Canad. Math. Bull. Vol. **37** (4), 1994 pp. 537–544

ON INFINITE LOCALLY FINITE GROUPS

AKBAR RHEMTULLA AND HOWARD SMITH

ABSTRACT. If G is a group such that every infinite subset of G contains a commuting pair of elements then G is centre-by-finite. This result is due to B. H. Neumann. From this it can be shown that if G is infinite and such that for every pair X, Y of infinite subsets of G there is some x in X and some y in Y that commute, then G is abelian. It is natural to ask if results of this type would hold with other properties replacing commutativity. It may well be that group axioms are restrictive enough to provide meaningful affirmative results for most of the properties. We prove the following result which is of similar nature.

If G is a group such that for each positive integer n and for every n infinite subset $X_1, \ldots, X_n$ of G there exist elements x_i of X_i, $i = 1, \ldots, n$, such that the subgroup generated by $\{x_1, \ldots, x_n\}$ is finite, then G is locally finite.

1. **Introduction.** If G is a group such that every infinite subset of G contains a commuting pair of elements then G is centre-by-finite. This result is due to B. H. Neumann [5]. From this it can be shown that if G is infinite and such that for every pair X, Y of infinite subsets of G there is some x in X and some y in Y that commute, then G is abelian. It is natural to ask if results of this type would hold with other properties replacing commutativity. It may well be that group axioms are restrictive enough to provide meaningful affirmative results for most of the properties. Questions of this nature are discussed in [4] where the property considered is a certain variety of soluble groups. In this paper we study the "local finiteness" property and obtain the following result.

If G is a group such that for each positive integer n and for every n infinite subsets $X_1, \ldots, X_n$ of G there exist elements x_i of X_i, $i = 1, \ldots, n$, such that the subgroup generated by $\{x_1, \ldots, x_n\}$ is finite, then G is locally finite.

We are not able to resolve the following question which seems to be considerably harder. Let G be a group such that for each positive integer n, every infinite subset X of G contains n element subset generating a finite subgroup of G. Does it follow that G is locally finite?

Although our primary objective is to study the local finiteness property, we can get two additional results with little additional work by dealing with a class Ω that is quotient, subgroup and locally closed. For any class Ω of groups denote by Ω^* the class of groups G satisfying the following hypothesis:

For each positive integer n and for every n infinite subsets $X_1, \ldots, X_n$ of G there exist elements x_i of X_i, $i = 1, \ldots, n$, such that the subgroup generated by $\{x_1, \ldots, x_n\}$ belongs to Ω.

The first author was partially supported by a grant from NSERC, Canada.
Received by the editors June 9, 1993.
AMS subject classification: 20F50.

Let F denote the class of all finite groups. It is clear that every group in the class $\Omega \cup F$ is contained in Ω^*. We are interested in the question as to whether the reverse inclusion holds. We establish a positive result for the class $L\mathbf{N}$ of locally nilpotent groups; the class $L\mathbf{S}$ of locally soluble groups and the class $L\mathbf{F}$ of locally finite groups. From these one can deduce positive results for locally supersoluble groups and for locally polycyclic groups, and we shall sketch their proofs. Our main result, in the above notation, is the following:

THEOREM. $(L\mathbf{N})^* = L\mathbf{N} \cup F$; $(L\mathbf{S})^* = L\mathbf{S} \cup \mathbf{F}$; $(L\mathbf{F})^* = L\mathbf{F}$.

An essential ingredient in the proof of the theorem is a special case of recent results in [1] by B. Hartley.

2. **Preliminary results.** We begin with an observation which has wider application than that required for our present purpose.

