langle \partial G^{ye} \rangle$, and therefore we have $1(G^y, e)$. □

PROPOSITION 3.4. *If x, y are parallel edges and $G \cup z$ is obtained by adding an edge z parallel to x and y , then $1(G, e) \Rightarrow 1(G \cup z, e)$.*

PROOF. This proof is very similar to the proof of Proposition 3.3. We have

$$(G \cup z)^e = xyzA + (yz + xz + xy)A^{34}.$$

The following polynomials are in the ideal $\langle \partial(G \cup z)^e \rangle$:

$$yzA^{34}, \quad xzA^{34}, \quad xyA^{34}, \quad xyzA^{34}.$$

This is because

$$yzA^{34} = (xyzA + (yz + xz + xy)A^{34}) - x(\partial/\partial x)(xyzA + (yz + xz + xy)A^{34}).$$

A similar argument works for xzA^{34}, xyA^{34} . Additionally,

$$xyzA^{34} = (xyzA + (yz + xz + xy)A^{34}) - yzA^{34} - xzA^{34} - xyA^{34}.$$

Suppose $1(G, e)$. We make the equation $xyz[x^1y^1]^* + (yz + xz + xy)[x^1y^0]^*$:

$$\begin{aligned} xyzA + (yz + xz + xy)A^{12,34} &= \sum_i p_{0i}(xyzA^{a_i} + (yz + xz + xy)A^{34a_i}) \\ &+ 2xyz(q_{10} - q_{01})A + 2xyz(q_{11} - q_{20} - q_{02})A^{34} \\ &+ (yz + xz + xy)(q_{10} + q_{01})A^{34}. \end{aligned}$$

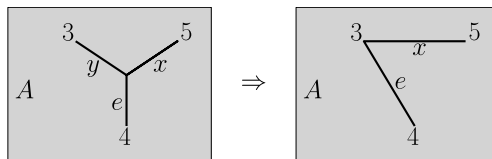
It follows that $1(G \cup z, e)$. □

4. Wheel graphs

We can give a full characterization of which edges of wheel graphs satisfy Condition 1. Specifically for wheels with more than 3 spokes, Condition 1 is false for all rim edges and true for all spoke edges.

We need a few lemmas to obtain the results. The next lemma is closely related to Proposition 2.3 and says that if two edges form a two-edge cut-set of G^e then contracting one of them is Condition 1 preserving.

LEMMA 4.1. *If a graph G has the form of the graph on the left below,*



with the equation

$$G = (ey + ex + xy)A + xA^{34} + yA^{45} + eA^{35},$$

then $1(G, e) \Rightarrow 1(G_y, e)$.

Note that here the central vertex connecting $x, y,$ and e is only connected to the rest of the graph A through $x, y,$ and e .

PROOF. The equation for $1(G, e)$ is

$$xyA + xA^{34} + yA^{45} = \sum_i P_i((x + y)A^{a_i} + A^{35a_i}) + QA + RA. \tag{4.1}$$

The equation for $1(G_y, e)$ is

$$xA + A^{34} = \sum_i S_i(xA^{a_i} + A^{35a_i}) + TA. \tag{4.2}$$

Set $y = 0$ in Equation (4.1). Then we have a solution for Equation (4.2) by selecting

$$S_i = P_i(y = 0); \quad T = Q(y = 0) + R(y = 0).$$

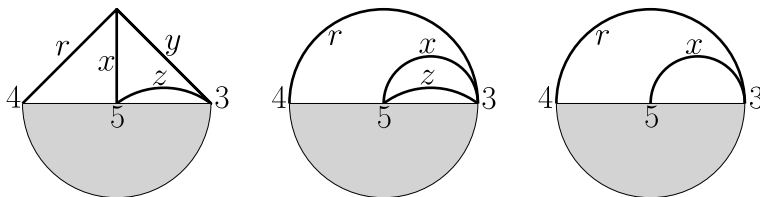
This completes the proof. □

PROPOSITION 4.2. *For the wheel graphs with $n > 3$ sides, Condition 1 is false for all rim edges.*

PROOF. This will be proved by induction. Let W_n be a wheel graph with n sides and let r be any rim edge. We will show that for $n > 3$ we have

$$1(W_n, r) \Rightarrow 1(W_{n-1}, r). \tag{4.3}$$

Assuming $n > 3$ guarantees that r is a regular edge of W_{n-1} .



Now assume $1(W_n, r)$. Then, looking at the diagram above, we see that r satisfies the conditions of Lemma 4.1, so we can contract the rim edge y beside r to get $1(W_{n,y}, r)$.

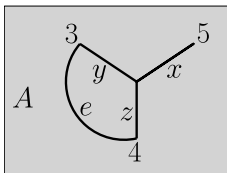
But then x and z are parallel in $W_{n,y}$, so by Proposition 3.3 we can delete z to get $1(W_{n,y}^z, r)$. But this is just Condition 1 for the wheel graph with one less side, as $W_{n,y}^z = W_{n-1}$. Therefore we have proved Equation (4.3). Then, taking the contrapositive,

$$\neg 1(W_{n-1}, r) \Rightarrow \neg 1(W_n, r)$$

for $n > 3$.

The base cases can be verified explicitly. By induction with $n = 4$ as the base case we obtain $\neg 1(W_n, r)$ for $n > 3$. □

LEMMA 4.3. *$1(G, e)$ is true for all graphs having the form*



with the equation

$$G = e[(yz + xz + xy)A + xA, +yA + zA + A] + (yz + xz + xy)A + (y + z)A.$$

PROOF. The equation $1(G, e)$ reads

$$\begin{aligned} & (yz + xz + xy)A + (y + z)A \\ &= \sum_i P_i [(yz + xz + xy)A^{a_i} + xA^{34a_i} + yA^{45a_i} + zA^{35a_i} + A^{345a_i}] \\ & \quad + Q[(y + z)A + A] + R[(x + z)A + A] + S[(x + y)A + A]. \end{aligned} \tag{4.4}$$

Now we will make use of Proposition 2.1 to try to guess what the solution polynomials to Equation (4.4) are.

The P_i are the logical polynomials to guess first because A^{345} occurs with them and nowhere else on the right-hand side of Equation (4.4). Then to equate the A^{345} terms, we will try

$$P_i = \frac{(y+z)a_i}{l-1}.$$

The loop number¹ of G , which we denote by l , is chosen as a reference value. Here, A (with four fewer edges and one less vertex than G) has loop number $l-3$, A^{34} (with one less vertex than A) has loop number $l-2$, and A^{345} has loop number $l-1$.

With this choice of the P_i , the $\sum_i P_i[\dots]$ in Equation (4.4) evaluates to

$$\frac{y+z}{l-1} [(yz+xz+xy)(l-3)A + x(l-2)A^{34} + y(l-2)A^{45} + z(l-2)A^{35} + (l-1)A^{345}].$$

Next, we show that the A^{34} terms work out. Notice that

$$\frac{l-2}{l-1} = 1 - \frac{1}{l-1}.$$

Then we endeavour to satisfy

$$(yz+xz+xy)A^{34} = x(y+z)\left(1 - \frac{1}{l-1}\right)A^{34} + QA^{34},$$

which can be accomplished by choosing

$$Q = yz + \frac{x(y+z)}{l-1}.$$

Finally, to make A^{45} and A^{35} work out, we choose

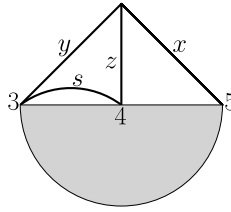
$$R = \frac{-y(y+z)(l-2)}{l-1}; \quad S = \frac{-z(y+z)(l-2)}{l-1}.$$

Then, with these choices, by construction, all of the terms involving A^{34} , A^{45} , A^{35} , and A^{345} equate on both sides of P_i, Q, R, S . Finally, then, as the reader can verify, the terms involving A all cancel, which gives a valid solution set P_i, Q, R, S . □

PROPOSITION 4.4. *For the wheel graphs with $n \geq 3$ sides, Condition 1 is true for all spoke edges.*

PROOF. This is an immediate consequence of Lemma 4.3. Let s be a spoke edge of W_n .

¹In graph theory language the loop number of a graph is the dimension of the cycle space of the graph; the term ‘loop number’ comes from physics.



With the edges as labelled, we see that (W_n, s) has the form specified by the lemma, and therefore $1(W_n, s)$ is true. □

The cancellation of terms involving A in Lemma 4.3 seems like something of a minor miracle. It is suggestive that making a Δ to Y transformation in the graph ought to preserve condition 1. In fact, this possibility is what suggested Lemma 4.3 to us. See Section 7 for further discussion of this point.