LEMMA 1. *Let Ω be a subgroup closed class of groups and suppose that G is in Ω^*. If G contains a subgroup A which is either infinite cyclic or isomorphic to C_{p^∞}, for some prime p, then every finitely generated subgroup of G is an Ω group.*

PROOF. Let n be a positive integer and let $g_1, \ldots, g_n$ be arbitrary elements of an Ω^* group G. Suppose $X = \langle x \rangle$ is an infinite cyclic subgroup of G and let p, q be distinct primes. Considering the sets $\{x^{p^i}\}$, $\{x^{q^j}\}$, $Xg_1, \ldots, Xg_n$, where i, j run through all positive integers, we see that, for some integers $u, v, r_1, \ldots, r_n$, $L = \langle x^{p^u}, x^{q^v}, x^{r_1}g_1, \ldots, x^{r_n}g_n \rangle$ belongs to Ω. Since $x \in L$, $\langle g_1, \ldots, g_n \rangle$ is also in Ω, as required.

Now suppose that G has a Prüfer subgroup $A = \langle t_1, t_2, \ldots \rangle$, where $t_1^p = 1$ and, for all $i \geq 1$, $t_{i+1}^p = t_i$. Partition the set $\{t_1, t_2, \ldots\}$ into disjoint infinite subsets U and V and consider the sets $Ug_1, \ldots, Ug_n$, $Vg_1, \ldots, Vg_n$. By hypothesis, there are elements t_{λ_i} of U and t_{μ_i} of V, $i = 1, \ldots, n$, such that $K = \langle t_{\lambda_1}g_1, \ldots, t_{\lambda_n}g_n, t_{\mu_1}g_1, \ldots, t_{\mu_n}g_n \rangle \in \Omega$. Now for each $i = 1, \ldots, n$, $k_i = (t_{\lambda_i}g_i)(t_{\mu_i}g_i)^{-1} = t_{\lambda_i}t_{\mu_i}^{-1} \in K$. But $\langle k_i \rangle = \langle t_{\lambda_1}, t_{\mu_i} \rangle$, and so $\langle g_1, \ldots, g_n \rangle \leq K$ and we are done.

From now on we assume that G is an infinite periodic group in the class $(L\mathbf{F})^*$. Our first objective is to show that every finite subgroup of G is contained in an infinite locally finite subgroup of G.

LEMMA 2. *Every infinite $(L\mathbf{F})^*$ group G has a nontrivial element whose centralizer in G is infinite.*

PROOF. Let $\pi = \{p$; p is a prime and G has elements of order $p\}$. Suppose the lemma is false. Take $p \in \pi$, and some $g \in G$ of order p. Let S be the set of conjugates of g in G, and let m denote the order of the centralizer of g in G. Then by a theorem of B. Hartley (Theorem B, [1]), there is an integer $f = f(m,p)$ such that for any finite subgroup H of G containing any element of S, $|H : \mathrm{Fitt}(H)| \leq f$, where $\mathrm{Fitt}(H)$ denotes the Fitting subgroup of H. If the set π is infinite then take $q \in \pi$, $q > f$, and h or order q. Now the set T of conjugates of h in G is infinite. For any given positive integer k, partition the set T into k infinite subsets T_i, $i = 1, \ldots, k$. Consider the sets $S, T_1, \ldots, T_k$.

By hypothesis, there exist a conjugate of g and k conjugates of h such that the subgroup H_k generated by these elements is finite. By taking a suitable conjugate of H_k, if necessary, we may assume that $h \in H_k$. By Hartley's theorem, the Sylow q-subgroup of H_k is normal in H_k. Let Z_k denote the centre of the Sylow q-subgroup Q_k of H_k, and $Z = \{Z_k\ ;\ k = 1, 2, \ldots\}$. Clearly Z centralizes h. Thus Z is finite, and hence, for infinitely many values of k, Z_k is some fixed subgroup of G. Now pick any nontrivial element of this subgroup and it is centralized by infinitely many Q_k's. This is a contradiction.

We may thus assume that the set π is finite consisting of primes $p = p_1, p_2, \ldots, p_r$. Let g_i be of order p_i, and S_i the set of conjugates of g_i. Then by partitioning the set S_i into k infinite subsets, using the $(L\mathbf{F})^*$ property and Hartley's theorem, we obtain subgroups L_k, all containing k conjugates of each of the elements g_i, and the index of $\mathrm{Fitt}(L_k)$ in L_k at most f for some integer f. Now for some prime p_j, and an infinite set of values of k, the Sylow p_j-subgroups of L_k are increasing in size with k. We take conjugates of these L_k and ensure that g_j lies in each of these. Now repeat the argument in the above paragraph and conclude that g_j or some element in the centre of the Sylow p_j-subgroup of L_k has infinite centralizer.