5. Simultaneous combinations in series–parallel graphs

Condition 1 itself is not ideally suited to the recursive constructions involved in building series–parallel graphs. Specifically, one frequently wants to combine expansions of the polynomials G and \bar{G} but lacks any information on how the coefficient polynomials relate. To work around this, we first consider a stronger condition where the coefficients are controlled.

DEFINITION 5.1. Let G be a source–terminal (series–parallel) graph and $e_1 \in G$ an edge. We say that *simultaneous combination* holds for (G, e_1) , or $S(G, e_1)$ holds, if there are polynomials A_j, B, C such that

$$G = \sum A_j G^{1j} + B \cdot G^1$$

$$\bar{G} = \sum A_j \bar{G}^{1j} + C \cdot \bar{G}^1.$$

We say that *simultaneous combination* holds for G , or $S(G)$ holds, if $S(G, e)$ holds for all $e \in G$.

Note that if $S(G, e)$ holds, then by Euler’s Theorem we can choose either $B = 0$ or $C = 0$ in the statement of Definition 5.1. The freedom to choose which is 0 is quite handy and explains why this symmetric, albeit redundant, definition was chosen.

PROPOSITION 5.2. *The simultaneous combination property $S(G, e)$ implies that Condition 1 holds for $e \in G$.*

PROOF. Immediate from the definitions. □

Simultaneous combinations are well behaved with respect to both series and parallel operations.

LEMMA 5.3. *Let H, H' be source–terminal graphs and let $e \in H$. If $S(H, e)$ then $S(H \star H', e)$ and $S(H \not\star H', e)$.*

The basic plan is to put the required linear combinations together using Euler’s theorem. This technique will be a theme in what follows, but since this is the first such argument, we will go into detail.

PROOF FOR $S(H \star H', e)$. Let $e = e_1$, let $n := \deg(\overline{H})$, let $m := \deg(\overline{H'})$, and let $G := H \star H'$. Then

$$G = H\overline{H'} + \overline{H}H'$$

$$\overline{G} = \overline{H}\overline{H'}$$

so

$$\langle G^{1j} \rangle = \langle H^{1j}\overline{H'} + \overline{H}^{1j}H', H^1\overline{H'}^j + \overline{H}^1H'^j \rangle$$

$$\langle \overline{G}^{1j} \rangle = \langle \overline{H}^{1j}\overline{H'}^j, \overline{H}^1\overline{H'}^j \rangle.$$

We note that $\deg(\overline{H}) = \deg(H) + 1$, so by Euler’s theorem the components of the vector

$$\begin{bmatrix} (n-2) & (n-1) \\ m & (m-1) \end{bmatrix} \begin{bmatrix} H^1\overline{H'} \\ \overline{H}^1H' \end{bmatrix}$$

are in $\langle G^{1j} \rangle$. Thus, so is

$$m \sum_{j \in H} x_j G^{1j} - (n-2) \sum_{j \in H'} x_j G^{1j} = (n+m-2) \overline{H}^1 H'.$$

We also have that

$$m \sum_{j \in H} x_j \overline{G}^{1j} - (n-2) \sum_{j \in H'} x_j \overline{G}^{1j} = m(n-1) \overline{H}^1 \overline{H'}^j - (n-2) m \overline{H}^1 \overline{H'}^j$$

$$= m \overline{H}^1 \overline{H'}^j.$$

For $j \in H$, pick A_j, C as in the statement of $S(H, e_1)$ (using the previously observed freedom to set $B = 0$). Then for

$$B_j := \begin{cases} A_j + \frac{C}{n+m-2}(mx_j) & \text{if } j \in H, \\ -\frac{(n-2)C}{n+m-2}x_j & \text{otherwise,} \end{cases}$$

$$C' := \left(1 - \frac{m}{n+m-2}\right)C,$$

we verify that we have $S(H \star H', e)$:

$$\sum_{j \in G} B_j G^{1j} = \sum_{j \in H} \left(A_j + \frac{C}{n+m-2}(mx_j) \right) G^{1j} + \sum_{j \in H'} \left(-\frac{(n-2)C}{n+m-2}x_j \right) G^{1j}$$

$$= \sum_{j \in H} A_j G^{1j} + \sum_{j \in H} \frac{C}{n+m-2}(mx_j) G^{1j} - \sum_{j \in H'} \frac{(n-2)C}{n+m-2}x_j G^{1j}$$

$$\begin{aligned} &= \sum_{j \in H} A_j G^{1j} + \frac{C}{n+m-2} \left(m \sum_{j \in H} x_j G^{1j} - (n-2) \sum_{j \in H'} x_j G^{1j} \right) \\ &= \sum_{j \in H} A_j (H^{1j} \overline{H'} + \overline{H}^{1j} H') + C \cdot \overline{H}^1 H'. \end{aligned}$$

By $S(H, e_1)$, we can simplify the remaining expression

$$\begin{aligned} &= \sum_{j \in H} A_j H^{1j} \overline{H'} + \sum_{j \in H} A_j \overline{H}^{1j} H' + C \cdot \overline{H}^1 H' \\ &= H \overline{H'} + H' (\overline{H} - C \overline{H}^1) + C \cdot \overline{H}^1 H' \\ &= G. \end{aligned}$$

Meanwhile,

$$\begin{aligned} \sum_{j \in G} B_j \overline{G}^{1j} &= \sum_{j \in H} \left(A_j + \frac{C}{n+m-2} (m x_j) \right) \overline{G}^{1j} + \sum_{j \in H'} \left(-\frac{(n-2)C}{n+m-2} x_j \right) \overline{G}^{1j} \\ &= \sum_{j \in H} A_j \overline{G}^{1j} + \sum_{j \in H} \frac{C}{n+m-2} (m x_j) \overline{G}^{1j} - \sum_{j \in H'} \frac{(n-2)C}{n+m-2} x_j \overline{G}^{1j} \\ &= \sum_{j \in H} A_j \overline{G}^{1j} + \frac{C}{n+m-2} \left(m \sum_{j \in H} x_j \overline{G}^{1j} - (n-2) \sum_{j \in H'} x_j \overline{G}^{1j} \right) \\ &= \sum_{j \in H} A_j (\overline{H}^{1j} \overline{H'}) + \frac{C}{n+m-2} \cdot m \overline{H}^1 \overline{H'}. \end{aligned}$$

By $S(H, e_1)$, we can simplify the remaining expression

$$\begin{aligned} &= \overline{H'} (\overline{H} - C \overline{H}^1) + \frac{mC}{n+m-2} \cdot \overline{H}^1 \overline{H'} \\ &= \overline{G} - \left(1 - \frac{m}{n+m-2} \right) C \cdot \overline{G}^1. \end{aligned}$$

So $S(G, e_1)$ holds. □

PROOF FOR $S(H \not\sim H', e)$. We use a similar argument for the series join. Let $e = e_1$, let $n := \deg \overline{H}$, let $m := \deg \overline{H'}$, and let $G := H \not\sim H'$. Then

$$\begin{aligned} G &= HH' \\ \overline{G} &= \overline{H}H' + H\overline{H'} \end{aligned}$$

so

$$\begin{aligned} \langle G^{1j} \rangle &= \langle H^{1j} H', H^1 H'^j \rangle \\ \langle \overline{G}^{1j} \rangle &= \langle \overline{H}^{1j} H' + H^{1j} \overline{H'}, \overline{H}^1 H'^j + H^1 \overline{H'}^j \rangle. \end{aligned}$$

We note that $\deg(\overline{H}) = \deg(H) + 1$, so by Euler’s theorem, the coordinates of

$$\begin{bmatrix} (n-1) & (n-2) \\ (m-1) & m \end{bmatrix} \begin{bmatrix} \overline{H}^1 H' \\ H^1 \overline{H'} \end{bmatrix}$$

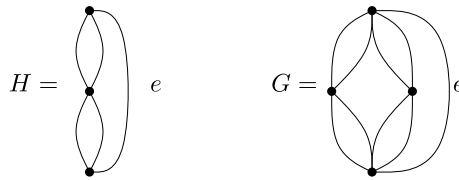


FIGURE 1. Graphs illustrating the gap between Condition 1 and S .

are in $\langle \bar{G}^{1j} \rangle$. Thus, so is

$$-(m-1) \sum_{j \in H} x_j \bar{G}^{1j} + (n-1) \sum_{j \in H'} x_j \bar{G}^{1j} = (n+m-2)H^1 \bar{H}'.$$

We also have that

$$\begin{aligned} & -(m-1) \sum_{j \in H} x_j G^{1j} + (n-1) \sum_{j \in H'} x_j G^{1j} \\ &= -(m-1)(n-2)H^1 H' + (n-1)(m-2)H^1 H' \\ &= (m-1)H^1 H'. \end{aligned}$$

For $j \in H$, pick A_j, B as in the statement of $S(H, e_1)$, this time suppressing $C = 0$. Then for

$$A'_j := \begin{cases} A_j - \frac{(m-1)B}{n+m-2}x_j & \text{if } j \in H, \\ \frac{(n-1)B}{n+m-2}x_j & \text{otherwise,} \end{cases}$$

$$B' := \left(1 - \frac{m-1}{n+m-2}\right)B,$$

we can verify that we have $S(H \sharp H', e)$ in exactly the same way as before. □

Simultaneous combination is stronger than Condition 1. For example, let G and H be as in Figure 1; then one can check that $1(H, e)$ is true but $S(H, e)$ is false. However, in some sense, simultaneous combination classifies when Condition 1 is stable under parallel join. In the example of Figure 1, we see that G is an extension of H by a parallel join but $1(G, e)$ is false. This is the content of Corollary 5.9.

First, we need a few observations on factorizations in Kirchhoff polynomials. Recall that a biconnected component of a graph G is a maximal connected subgraph of G that has no cut vertex. The terms block and biconnected component are synonymous.

LEMMA 5.4. *The nontrivial factors of the Kirchhoff polynomial of a loopless graph G correspond to the biconnected components of G which are not isomorphic to K_2 .*

Note that a tree has only copies of K_2 as biconnected components corresponding correctly to the Kirchhoff polynomial being 1. Also, the result can easily be extended to graphs with self-loops by noting that a self-loop contributes a factor of its variable and any such factors arise in this manner.

PROOF. First, note that spanning trees in the different biconnected components meet only at cut vertices, so they are independent. Thus, the Kirchhoff polynomial of G is the product of the Kirchhoff polynomials of these components.

Next, we need to show that there are no other factors. Suppose G has no cut vertices but assume for a contradiction that $G = P \cdot Q$ for some nonconstant P and Q . Note, *a priori*, that we do not know whether P, Q are Kirchhoff polynomials of a graph. The Kirchhoff polynomial G is linear in each of the edge variables, so any factorization gives a partition of the edges of G . Call the two subgraphs that the factorization into $P \cdot Q$ induces A , whose edges are red, and B , whose edges are blue. Let $H^0(X)$ denote the set of connected components of the graph X . Let Υ be the graph whose vertices are elements of $H^0(A) \cup H^0(B)$ and whose edges are shared vertices in G .

Choose spanning trees T_a for each $a \in H^0(A)$, and notice $\cup_a T_a$ can always be extended to a spanning tree of G using (necessarily) blue edges. Thus,

$$\lambda \left(\prod_{a \in H^0(A)} \prod_{e \notin T_a} e \right)$$

is a monomial of P , and $\lambda \neq 0$, since otherwise this product will never show up in the polynomial G , contradicting that the spanning trees T_a can be extended to a spanning tree of G . Similarly,

$$\mu \left(\prod_{b \in H^0(B)} \prod_{e \notin T_b} e \right)$$

is a nonzero monomial of Q . But now

$$\lambda \mu \left(\prod_{a \in H^0(A)} \prod_{e \notin T_a} e \right) \left(\prod_{b \in H^0(B)} \prod_{e \notin T_b} e \right)$$

is a monomial of G . In particular, $\cup_a T_a \cup \cup_b T_b$ must be a spanning tree, and hence acyclic, so Υ must be a tree. In particular, G has at least one cut vertex. \square

COROLLARY 5.5. *If G is biconnected, does not have self-loops, and is not K_2 , then the Kirchhoff polynomial of G is irreducible and nonconstant in every edge variable.*

PROOF. By Lemma 5.4 the irreducibility is immediate. If G is biconnected and not K_2 then it does not have a bridge, so each edge is avoided in at least one spanning tree. \square

COROLLARY 5.6. *Let H be a connected subgraph of a biconnected graph G that does not have self-loops. If $\gcd(H, G) \neq 1$, then $H = G$ as graphs.*

PROOF. Since $\gcd(H, G) \neq 1$, we have that neither G nor H is a tree. As H is connected, its Kirchhoff polynomial is nonzero. Since the Kirchhoff polynomial of G is irreducible, by Corollary 5.5 we see that G and H have the same Kirchhoff polynomials. In particular, H is nonconstant in all of the edge variables of G , so the subgraph H must contain all of the edges of G . \square

We remark that the condition that H be a subgraph of G in the previous corollary is essential.

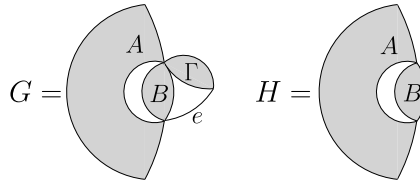


FIGURE 2. Graphs used for proof of Proposition 5.8.

LEMMA 5.7. *Let G be a connected graph with terminal vertices s and t . If the polynomials G and \overline{G} have a nontrivial common factor then either G has a biconnected component which is connected to the rest of G at a single vertex and includes neither s nor t except possibly one of them as the cut vertex, or G has a self-loop.*

PROOF. If G has a self-loop, the result is immediate, so we may assume otherwise. Let Γ be the graph G with s and t identified and note \overline{G} is the Kirchhoff polynomial Γ , which by the gcd condition is nontrivial. Additionally, note that there is a natural morphism of graphs $\varphi: G \rightarrow \Gamma$ induced by identifying s and t . Let v be the vertex in Γ that is the image of both s and t under φ . Observe that the restriction $\varphi: G \setminus \{s, t\} \rightarrow \Gamma \setminus v$ is an isomorphism.

Observe that $\deg(\overline{G}) = \deg(G) + 1$, so \overline{G} and G having a factor in common implies that \overline{G} has a nontrivial factor. Furthermore, Γ is not K_2 , and so by Corollary 5.5 we see that Γ has at least one cut vertex.

Let $\{Y_1, \dots, Y_n\}$ be the biconnected components of Γ . If Γ has a cut vertex w which is not v then there is a Y_j not containing v . Thus $\varphi^{-1}(Y_j)$ is a biconnected component of $G \setminus \{s, t\}$. There is at most one neighbour of v in Y_j , namely w , so it follows that $\varphi^{-1}(Y_j)$ is a biconnected component of G that does not contain either s or t .

Otherwise, the unique cut vertex of Γ is v . Suppose G has no biconnected component as described in the statement. Then either G is biconnected or G is a series join of at least two biconnected components running from s to t . In this latter case, Γ is a cycle of biconnected components and hence has no cut vertex, giving a contradiction. Finally, suppose the graph G is biconnected. Since $\gcd(G, \Gamma) \neq 1$, we see that G is not K_2 , and so by Corollary 5.5 the polynomial G is irreducible. Thus by looking at the degrees, we see $\overline{G} = G \cdot L$ where $\deg L = 1$. Also, by Corollary 5.5, the polynomial G is nonconstant in every edge variable, so any variable in L appears quadratically in \overline{G} , which is a contradiction. Thus, G has a biconnected component as described in the statement. □

PROPOSITION 5.8. *Let G and H be series-parallel graphs of the form illustrated in Figure 2 and suppose Condition 1 holds for e in $G \star H$. Then $S(G, e)$ holds.*

PROOF. Let $e = e_1$ and note that series-parallel graphs do not contain self-loops.

In what follows, we identify the edge variables of H with their counterparts in $H \hookrightarrow G$. Let $\Gamma := G \setminus (H \cup \{e\})$. Note $G^1 = H\Gamma$ and $\overline{G}^1 = \overline{H}\Gamma$. Since Condition 1

holds, we may write $G \star H$ as

$$\begin{aligned} G\bar{H} + \bar{G}H &= \sum_{j \in H \subset G} B'_j H^j \Gamma \bar{H} + \sum_{j \in \Gamma} C_j H \Gamma^j \bar{H} + \sum_{j \in H} B'_j H \Gamma \bar{H}^j \\ &\quad + \sum_{j \in H \subset G} B'_j \bar{H}^j \Gamma H + \sum_{j \in \Gamma} C_j \bar{H} \Gamma^j H + \sum_{j \in H} B'_j \bar{H} \Gamma H^j + DH \Gamma \bar{H} \\ &= \bar{H} \left(\sum_{j \in H} B_j G^{1j} + \sum_{j \in \Gamma} C_j G^{1j} \right) \\ &\quad + H \left(\sum_{j \in H} B_j \bar{G}^{1j} + \sum_{j \in \Gamma} C_j \bar{G}^{1j} \right) + DH \Gamma \bar{H} \end{aligned}$$

where $B_j = B'_j + B''_j$. Since H is series-parallel, it does not have a cut vertex with a biconnected component joining only at that vertex and containing neither s nor t except possibly as the cut vertex, so $\gcd(H, \bar{H}) = 1$. Thus, from rearranging and factoring we get

$$\bar{H} \mid \bar{G} - \left(\sum_{j \in H} B_j \bar{G}^{1j} + \sum_{j \in \Gamma} C_j \bar{G}^{1j} \right) \tag{5.1}$$

$$H \mid G - \left(\sum_{j \in H} B_j G^{1j} + \sum_{j \in \Gamma} C_j G^{1j} \right). \tag{5.2}$$

If Γ were also to divide each of the expressions above, then by disjointness of the variables of H, Γ we see $\bar{G}^1 = \bar{H}\Gamma$ divides the first expression and $G^1 = H\Gamma$ divides the second. Hence, $S(G, e)$ would hold. We proceed to prove exactly this.