LEMMA 3. *Every infinite $(L\mathbf{F})^*$ group G has an infinite abelian subgroup.*

By Lemma 2, G has an infinite subgroup G_1 whose center Z_1 is nontrivial. If Z_1 is infinite then pick this to be the required subgroup. If Z_1 is finite then consider the group $H_1 = G_1/Z_1$. Now H_1 is an infinite $(L\mathbf{F})^*$ group. We may now repeat the above argument with H_1 in place of G and, continuing this process, we obtain an ascending chain of subgroups $1 \neq Z_1 < Z_2 < \cdots$ whose union W is an infinite ZA group, and hence has an infinite abelian subgroup.

LEMMA 4. *Every finite subgroup of an infinite $(L\mathbf{F})^*$ group G is contained in an infinite locally finite subgroup of G.*

Let F be a finite subgroup of G and let A be an infinite abelian subgroup of G. Suppose $F = \{f_1, \ldots, f_n\}$ and consider the sets $A, Af_1, \ldots, Af_n$. Then for some $a, a_1, \ldots, a_n$ in A, $\langle a, a_1f_1, \ldots, a_nf_n\rangle$ is finite, as therefore is $[F, \langle a\rangle]$. Since A may be replaced, in the above argument, by any infinite subset of A, it follows that $[F, \langle a\rangle]$ is finite for all a in some cofinite subset A_1 of A. Choose a_1 in $A_1 \setminus F$. Then $F_1 = \langle F, a_1\rangle$ is finite. Similarly, let A_2 be a cofinite subset of A_1 such that $[F_1, \langle a\rangle]$ is finite for all a in A_2 and choose a_2 in $A_2 \setminus F_1$. Then $F_2 = \langle F, a_1, a_2\rangle$ is finite. Continuing, we obtain an infinite subgroup $H = \langle F, a_1, a_2, \ldots\rangle$ which is locally finite.

Observe that if $G \in (L\mathbf{S})^*$ then $\langle a_1, a_2, \ldots\rangle^F$ is locally soluble. But H is in $(L\mathbf{S})^*$ and so all finite images of H (indeed all images H/N, where N is infinite) are soluble. Thus H is locally soluble, establishing the following.

LEMMA 5. *Every finite subgroup of an infinite $(L\mathbf{S})^*$ group G is contained in an infinite locally soluble subgroup of G.*

The next major result that we want to establish is that an infinite $(L\mathbf{F})^*$ group G is locally finite. If G is locally finite then there is nothing to show, so assume that G is an infinite, finitely generated periodic $(LF)^*$ group. Apply Zorn's Lemma to obtain a normal subgroup N maximal subject to G/N being infinite. Then G/N is a $(LF)^*$ group; and, in order to obtain a contradiction, we may assume that every non-trivial normal subgroup of G has finite index.

DEFINITION. We shall call G a critical $(LF)^*$ group if G is a finitely generated infinite periodic $(LF)^*$ group in which every nontrivial normal subgroup is of finite index in G.

LEMMA 6. *If H is an infinite locally nilpotent subgroup of a critical $(LF)^*$ group G and F is a finite subgroup of H, then the centralizer $C_H(F)$ of F in H is infinite.*

PROOF. For suppose that F is finite and that $H = \langle F, h_1, h_2, \ldots \rangle$ is an infinite locally nilpotent subgroup of G such that $C_H(F)$ is finite. Then the descending chain $C_H(F) \geq C_H(\langle F, h_1 \rangle) \geq C_H(\langle F, h_1, h_2 \rangle) \geq \cdots$ of nontrivial subgroups must terminate in a nontrivial subgroup Z_1, say, which is clearly the centre of H. Since Z_1 is finite, so is $C_{H/Z_1}(FZ_1/Z_1)$. By repeating this argument we may obtain an infinite hypercentral subgroup KF of H. Let A be a maximal normal abelian subgroup of KF. Then A is infinite. If B is any π' subgroup of A, where π is the set of primes dividing the order of F, then $[B, F] = 1$, and so B is finite. Thus, using Lemma 1, A contains an infinite, characteristic subgroup D of exponent p, for some prime $p \in \pi$. Then DF is nilpotent and so $C_D(F)$ is infinite.