Notice for $j \in H$ that Γ remains a 1-cut component of $G \setminus \{1, j\}$, so $\Gamma \mid \bar{G}^{1j}$ and $\Gamma \mid G^{1j}$. Since H is connected we get by the 2-vertex-cut formula that

$$\begin{aligned} G &= H(x_e \Gamma + \bar{\Gamma}) + H \overset{uv}{\Gamma} \\ \bar{G} &= \bar{H}(x_e \Gamma + \bar{\Gamma}) + \overset{st, uv}{H} \Gamma \end{aligned}$$

where u, v are the source and terminal of $\Gamma \setminus e$. So the right-hand sides of (5.1) and (5.2) modulo $\langle \Gamma \rangle$ become

$$\begin{aligned} H\bar{\Gamma} - \sum_{j \in \Gamma} C_j G^{1j} + \langle \Gamma \rangle \\ \bar{H}\bar{\Gamma} - \sum_{j \in \Gamma} C_j \bar{G}^{1j} + \langle \Gamma \rangle, \end{aligned}$$

or, further simplifying with our explicit expressions of \bar{G}, G ,

$$\begin{aligned} H\bar{\Gamma} - \sum_{j \in \Gamma} C_j H \Gamma^j + \langle \Gamma \rangle \\ \bar{H}\bar{\Gamma} - \sum_{j \in \Gamma} C_j \bar{H} \Gamma^j + \langle \Gamma \rangle. \end{aligned}$$

Again, since H, \bar{H} are coprime to Γ it suffices to verify

$$\Gamma \bar{\Gamma} - \sum_{j \in \Gamma} C_j \Gamma^j.$$

Returning to the original expression, we see that

$$\begin{aligned} G \star H + \langle \Gamma \rangle &= \sum_{j \in H} B_j G^{1j} \bar{H} + \sum_{j \in \Gamma} C_j G^{1j} \bar{H} \\ &\quad + \sum_{j \in H} B_j \bar{G}^{1j} H + \sum_{j \in \Gamma} C_j \bar{G}^{1j} H + \langle \Gamma \rangle \\ &= \sum_{j \in \Gamma} C_j G^{1j} \bar{H} + \sum_{j \in \Gamma} C_j \bar{G}^{1j} H + \langle \Gamma \rangle \\ &= \sum_{j \in \Gamma} C_j \Gamma^j H \bar{H} + \sum_{j \in \Gamma} C_j \Gamma^j \bar{H} H + \langle \Gamma \rangle \\ &= 2 \sum_{j \in \Gamma} C_j \Gamma^j H \bar{H} + \langle \Gamma \rangle. \end{aligned}$$

Meanwhile, from Figure 2 we see by the two-vertex cut formula that

$$\begin{aligned} G \star H + \langle \Gamma \rangle &= (\bar{\Gamma} + x_e \Gamma)(H \star H) + \Gamma(H \overset{st,uv}{H} + \bar{H} \overset{uv}{H}) + \langle \Gamma \rangle \\ &= \bar{\Gamma}(H \star H) + \langle \Gamma \rangle \\ &= 2\bar{H}H\bar{\Gamma} + \langle \Gamma \rangle. \end{aligned}$$

Thus, taking the difference of the two previous calculations,

$$0 = 2\bar{H}H\left(\bar{\Gamma} - \sum_{j \in \Gamma} C_j \Gamma^j\right) + \langle \Gamma \rangle.$$

Since both \bar{H} and H are coprime to Γ , we may cancel these terms to obtain the result. □

COROLLARY 5.9. *Suppose G is a series-parallel graph such that $S(G, e)$ does not hold but Condition 1 does hold for e . Then there is a series-parallel graph H such that Condition 1 fails for e in $G \star H$.*

PROOF. Note that G is series-parallel and so does not have self-loops. Condition 1 holds for e in G , so e is regular, and so at some point in the construction of G by series and parallel operations we have the subgraph $B \star (C_1 \dot{\wedge} e \dot{\wedge} C_2)$, where one or both of C_1 and C_2 may be empty. Without loss of generality, we may instead consider $B \star (C \dot{\wedge} e)$ where $C = C_1 \dot{\wedge} C_2$, since the Kirchhoff polynomial is invariant under this transformation. By Proposition 2.3, we may subdivide e if necessary to ensure that C is nonempty (if e is subdivided, the two halves are the two-edge cut-set). Therefore G is of the form to apply Proposition 5.8. Let H be as in Proposition 5.8. Since $S(G, e)$ does not hold by assumption, we see by Proposition 5.8 that Condition 1 does not hold for e in $G \star H$. □

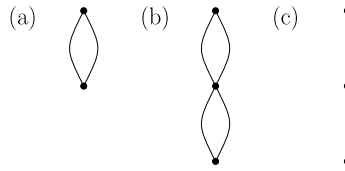


FIGURE 3. Graph (a) satisfies condition T . Graph (b) does not satisfy condition T , which demonstrates that condition T is not well behaved with respect to series join. Finally, graph (c) does not satisfy condition T for the trivial reason that the derivatives of its Kirchhoff polynomial are identically 0.

6. Carving out a series-parallel class for Condition 1

As we saw in the previous section, simultaneous combination is well behaved with respect to series and parallel joins, implies Condition 1, and is strictly stronger than Condition 1. Furthermore, the way in which it is stronger is itself well behaved under parallel join in the sense made precise in Corollary 5.9.

Consequently, characterizing a class of series-parallel graphs which have the simultaneous combination property would give a class of series-parallel graphs whose edges all satisfy Condition 1. Because of the nice behaviour with respect to series and parallel joins, one would expect that we could give a recursively defined family of series-parallel graphs with the simultaneous combination property. Unfortunately, simultaneous combination is not well suited to pulling out a good base case for implementing this plan. We need instead a variant on simultaneous combination to generate a larger class of graphs that satisfy $S(G)$. The point of this condition shows up most strongly in Corollary 6.6; specifically, it captures when a parallel join with edge e will satisfy $S(G, e)$. As an added benefit, we are then able to apply these technical results to identify a combinatorial condition that ensures $S(G)$.

DEFINITION 6.1. Let G be a series-parallel graph. We say $T(G)$ holds if there are polynomials A_j, C such that

$$\begin{aligned} \bar{G} &= \sum A_j G^j \\ C \cdot \bar{G} &= \sum A_j \bar{G}^j \in \langle \bar{G} \rangle. \end{aligned}$$

We give some examples of condition T for small graphs in Figure 3. It is helpful to compare these examples to the example in Figure 1. Condition 1 is false for the edge $e \in G$ in Figure 1. By Corollary 5.9, we see that $S(H, e)$ does not hold, where H is the graph from Figure 1. Finally, we will see by Lemma 6.7(i) that condition T fails for the graph in Figure 3b, explaining why $S(H, e)$ fails.

Condition T is well behaved with respect to parallel join, and we can understand both joins with paths.

LEMMA 6.2. If H, H' are series-parallel graphs and $T(H)$ then $T(H \star H')$.

PROOF. Let $G := H \star H'$ and note that $G = H\overline{H'} + \overline{H}H'$. Let $n := \deg \overline{H}$ and let $m := \deg(\overline{H'})$. Thus,

$$\langle G^j \rangle = \langle H^j\overline{H'} + \overline{H}^jH', H\overline{H'}^j + \overline{H}H'^j \rangle.$$

Now,

$$\overline{H}H' = -\frac{1}{n+m-1} \left((m-1) \sum_{j \in H} x_j G^j - n \sum_{j \in H'} x_j G^j \right).$$

By $T(H)$, we may choose $B_j, j \in H$ and C such that

$$\sum_{j \in H} B_j G^j = \sum_{j \in H} B_j (H^j\overline{H'} + \overline{H}^jH') = \overline{H}\overline{H'} + C\overline{H}H'.$$

Let

$$B'_j := \begin{cases} B_j - \frac{C(m-1)}{n+m-1}x_j & \text{if } j \in H, \\ \frac{nC}{n+m-1}x_j & \text{otherwise.} \end{cases}$$

We verify that $T(G)$ is satisfied with a calculation.

$$\begin{aligned} \sum_{j \in G} B'_j G^j &= \sum_{j \in H} \left(B_j - \frac{C(m-1)}{n+m-1}x_j \right) G^j + \sum_{j \in H'} \left(\frac{nC}{n+m-1}x_j \right) G^j \\ &= \overline{H}\overline{H'} + C\overline{H}H' - C\overline{H}H' \\ &= \overline{G}. \end{aligned}$$

Meanwhile,

$$\begin{aligned} \sum_{j \in G} B'_j \overline{G}^j &= \sum_{j \in H} \left(B_j - \frac{C(m-1)}{n+m-1}x_j \right) \overline{H}^j\overline{H'} + \sum_{j \in H'} \left(\frac{nC}{n+m-1}x_j \right) \overline{H}\overline{H'}^j \\ &= \overline{H}' \left(\sum_{j \in H} B_j \overline{H}^j \right) + \frac{C}{n+m-1} \left(\sum_{j \in H'} nx_j \overline{H}\overline{H'}^j - \sum_{j \in H} (m-1)x_j \overline{H}^j\overline{H}' \right) \\ &= \overline{H}' \left(\sum_{j \in H} B_j \overline{H}^j \right) + \frac{C}{n+m-1} (nm\overline{H}\overline{H}' - n(m-1)\overline{H}\overline{H}'), \end{aligned}$$

which by $T(H)$ is in the ideal $\langle \overline{H}\overline{H}' \rangle$. □

Condition T has special behaviour for paths which we illustrate with the following lemmas.

LEMMA 6.3. *Let H be a series-parallel graph such that $T(H)$. Then $T(H \sharp K_2)$.*

PROOF. Let e be the edge of K_2 , let $n := \deg H$, and let A_j, C be polynomials such that

$$\sum_{j \in H} A_j H^j = \overline{H}$$

$$\sum_{j \in H} A_j \bar{H}^j = C \cdot \bar{H}.$$

Note that for the polynomials, $H \sharp K_2 = H$ and $\overline{H \sharp K_2} = \bar{H} + x_e H$. Now let

$$C' := \left(C + x_e + \frac{n+1}{n} x_e \right)$$

and observe that

$$\sum_{j \in H} \left(A_j + \frac{x_e}{n} x_j \right) H^j + x_e (C' - x_e) \frac{\partial}{\partial x_e} (H) = \bar{H} + x_e H + 0$$

together with

$$\begin{aligned} & \sum_{j \in H} \left(A_j + \frac{x_e}{n} x_j \right) (\bar{H} + x_e H)^j + x_e (C' - x_e) \frac{\partial}{\partial x_e} (\bar{H} + x_e H) \\ &= C \cdot \bar{H} + x_e \bar{H} + \frac{n+1}{n} x_e \bar{H} + x_e^2 H + x_e (C' - x_e) H \\ &= \left(C + x_e + \frac{n+1}{n} x_e \right) \bar{H} + C' x_e H \\ &= C' (\bar{H} + x_e H) \end{aligned}$$

proves the result. □

LEMMA 6.4. *Let H be a series-parallel graph and Γ a path. Then $T(H \star \Gamma)$.*

PROOF. Let $G := H \star \Gamma$, let $n := \text{deg}(G)$, and let z_1, \dots, z_m be the edge variables for Γ . Now,

$$\bar{G} := (z_1 + \dots + z_m) \bar{P}, \quad G = (z_1 + \dots + z_m) H + \bar{H},$$

from which it is clear that

$$\bar{G} = (z_1 + \dots + z_m) \left(\frac{1}{n} \sum_{j \in G} x_j G^j - \sum_{j \in \Gamma} z_j G^j \right)$$

and

$$\begin{aligned} & (z_1 + \dots + z_m) \left(\frac{1}{n} \sum_{j \in G} x_j \bar{G}^j - \sum_{j \in \Gamma} z_j \bar{G}^j \right) \\ &= (z_1 + \dots + z_m) \left(\frac{n+1}{n} \bar{G} - \bar{G} \right) \\ &\in \langle \bar{G}^j \rangle, \end{aligned}$$

so we have $T(G)$. □

Now we are positioned to see what condition $T(H)$ is really for, namely, guaranteeing $S(G, e)$ when G is H parallel joined with e and, more generally, for series joins of such H s.

LEMMA 6.5. *Let \mathcal{H} be a finite collection of series–parallel graphs such that $T(H)$ holds for each $H \in \mathcal{H}$. Let G be the series join of all $H \in \mathcal{H}$. Then $\overline{G} \in \langle G^j \rangle$.*

PROOF. We have that

$$G = \prod_{H \in \mathcal{H}} H, \quad \overline{G} = \sum_{H \in \mathcal{H}} \overline{H} \prod_{H' \neq H} H'.$$

So,

$$\langle G^j \rangle = \bigoplus_{H \in \mathcal{H}} \left\langle H^j \prod_{H' \neq H} H' : j \in H \right\rangle.$$

By $T(H_i)$, we may choose B_j^H such that

$$\sum_{H \in \mathcal{H}} \sum_{j \in H} B_j^H G^j = \sum_{H \in \mathcal{H}} \overline{H} \prod_{H' \neq H} H'. \quad \square$$

The next corollary is an easy consequence of Lemma 6.5 but is not essential to the rest of the paper.

COROLLARY 6.6. *Let \mathcal{H} be a finite collection of series–parallel graphs such that $T(H)$ holds for each $H \in \mathcal{H}$. Let G be the series join of all $H \in \mathcal{H}$. Then Condition 1 holds for (the edge of) K_2 in $K_2 \star G$.*

PROOF. Let $\Gamma := K_2 \star G$ and e be the edge of K_2 . Then

$$\Gamma := x_e G + \overline{G}.$$

By the previous lemma, Condition 1 holds for e in Γ . □

Next we look at the base cases for building series–parallel graphs with the simultaneous combination property. A notable obstacle in classifying series–parallel graphs satisfying simultaneous combination is the fact that a series–parallel graph G can decompose as $H \star H'$ where one of H or H' has edges which are not regular. Lemma 6.7 shows how we can sometimes overcome this obstacle. The distinct cases of Lemma 6.7 are due to the fact that condition T and simultaneous combination are sensitive to the marking of source and terminal.

LEMMA 6.7. *Let $H := H_1 \dot{\star} \dots \dot{\star} e_1 \dot{\star} \dots \dot{\star} H_r$ be a series–parallel graph where $e_1 \in H$ is a bridge, and let Γ be a series–parallel graph. If any of:*

- (i) $T(\Gamma)$ holds and H is a path;
- (ii) $T(\Gamma)$ holds and $T(H_i)$ holds for all i ;
- (iii) Γ is a path and $T(H_i)$ holds for all i ;

are satisfied, then $S(H \star \Gamma, e_1)$ and $T(H \star \Gamma)$.

PROOF. Note that $T(\Gamma)$ implies $T(\Gamma \star H)$, so this condition is satisfied in cases (i) and (ii). Let $G := H \star \Gamma$. We prove that $S(G, e_1)$ holds case by case.

(i) Note that

$$G = (x_1 + \dots + x_n)\Gamma + \bar{\Gamma}, \quad \bar{G} = (x_1 + \dots + x_n)\bar{\Gamma}.$$

Let A_j, C be the polynomials as in $T(\Gamma)$. Then

$$\sum_{j \in \Gamma} A_j G^{1j} + (x_1 + \dots + x_n)G^1 = G$$

and

$$\sum_{j \in \Gamma} A_j \bar{G}^{1j} + ((x_1 + \dots + x_n) - C)\bar{G}^1 = \bar{G}$$

so we are done.