LEMMA 7. *If G is a critical $(LF)^*$ group then every infinite locally nilpotent subgroup of G is abelian.*

PROOF. Suppose H is a nonabelian infinite locally nilpotent subgroup of G. Then for some h_1, h_2 in H, $d = [h_1, h_2] \neq 1$. Let $H_0 = \langle h_1, h_2 \rangle$. Then H_0 is a finite subgroup of H. By Lemma 6, $C_H(H_0)$ is infinite and hence contains an infinite abelian subgroup A. Let $g_1, \ldots, g_n$ be arbitrary elements of G, and consider the sets $Ah_1, Ah_2, Ag_1, \ldots, Ag_n$. For some $a, b, a_1, \ldots, a_n$ in A we have $T = \langle ah_1, bh_2, a_1g_1, \ldots, a_ng_n \rangle$ is finite. Clearly T contains the element $d = [ah_1, bh_2]$ and hence the subgroup $\langle d, d^{g_1}, \ldots, d^{g_n} \rangle$ is finite. Since n and the elements g_i were arbitrary, it follows that $\langle d^G \rangle$ is locally finite, and hence G is locally finite, and thus finite; a contradiction.

LEMMA 8. *If G is a critical $(LF)^*$ group then every infinite p-subgroup of G is abelian.*

PROOF. Suppose H is an infinite p-subgroup of G. By Lemma 4 and Lemma 7, every finite subgroup of H is abelian. Hence given any infinite subsets X_1, X_2 of H, there exist $x_1 \in X_1$, $x_2 \in X_2$ such that x_1 and x_2 commute. But this implies that every infinite subset of H contains a commuting pair of elements and so, by a theorem of B. H. Neumann in [5] H is centre-by-finite and thus locally nilpotent. This completes the proof.

LEMMA 9. *Let G be a critical $(LF)^*$ group. If h_1, h_2 are non-commuting elements of G then $C_G(h_1) \cap C_G(h_2)$ is finite.*

PROOF. Let $d = [h_1, h_2]$, $C = C_G(h_1) \cap C_G(h_2)$ and let $g_1, \ldots, g_n$ be arbitrary elements of G. If C is infinite, let A be an infinite abelian subgroup of C and consider the sets $Ah_1, Ah_2, Ag_1, \ldots, Ag_n$ and argue exactly as in the proof of Lemma 7 to obtain a contradiction.

LEMMA 10. *Let G be a critical (LF)* group. Then all nontrivial maximal p-subgroups of G are infinite.*

PROOF. First let us note that, for any prime q, if Q is an infinite q-subgroup of G then every maximal q-subgroup R is infinite. For, by Lemma 8, Q is abelian, and if R were finite, then by replacing A by Q and F by R in the proof of Lemma 4 we would obtain an infinite locally finite subgroup L of G containing R and also an infinite subgroup of Q. Thus all maximal q-subgroups of L are infinite (see Lemma 1.D.12 in [3]), and R is not a maximal q-subgroup of G, contrary to the choice of R.

Now suppose the result is false. Let H be a finite maximal p-subgroup of G for some prime p. If for some nontrivial element h of H the centralizer C of $\langle h\rangle$ is infinite, then by Lemma 3 and the remark above, C contains an infinite abelian p'-subgroup A. Now $\langle A, h\rangle$ is abelian and, for any finite set $\{u, v, \ldots, w\}$ of G, by considering the sets $Ah, Au, Av, \ldots, Aw$ we obtain, by hypothesis, a finite group generated by $ah, bu, cv, \ldots, dw$ for some $a, b, c, \ldots, d$ in A. Now the subgroup generated by h and its conjugates under $u, v, \ldots, w$ is contained in this subgroup since the order of a is coprime to the order of h. From this it follows that the normal closure of the subgroup $\langle h\rangle$ in G is locally finite, and hence so is G.