(ii) We observe that the claim is equivalent to proving $S((H_1 \not\ell \dots \not\ell H_r \not\ell e_1) \star \Gamma, e_1)$, since the relevant polynomials associated to these two graphs are identical. Let us relabel $H := H_1 \not\ell \dots \not\ell H_r$ and $G := (H \not\ell e_1) \star \Gamma$. Now,

$$G = \Gamma(Hx + \bar{H}) + H\bar{\Gamma}$$

$$\bar{G} = \bar{\Gamma}(Hx + \bar{H}).$$

Choose B_j, D as in $T(\Gamma)$. Since $T(H_i)$ for all i , we have by Lemma 6.5 that there are C_j such that

$$\sum_{j \in H} C_j H^j = \bar{H}.$$

Letting $m := \deg \Gamma$, we verify that

$$D' := -\left(D + \frac{x}{m}\right), \quad A_j := \begin{cases} \frac{xx_j}{m} + B_j & \text{if } j \in \Gamma, \\ C_j & \text{otherwise,} \end{cases}$$

are certificates for $S(H \star \Gamma, x)$:

$$\sum_{j \in G} A_j G^{1j} = \sum_{j \in H} C_j \Gamma H^j + \sum_{j \in \Gamma} \left(\frac{xx_j}{m} + B_j\right) \Gamma^j H$$

$$= \Gamma \bar{H} + x\Gamma H + \bar{\Gamma} H.$$

Meanwhile,

$$\sum_{j \in G} A_j \bar{G}^{1j} - D\bar{G}^1 = \sum_{j \in H} C_j \bar{\Gamma} H^j + \sum_{j \in \Gamma} \left(\frac{xx_j}{m} + B_j\right) \bar{\Gamma}^j H - \left(D + \frac{x}{m}\right) \bar{\Gamma} H$$

$$= \bar{\Gamma} \bar{H} + x\bar{\Gamma} H.$$

(iii) Again we observe that the claim is equivalent to proving

$$S((H_1 \not\ell \dots \not\ell H_r \not\ell e_1) \star \Gamma, e_1),$$

so we relabel $H := H_1 \not\ell \dots \not\ell H_r$. Write $\bar{\Gamma} = z_1 + \dots + z_m$ and note

$$G = H(z_1 + \dots + z_m) + (x_e H + \bar{H})$$

$$\bar{G} = (x_e H + \bar{H})(z_1 + \cdots + z_m).$$

Since $T(H_i)$ holds for all i , we may choose A_j as in Lemma 6.5. Now let

$$B_j := \begin{cases} A_j & \text{if } j \in H, \\ 0 & \text{otherwise,} \end{cases}$$

$$C := (z_1 + \cdots + z_m + x_e), \quad D := x_e.$$

Then

$$\sum_{j \in G} B_j G^{e_j} + CG^e = \bar{H} + (z_1 + \cdots + z_m + x_e)H,$$

$$\sum_{j \in G} B_j \bar{G}^{e_j} + D\bar{G}^e = (z_1 + \cdots + z_m)\bar{H} + x_e(z_1 + \cdots + z_m)H,$$

and so $S(G, e_1)$ holds. In this case we also need to show $T(G)$, but this follows from Lemma 6.4. □

Cycles are graphs which behave very nicely under series and parallel joins despite the fact that Condition 1 is false for every edge. It will be convenient to use cycles as building blocks for larger graphs satisfying simultaneous combination.

LEMMA 6.8. *Let H be a series-parallel graph which is a cycle and Γ be any series-parallel graph. Then $T(H)$, $T(H \star \Gamma)$, and $S(H \star \Gamma, e)$ holds for any edge $e \in H$. If Γ is not a path then $S(H\sharp\Gamma, e)$ as well.*

PROOF. Write $H := \mathcal{P}\mathcal{P}'$ with $\mathcal{P}, \mathcal{P}'$ paths. Let Γ be a series-parallel graph. By Lemma 6.4, we get that all of $T(\mathcal{P} \star \Gamma)$, $T(\mathcal{P}' \star \Gamma)$, and $T(H)$ hold. Thus, if $e \in \mathcal{P}$ we have $S(\mathcal{P} \star (\Gamma \star \mathcal{P}'), e)$ by Lemma 6.7. Moreover, $T(\Gamma \star \mathcal{P})$ implies $T(\Gamma \star H)$ by Lemma 6.2.

Finally we consider series join. Let $G := H\sharp\Gamma$, let x_i denote the edge variables of \mathcal{P} , and let y_i denote the edge variables of \mathcal{P}' . Note

$$\bar{G} = (x_1 + \cdots + x_n)(y_1 + \cdots + y_m)\Gamma + (x_1 + \cdots + x_n + y_1 + \cdots + y_m)\bar{\Gamma}$$

$$G = (x_1 + \cdots + x_n + y_1 + \cdots + y_m)\Gamma$$

$$= H\Gamma.$$

Then

$$\langle G^{1j} \rangle = \langle 0, \Gamma^j \rangle$$

$$\langle \bar{G}^{1j} \rangle = \langle 0, \Gamma, (y_1 + \cdots + y_m)\Gamma^j + \bar{\Gamma}^j \rangle.$$

Let $k := \deg \Gamma$. We verify that the choice of

$$B_j := \begin{cases} \frac{1}{k} Hx_j & \text{if } j \in \Gamma, \\ \bar{H} - (y_1 + \cdots + y_m)\left(1 - \frac{1}{k}\right)H & \text{if } x_j = y_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$C := -\frac{1}{k}H$$

works as a certificate for $S(G, e_1)$. We see that

$$\sum_{j \in G} B_j G^{1j} = \sum_{j \in \Gamma} \frac{1}{k} H x_j \Gamma^j + 0$$

and that

$$\begin{aligned} \sum_{j \in G} B_j \bar{G}^{1j} + C \bar{G}^1 &= \sum_{j \in \Gamma} \frac{1}{k} H x_j ((y_1 + \dots + y_m) \Gamma^j + \bar{\Gamma}^j) \\ &\quad + \left(\bar{H} - (y_1 + \dots + y_m) \left(1 - \frac{1}{k} \right) H \right) \Gamma - \frac{1}{k} H ((y_1 + \dots + y_m) \Gamma + \bar{\Gamma}) \\ &= (y_1 + \dots + y_m) H \Gamma + \frac{k+1}{k} H \bar{\Gamma} + \bar{H} \Gamma - (y_1 + \dots + y_m) \left(1 - \frac{1}{k} \right) H \Gamma \\ &\quad - \frac{1}{k} H ((y_1 + \dots + y_m) \Gamma + \bar{\Gamma}) \\ &= H \bar{\Gamma} + \bar{H} \Gamma + \left(1 - \left(1 - \frac{1}{k} \right) - \frac{1}{k} \right) (y_1 + \dots + y_m) H \Gamma. \end{aligned}$$

So we are done. □

COROLLARY 6.9. *Let H be a cycle, let \mathcal{P} be a path, let $M := H \star \mathcal{P}$, and let Γ be a series-parallel graph. Then:*

- (i) $T(M)$;
- (ii) $T(M \star \Gamma)$;
- (iii) $S(M \star \Gamma, e)$ for all $e \in H$; and
- (iv) if $T(\Gamma)$ then $S(M \star \Gamma, e)$ for all $e \in M$.

PROOF. We obtain $T(M)$ by repeatedly invoking Lemma 6.3 and $T(M \star \Gamma)$ from Lemma 6.2. Define $G := M \star \Gamma$. If $e \in \mathcal{P}$ and $T(\Gamma)$ then we have $S(G, e)$ from Lemma 6.7. Therefore (iv) follows from (iii) and all that is left to prove is (iii). If $e \in H$, let $n := \text{deg}(\bar{\Gamma})$ and write

$$\begin{aligned} \bar{H} &= (e + x_1 + \dots + x_r)(y_1 + \dots + y_m) \\ H &= e + x_1 + \dots + x_r + y_1 + \dots + y_m \\ \bar{\mathcal{P}} &= (z_1 + \dots + z_k), \quad \mathcal{P} = 1. \end{aligned}$$

Then

$$\begin{aligned} G &= H \bar{\Gamma} + (\bar{H} + H(z_1 + \dots + z_k)) \Gamma \\ \bar{G} &= \bar{M} \bar{\Gamma} = (\bar{H} + H(z_1 + \dots + z_k)) \bar{\Gamma} \\ G^e &= \bar{\Gamma} + (y_1 + \dots + y_m + z_1 + \dots + z_k) \Gamma \\ \bar{G}^e &= (y_1 + \dots + y_m + z_1 + \dots + z_k) \bar{\Gamma}. \end{aligned}$$

We notice that $G^{ey_1} = \Gamma$, $\overline{G}^{ey_1} = \overline{\Gamma}$,

$$\begin{aligned} & \sum_{j \in \Gamma} x_j HG^{ej} - (n - 1)HG^e \\ &= nH\overline{\Gamma} + (n - 1)H(y_1 + \dots + y_m + z_1 + \dots + z_k)\Gamma - (n - 1)HG^e \\ &= H\overline{\Gamma}, \end{aligned}$$

and

$$\begin{aligned} \sum_{j \in \Gamma} x_j H\overline{G}^{ej} &= nH(y_1 + \dots + y_m + z_1 + \dots + z_k)\overline{\Gamma} \\ &= nH\overline{G}^e. \end{aligned}$$

So it follows that

$$A_j := \begin{cases} x_j H & \text{if } j \in \Gamma, \\ \overline{H\mathcal{L}\mathcal{P}} & \text{if } j = y_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$B := -(n - 1)H, \quad C := -nH$$

serve as witnesses for $S(G, e)$. □

We are now ready to give a class of series-parallel graphs which satisfy Condition 1. A key class of graphs is those which have a planar embedding where one of the faces gives a Hamiltonian cycle, that is, the cycle defined by the face includes all the vertices of the graph.

For convenience in what follows, we make the following definition.

DEFINITION 6.10. We call the operation $\star^e(G) := G \star e$ the *restricted parallel join*. Similarly, we call $\mathcal{L}^e(G) := e\mathcal{L}G$ and $\mathcal{L}_e(G) := G\mathcal{L}e$ the *restricted series joins*.

REMARK 6.11. Let G be a graph with a planar embedding that has a Hamiltonian cycle as a facial cycle and let s, t be two vertices which are consecutive on this cycle (joined by the edge e). We may always view G as a noncrossing arc diagram with the Hamilton path from s to t as a horizontal line segment whose left endpoint is s and right endpoint t , and all other edges as arcs above this line segment. Since the aforementioned Hamilton path together with the edge e form the Hamiltonian face, the arc e is always the outermost arc. In such an arc diagram the Hamiltonian face is the unbounded face.

LEMMA 6.12. Let G be a graph that has a planar embedding with a Hamiltonian cycle as a facial cycle and let $s, t \in G$ be vertices consecutive on the Hamiltonian face. Then there is an isomorphism ϕ to a series-parallel graph (\widetilde{G}, s', t') built out of \star^e and \mathcal{L} such that $\phi(s) = s'$ and $\phi(t) = t'$. Conversely, if (G, s, t) is a series-parallel graph built out of \star^e and \mathcal{L} with no cut vertex, then G has a planar embedding with a Hamiltonian cycle as a facial cycle, where s, t are consecutive on the Hamiltonian face.

PROOF. (\Rightarrow) Let $\pi : G \rightarrow \mathbb{C}$ be a planar embedding with a Hamiltonian face and let $s, t \in G$ be two consecutive vertices on the Hamiltonian face. Then, as in Remark 6.11, we may view G as a noncrossing arc diagram with s the leftmost vertex, t the rightmost vertex, and e the outermost arc which completes the Hamiltonian face.

Remove the arc e from G . Then the biconnected components of the resulting graph are either single edges or are embedded in the plane as noncrossing arc diagrams with an arc connecting the leftmost and rightmost vertices (this is still true even when e has a parallel edge in G). Let H be one of these biconnected components.

If H is just a single edge then it is a series–parallel graph. Otherwise, H is already embedded as a noncrossing arc diagram such that all of the vertices of H lie on a horizontal path with an arc connecting the leftmost vertex s' and rightmost vertex t' . In particular, (H, s', t') is a smaller graph with a marked pair of vertices satisfying the hypothesis, so by an inductive argument we see that H is a series–parallel graph with source s' and terminal t' , built using only the prescribed operations. We now recover G in two steps:

- (i) taking the series join of all biconnected components;
- (ii) taking the (restricted) parallel join with the single arc e .

In particular, G is series–parallel with source the leftmost vertex of the first biconnected component, which is s , and terminal the rightmost vertex of the last biconnected component, which is t .

(\Leftarrow) Let (G, s, t) be a series–parallel graph satisfying the criterion of the lemma. We prove the result by induction on $\min\{\text{ht } \Upsilon\}$, with Υ running over the decomposition trees for (G, s, t) . Fix Υ to be a decomposition tree of smallest height for G .

If $\text{ht } \Upsilon = 0$ then $G = K_2$. If $\text{ht } \Upsilon = 1$ then since G has no cut vertex, $G = K_2 \star K_2$, for which the result is clear. Assuming the result for all $\text{ht } \Upsilon \leq k$, we prove the result for height $k + 1$. Since G has no cut vertex, the root of Υ cannot be a \sharp operation. Thus $G = H \star e$, where

$$H = H_1 \sharp H_2 \sharp \cdots \sharp H_n$$

for some $n \geq 1$, and each H_i is either K_2 or does not have a cut vertex. But each H_i is built out of only $\{\sharp, \star^e\}$ and has height strictly smaller than $k + 1$, so by the induction hypothesis, any H_i has an arc diagram embedding as in Remark 6.11. Concatenating the arc diagrams of the H_i and adding e as an outer arc above the rest shows that G has a face which is a Hamiltonian cycle, namely, the outer face given by the horizontal path in the arc diagram and the arc e . In particular, s and t are consecutive on this face. □

We recall the definition of the Υ -dual from Definition 2.8 to state the next result, which gives a criterion for condition T on series–parallel graphs.

COROLLARY 6.13. *Let G be a graph with a planar embedding with a Hamiltonian face and let $s, t \in G$ be consecutive on the Hamilton face. Invoking the previous lemma, fix a decomposition tree Υ for (G, s, t) . If $V(G) = \{s, t\}$ then the Υ -dual is a path. Otherwise, the Υ -dual is a series-parallel graph satisfying condition T .*

PROOF. Let G^\vee be the Υ -dual of G . By definition of the Υ -dual, we see that G^\vee is a series-parallel graph built out of $\{\star, \ell^e, \ell_e\}$. In particular, condition T is stable under these operations by Lemmas 6.3 and 6.2. If G^\vee is not a path then Υ^\vee contains at least one \star operation, so there must be one furthest from the root (under the obvious partial order). The subtree rooted at this \star builds a cycle, so, in particular, T holds for this cycle. As previously mentioned, T then extends to G^\vee . Otherwise, Υ contains only \star operations, so $V(G) = \{s, t\}$. □

We are finally ready to produce a combinatorial condition on a graph which implies simultaneous combination and hence Condition 1 for every edge.

COROLLARY 6.14. *Let H_i^\vee for $2 \leq i \leq n$ be graphs which either have a Hamiltonian face with two consecutive marked vertices or are paths. View each of these as a series-parallel graph and let $G^\vee := H_1^\vee \ell \cdots \ell H_n^\vee$. Fix an arbitrary decomposition tree Υ^\vee for G^\vee and let G be the Υ^\vee -dual. If G^\vee has at least four vertices then $S(G, e)$ holds for every $e \in G$.*

REMARK 6.15. As in Remark 6.11, we may view each of the H_i^\vee as noncrossing arc diagrams with a horizontal Hamilton path. Thus, we may view G^\vee as a noncrossing arc diagram with the Hamilton path from s to t a horizontal line segment whose left endpoint is s and right endpoint t , and all other edges as arcs above this line segment.

PROOF. By fixing Υ^\vee for G^\vee we implicitly fix a decomposition tree Υ_i^\vee for each H_i^\vee . Indeed, Υ_i^\vee is the rooted subtree of Υ^\vee whose leaves are exactly those labelled with the edges of H_i^\vee . Note by definition of the Υ^\vee -dual that $G = H_1 \star \cdots \star H_n$. If H_i^\vee has at most three vertices then Υ_i^\vee has at most one ℓ operation. Thus, Υ_i has at most one \star operation, so H_i has at most one cycle. With this in mind, we divide the proof into three cases: (1) each H_i is a path; (2) each H_i has at most one cycle; (3) H_1^\vee has at least four vertices.

- (1) If each H_i is a path then each H_i^\vee has two vertices, by Corollary 6.13. As G^\vee has at least 4 vertices, we have $n \geq 3$. In particular, $H_1 \star H_2$ is a cycle, so by Lemmas 6.8 and 6.7 we have both $S(G)$ and $T(G)$.
- (2) If H_1 has exactly one cycle then $H_1 = C \ell \mathcal{P}$ with C a cycle and \mathcal{P} a path. Thus we have $T(H_1)$ by Corollary 6.9. If H_2 is a path then by Lemma 6.7 we have $S(H_1 \star H_2, e)$ for all $e \in H_2$. By Corollary 6.9, we have $S(H_1 \star H_2, e)$ for all $e \in H_1$ and $T(H_1 \star H_2)$. We apply the same techniques to the rest of the H_i to get $S(G)$ and $T(G)$.
- (3) As H_1^\vee has arcs connecting its source and terminal we have that $H_1^\vee = \Gamma^\vee \star e_1 \star \cdots \star e_k$. Since H_1^\vee has at least four vertices so, too, does Γ^\vee , so it is a graph satisfying the hypothesis of Corollary 6.14. Thus, from an inductive

argument on $\text{ht } \Upsilon^V$ we may assume that Γ satisfies both conditions S and T . Since $H_1 = \Gamma \sharp e_1 \sharp \dots \sharp e_k$, we have by Lemma 6.3 that $T(H_1)$ and by Lemma 5.3 that $S(H_1, e)$ for all $e \in \Gamma$.