We may thus assume that the centralizer of every nontrivial element of H is finite. Consider any $h \in H$ of order p, and let m denote the order of its centralizer in G. Let $f = f(m,p)$ as used in Lemma 2. If G has an element g of order a prime $q > f$, then let Q be a maximal q-subgroup of G containing g. There are two cases to be considered.

(1) Q is infinite. Then let R be another maximal q-subgroup (say a suitable conjugate of Q) not containing g. Partition R into k infinite subsets R_i, $i = 1, \ldots, k$, where k is any positive integer. Let T be the set of conjugates of h and let T_0 be the set of all conjugates of g by elements of R. Since R is abelian by Lemma 8, and $\langle R, g\rangle$ is nonabelian, it follows from Lemma 9 that $C_R(g)$ is finite and hence the set T_0 is infinite. By considering the sets $T, T_0, R_1, \ldots, R_k$ we obtain a finite group $\langle h_1, g_1, r_1, \ldots, r_k\rangle$ for some conjugate h_1 of h, some conjugate g_1 of g and some r_i in R_i. Since the centralizer of h_1 in this subgroup has order at most m, it follows from Hartley's theorem that the Fitting subgroup of this group contains the Sylow-q-subgroup, and hence $\langle g_1, r_1, \ldots, r_k\rangle$ is a q-subgroup. By Lemma 8, and the fact that all maximal q-subgroups are infinite, this is abelian. This shows us that the centralizer of g in R is infinite and this is a contradiction.

(2) Q is finite. In this case take R to be the set of all conjugates of g, partitioned into k disjoint infinite subsets with k larger than size of Q. From these sets and the set T of conjugates of h in G we obtain, by hypothesis, a q-subgroup of size at least k, a contradiction.

We may now assume that G is a π-group where π is a finite set of primes. Then it follows that for at least one prime q, maximal q-subgroups of G are infinite, for G has infinite locally finite π-subgroups. Let Q and R be two distinct maximal q-subgroups of

G. Suppose $A = Q \cap R$ is infinite and choose h_1 in Q, h_2 in R such that $d = [h_1, h_2] \neq 1$. Let $g_1, \ldots, g_n$ be arbitrary elements of G, and consider the sets $Ah_1, Ah_2, Ag_1, \ldots, Ag_n$. For some $a, b, a_1, \ldots, a_n$ in A we have $T = \langle ah_1, bh_2, a_1g_1, \ldots, a_ng_n \rangle$ is finite. Clearly T contains the element $d = [ah_1, bh_2]$ and hence the subgroup $\langle d, d^{g_1}, \ldots, d^{g_n} \rangle$ is finite. Since n and the elements g_i were arbitrary, it follows that $\langle d^G \rangle$ is locally finite, and hence G is locally finite, and thus finite; a contradiction.

Thus we may assume that $Q \cap R$ is finite. Since Q and R are abelian q-groups, not containing a Prüfer q-subgroup by Lemma 1, Q has an infinite elementary abelian q-subgroup that intersects trivially with R. Let F be a finite subgroup of this group with the order of F exceeding $f = f(m, q)$. Take T to be the set of all conjugates of h by elements of R and $T_1, T_2, \ldots, T_n$ to be the sets Rg_i where $g_1, \ldots, g_n$ are the elements of F, and R_i, $i = 1, \ldots, k$, a partition of R into k infinite subsets where k is any positive integer.

By hypothesis, we obtain a finite group $K = \langle h_1, a_1g_1, \ldots, a_ng_n, r_1, \ldots, r_k \rangle$ for some conjugate h_1 of h, some a_i, $i = 1, \ldots, n$, in R, and r_j in R_j, $j = 1, \ldots, k$. Let S be a Sylow q-subgroup of K containing $\langle r_1, \ldots, r_k \rangle$. By Hartley's theorem, the index of Fitt(K) in K is at most f, so that some $(a_ig_i)(a_jg_j)^{-1}$ lies in Fitt(K) and hence commutes with Fitt(K)$\cap \langle r_1, \ldots, r_k \rangle$. Thus $g_ig_j^{-1}$ commutes with Fitt(K)$\cap \langle r_1, \ldots, r_k \rangle$. Using the pigeon-hole principle, we get some element in F centralizing an infinite subgroup of R. This is not possible by Lemma 9.