Now, by symmetry and the previous cases, either H_2 is a path or it satisfies $T(H_2)$, so by Lemma 6.7 we have that $S(H_1 \star H_2, e)$ for all $e \in \mathcal{P}$. Since we already have $S(H_1, e)$ for all $e \in \Gamma$, we have $S(H_1 \star H_2, e)$ for all $e \in H_1$. By $T(H_1)$, we have $T(H_1 \star H_2)$, so, in particular, if H_2 is a path or has exactly one cycle then we have $S(H_1 \star H_2)$; on the other hand, if H_2^V has at least four vertices then by symmetry we obtain $S(H_1 \star H_2)$ anyway. We can repeat the symmetry argument to get $S(G)$ and $T(G)$. □

In fact, we can conclude substantially more. We can take *any* graph and replace an edge with a piece as in Corollary 6.14 and conclude that Condition 1 holds for edges from the piece. The key is that Lemmas 5.3 and 6.8 hold for any source–terminal graphs, not just series–parallel graphs, since the proofs never use more than that, so we can do a parallel join with any graph once we have a piece satisfying S .

If G is a graph, $(u, v) \in G$ is an edge, and H is a source–terminal graph, then we can construct a new graph by deleting (u, v) and gluing the source (respectively, terminal) of H to u (respectively, v). In effect, we replace the edge (u, v) with H .

PROPOSITION 6.16. *Let H be a graph which either is a cycle or satisfies the hypothesis of Corollary 6.14. Let G be any graph, and replace any edge $(u, v) \in G$ which is not a self-loop with H ; call the new graph \widetilde{G} . Then Condition 1 holds for any $e \in H \subseteq \widetilde{G}$ provided the loop number of \widetilde{G} is at least 2.*

Note that this proposition also gives an alternate (albeit considerably less elementary) proof for Proposition 3.1, by replacing an edge by the cycle consisting of two parallel edges.

PROOF. If H is just a cycle, G has at least one cycle, and (u, v) is not a bridge, then by Lemma 6.8 we obtain simultaneous combination for e and thus Condition 1. Otherwise, we have $S(H, e)$ by Corollary 6.14, so this implies $S(\widetilde{G}, e)$ and thus Condition 1. □

To underscore the freedom available from the fact that G is unrestricted, note that if we choose G to be any series–parallel graph then we can construct a new series–parallel graph \widetilde{G} such that $S(\widetilde{G})$ holds, by applying Proposition 6.16 to every edge of G . We can also feed these graphs through Proposition 6.16 into any other graph Γ , *et cetera*.

We provide an example of this procedure using the graphs in Figure 4. Recall that the wheel graph W_4 satisfies Condition 1 for spoke edges and does not satisfy Condition 1 for rim edges. The graph H^V is the series join of graphs satisfying the conditions of Corollary 6.14. We see that H is an Υ -dual of H^V and has at least four vertices, so in particular we have $S(H, e)$ for all $e \in H$ by Corollary 6.14.

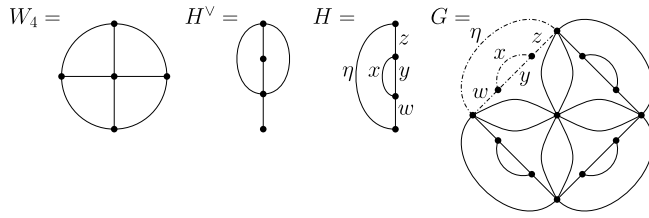


FIGURE 4. Example for Proposition 6.16.

Replacing the rim edges of W_4 with \overline{H} and doubling the spoke edges gives the graph G . By Proposition 6.16, we see that $S(G, e)$ holds if e lies in a copy of H . Additionally, if e is not in a copy of H then e is a double edge. Therefore, Condition 1 holds for every edge of G .

We can verify that Condition 1 holds explicitly in this example for the edge $\eta \in G$. Calculations were done with Magma [6], and a script is provided at [14]. For H we have

$$H = (x + y)\eta + (xy + (x + y)(z + w))$$

$$\overline{H} = \eta(xy + (x + y)(z + w)).$$

Let $n := \deg(\overline{H}) = 3$ and let

$$A_x := \frac{1}{2}yz + \frac{1}{2}yw - \frac{1}{2}y\eta,$$

$$A_y := xy + xz + xw + x\eta + \frac{1}{2}yz + \frac{1}{2}yw + \frac{3}{2}y\eta,$$

$$A_z := 0,$$

$$A_w := -xy - xz - xw - x\eta - \frac{3}{2}yz - \frac{3}{2}yw + \frac{1}{2}y\eta - z^2 - 2zw - w^2,$$

$$A_\eta := 0,$$

$$C := y.$$

One can check that

$$H = \sum_{j \in H} A_j H^{\eta_j}$$

$$\overline{H} = \sum_{j \in H} A_j \overline{H}^{\eta_j} + C \overline{H}^\eta.$$

That is, we have satisfied $S(H, \eta)$. Let Γ be the undotted subgraph of G . Note that Γ has 11 vertices and 23 edges, so each spanning tree excludes 13 edges. That is, $\deg(\Gamma) = 13$ and $\deg(\overline{\Gamma}) = 14 =: m$. We note $G = H \star \Gamma$, so

$$G = ((x + y)\eta + (xy + (x + y)(z + w)))\overline{\Gamma} + \eta(xy + (x + y)(z + w))\Gamma$$

$$\overline{G} = \eta(xy + (x + y)(z + w))\overline{\Gamma}.$$

The explicit forms of Γ and $\bar{\Gamma}$ are too large to be included in print. However, for the computations checking $S(G, \eta)$, it is enough to know that they are homogeneous of degrees $m - 1$ and m respectively. Setting

$$B_j := \begin{cases} A_j + \frac{C}{n+m-2}(mx_j) & \text{if } j \in \{x, y, z, w, \eta\}, \\ -\frac{(n-2)C}{n+m-2}x_j & \text{otherwise,} \end{cases}$$

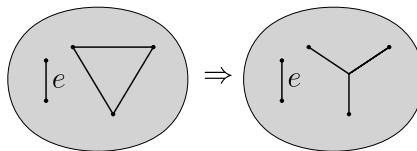
$$C' := \left(1 - \frac{m}{n+m-2}\right)C,$$

we obtain polynomials which verify $S(G, \eta)$. In particular, Condition 1 is satisfied for $\eta \in G$.

7. Conclusion

Our results significantly increase the number of graphs and edges for which we know whether or not Condition 1 holds. This means that there are many more graphs for which the tools of [1] can be applied.

After our investigations of multiple edges, wheel graphs, and series–parallel graphs, we are left with some questions. Lemma 4.3 was inspired by the possibility that Condition 1 might carry through Δ to Y transformations. Consider graphs containing the following structure.



Suppose we have a graph G_Δ , as on the left in the above diagram, with three edges forming a Δ shape and a distinct fourth edge e . The assertion that Δ implies Y for Condition 1 would mean that if $1(G_\Delta, e)$ is true then we could replace the Δ -forming edges with edges forming a Y , to make the graph G_Y , and $1(G_Y, e)$ would still be true. We suspect that this is the case.

The following heuristic has been very useful in this paper. The equations for Condition 1 are unstable under natural operations such as gluing two graphs along $n > 1$ vertices or adding parallel edges. However, properties or transformations one suspects to imply condition 1 are sometimes preserved by these operations. For example, when we investigated whether or not Condition 1 was true for all regular edges in series–parallel graphs, we found extra conditions that were well behaved under the operation of gluing two series–parallel graphs together, which could be related to Condition 1. A study of these conditions then led us to the example of Figure 1, which was a counterexample to our initial hopes. The relationship

between Condition 1 and the $\Delta - Y$ transformation could be approached using this idea. Furthermore, Aluffi observes [1, page 5] that, in general, ‘condition 1 depends on the global features of the graph’; we can see the simultaneous combination conditions as serving to correct this, enabling us to obtain local results like Proposition 6.16 even for Condition 1 itself.

One could hope for a full characterization of graph–edge pairs satisfying Condition 1. We have been looking for a structural graph-theoretic characterization, but one could also ask about the computational question – what is the computational complexity of checking Condition 1 on a graph–edge pair? Continuing our series–parallel investigations, both the structural and computational questions could be asked for specific classes of graphs. It would also be interesting to study the proportion of edges that satisfy condition 1 for large graphs. We suspect that this proportion may asymptotically approach zero, simply because as graphs become larger the condition becomes more complicated and difficult to satisfy; however, we would like to have a more rigorous analysis of this problem.

Finally, one could consider Aluffi’s Condition 2. Condition 2 seems to be much more geometric and less graph-theoretic, so we expect it to be less amenable to this sort of analysis.

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