LEMMA 11. *Let G be a critical* (**LF**)* *group. Then the order of every element of G is a prime.*

PROOF. Suppose false; let g be an element of G of order pq, where p, q are primes ($p = q$ allowed). Write y for g^q so that y is of order p. As in the proof of Lemma 4 embed $\langle g \rangle$ in an infinite locally finite subgroup H of $\langle g, S \rangle$ where S is a Sylow-p-subgroup of G containing y, and H contains an infinite subgroup of S. Let K be a Sylow-p-subgroup of H containing y. Now let T be the set of all conjugates of g by elements of K. If the set T is finite then the centralizer of g in K is infinite, and hence, by Lemma 1, there exists an infinite elementary abelian subgroup C of K centralizing $\langle g \rangle$. Since the p-th power of cg is the same (say z), independent of choice of c in C, considering the sets $Cg, Cu, \ldots, Cw$, we find that the subgroup generated by z and conjugates of z by $u, \ldots, w$ is finite for all $u, \ldots, w$ in G. Hence the normal closure of $\langle z \rangle$ in G is locally finite and we are done. We may thus assume that T is infinite. Now for every element of the infinite set T, the q-th power is y. So if we were to consider the sets $T, Ku, \ldots, Kw$, we would get the normal closure of $\langle y \rangle$ in G to be locally finite, another contradiction.

3. **Proof of the Theorem.** We begin with a result that is probably known but we could not find a suitable reference.

LEMMA 12. *Let G be a finite group in which the order of every element is a prime, and the Sylow subgroups are all abelian. Then G is isomorphic to* A_5 *or G is abelian-by-cyclic.*

PROOF. The proof that G is simple or soluble goes back to Weisner in [7]. If G is simple then it is of type $\mathrm{PSL}(2,q)$, where q is a power of two (see Corollary 2.8, p. 182 of [2]). Since such groups have cyclic subgroups of order $q-1$ and $q+1$, q must equal four and G is A_5. If G is soluble then it has a normal subgroup H of index p, for some prime p. Now each element of G not in H must be of order p. This is forced by the hypothesis of the lemma. The result now follows from Theorem 4.24 in [6].

LEMMA 13. *There is no critical* $(L\mathbf{F})^*$ *group.*

PROOF. Suppose that G is such a group. If H is a finite subgroup of G, then every element of H is of a prime order by Lemma 11. Also any Sylow p-subgroup P of H is contained in some maximal p-subgroup of G which, by Lemma 10, is infinite and thus, by Lemma 8, is abelian. It follows from Lemma 12 that H is isomorphic to A_5 or it is metabelian. Now H can be embedded in an infinite locally finite subgroup of G by Lemma 4. Thus, if H is simple, then $H = G$ and G is finite, a contradiction. We may thus assume that every finite subgroup of H is metabelian. If X_1, X_2, X_3, X_4 are any infinite subsets of G then by hypothesis, there exist x_i in X_i such that the subgroups generated by these is finite, and hence metabelian. Then, by the main result of [4], G is metabelian. Since it is finitely generated and periodic, it is finite, a contradiction.

The proof that $(L\mathbf{F})^*$ groups are locally finite now follows from Lemma 1 and Lemma 13. That $(L\mathbf{S})^* = L\mathbf{S} \cup F$ can be seen with the additional information from Lemma 5. The result for locally polycyclic groups follows in the same way. We shall now assume that $(L\mathbf{S})^* = L\mathbf{S} \cup \mathbf{F}$.

In order to show that $(L\mathbf{N})^* = L\mathbf{N} \cup \mathbf{F}$ we may assume G to be an infinite locally finite soluble $(L\mathbf{N})^*$ group. If G is not locally nilpotent then there is a finite, non-nilpotent subgroup F of G. If G contains an infinite, normal abelian subgroup A, then by Lemma 1, A is reduced and therefore contains an infinite characteristic subgroup B which is residually finite. The infinite $(L\mathbf{N})^*$ group BF is also residually finite, while all its finite images are nilpotent. Since BF is also locally finite, it is locally nilpotent, a contradiction. Assuming that all normal abelian subgroups of G are finite, let K be the intersection of all normal subgroups N of G such that G/N is locally nilpotent. Thus G/K is locally nilpotent, and $K \neq 1$. Then every proper G-invariant subgroup N of K is finite—since G/N is locally nilpotent otherwise. Hence its centralizer contains K. In other words, it is central in K. The centre Z of K is finite and K/Z is a chief factor of G and therefore abelian. Now suppose K is infinite. Then K/Z is infinite, and G/Z is locally nilpotent, a contradiction.

Thus K must be finite. Clearly we may assume K is abelian of prime exponent p. Let $C = C_G(K)$. Then C is locally nilpotent. Since G/K (but not G) is locally nilpotent, there is an element h of G such that h has q-power order for some prime q different from p and $\langle K, h\rangle$ is not nilpotent. We may assume that $F = \langle K, h\rangle$. If The p'-component L of C is infinite then G/L is locally nilpotent and so $G \in L\mathbf{N}$, a contradiction. Certainly, therefore, the p-component P of C is infinite. We may assume that $G = \langle P, F\rangle$ and hence that G/K is a direct product of P/K and F/K. Then F is in the FC-centre of G and the centralizer of F in P is infinite, and so contains an infinite abelian subgroup D.

Now replace P by D and assume that $G = DF$, where D is an abelian p-group F is a finite non-nilpotent group and $[D, F] = 1$. But in this set-up, D is an infinite normal abelian subgroup of G and, as before, this gives the required contradiction.

The proof for the locally supersoluble case is analogous to the locally nilpotent case up to the point where we are able to assume that G/K is locally supersoluble and K is a finite abelian group of exponent p. Since G is not locally supersoluble, there exists a finite subgroup F that is not supersoluble. Now $G'K/K$ is locally nilpotent and G' is infinite—else F centralizes an infinite abelian subgroup of G. Thus by considering the centralizer of K in G' we may pass to the case where $G = HF$ and H is a locally nilpotent p-group which is normal in G and centralized by K. As in the proof of Lemma 6 we have that every finite F-invariant subgroup of H has infinite centralizer. If A is a maximal normal abelian subgroup of H then $A^G = A^F$ is nilpotent. Replace H by A^G. The centre of H is finite and so H has finite exponent. Passing to the centralizer of the last finite term of the upper central series of H and then to a suitable subgroup we may assume that $H/Z(H)$ is elementary abelian. Using the local supersolubility of $G/Z(H)$ we easily construct an infinite F-invariant abelian subgroup of H, thus obtaining a contradiction.

REFERENCES

1. B. Hartley, *A general Brauer-Fowler Theorem and centralizers in locally finite groups,* Pacific J. Math. **152**(1992), 101–117.

2. B. Huppert and N. Blackburn, *Finite Groups III,* Grundlehren Math. Wiss. **243**, Springer-Verlag, Berlin, Heidelberg, New York,1982.

3. O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups,* North Holland, American Elsevier, Amsterdam, London, New York, 1973.

4. P. S. Kim, A. Rhemtulla and H. Smith, *A characterization of infinite metabelian groups,* Houston J. Math. **17**(1991), 429–437.

5. B. H. Neumann, *On a problem of Paul Erdős on groups,* J. Austral. Math. Soc. **21**(1976), 467–472.

6. M. Suzuki, *Group Theory II,* Grundlehren Math. Wiss. **248**, Springer-Verlag, Berlin, Heidelberg, New York, 1986.

7. L. Weisner, *Groups in which the normalizer of every element except identity is abelian,* Bull. Amer. Math. Soc. **31**(1925), 413–416.

University of Alberta
Edmonton, Alberta
T6G 2G1

Bucknell University
Lewisburg, Pennsylvania 17837
U.S.A.