

ON GROUPS WITH DEFINABLE F -GENERIC DEFINABLE IN P -ADICALLY CLOSED FIELDS

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Abstract. The aim of this paper is to develop the theory of groups definable in the p -adic field \mathbb{Q}_p , with “definable f -generics” in the sense of an ambient saturated elementary extension of \mathbb{Q}_p . We call such groups *definable f -generic* groups.

So, by a “definable f -generic” or *dfg* group we mean a definable group in a saturated model with a global f -generic type which is definable over a small model. In the present context the group is definable over \mathbb{Q}_p , and the small model will be \mathbb{Q}_p itself. The notion of a dfg group is dual, or rather opposite to that of an fsg group (group with “finitely satisfiable generics”) and is a useful tool to describe the analogue of torsion-free ω -minimal groups in the p -adic context.

In the current paper our group will be definable over \mathbb{Q}_p in an ambient saturated elementary extension \mathbb{K} of \mathbb{Q}_p , so as to make sense of the notions of f -generic type, etc. In this paper we will show that every definable f -generic group definable in \mathbb{Q}_p is virtually isomorphic to a finite index subgroup of a trigonalizable algebraic group over \mathbb{Q}_p . This is analogous to the ω -minimal context, where every connected torsion-free group definable in \mathbb{R} is isomorphic to a trigonalizable algebraic group [5, Lemma 3.4]. We will also show that every open definable f -generic subgroup of a definable f -generic group has finite index, and every f -generic type of a definable f -generic group is almost periodic, which gives a positive answer to the problem raised in [28] of whether f -generic types coincide with almost periodic types in the p -adic case.

§1. Introduction. In the recent years there has been growing interest in the interaction between topological dynamics and model theory. This approach was introduced by Newelski [22], then developed in a number of papers, including [4, 15, 27, 28, 38], and now called definable topological dynamics. Definable topological dynamics studies the action of a group G definable in a structure M on its type space $S_G(M)$ and tries to link the invariants suggested by topological dynamics (e.g., enveloping semigroups, minimal subflows, Ellis groups, ...) with model-theoretic invariants.

This framework allows us to generalize stable group theory to some unstable groups with tame behaviour. Under the assumption that the theory T has NIP, the class of definably amenable groups is a reasonable choice for the class of stable-like groups in NIP context since Newelski’s conjecture [22] holds. Namely, the smallest type-definable subgroup of G of bounded-index, written G^{00} , exists and G/G^{00} is isomorphic to its Ellis group (see [4]). In the stable case, this coincides with the space (group) of generic types of G .

We will now be assuming that our ambient theory T is NIP.

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As generic types may not always exist in the NIP environment, other notions were introduced to describe the generic-like formulas and types. The notion of f -generic was first introduced in [13] and then slightly modified in [4]. We let \mathbb{M} be a so-called “monster” model (of whatever complete theory we are working with). Recall that this means that \mathbb{M} is κ -saturated and κ -strongly homogeneous for some suitably large κ . If the reader is willing to make some set-theoretic assumptions, they can just take \mathbb{M} to be κ -saturated and of cardinality κ for suitable κ . Let G be a group definable in \mathbb{M} . A definable subset (or formula) $X \subseteq G$ (definable over M) is said to be f -generic if for every $g \in G$, the left translate gX does not fork over M . A global type $p \in S_G(\mathbb{M})$ is f -generic if every formula in p is f -generic. A global f -generic type p is strongly f -generic if every G -translate of p does not fork over a fixed small submodel M (equivalently by NIP is invariant under automorphisms fixing M).

One of the nice observations in [4] was that an NIP group is definably amenable if and only if it admits a global strongly f -generic type. Now there are two extreme case for a global strongly f -generic type p in NIP environment:

- (i) p (as well as every left translate) is finitely satisfiable in some fixed small model M . In this case p is actually generic, meaning that for every formula ϕ in p , finitely many translates of ϕ cover G . We call p a finitely satisfiable generic, abbreviated as *fsg*.
- (ii) p (so also all its translates) is definable over some fixed small model M , and we call p a definable f -generic, abbreviated as *dfg*.

A definable group G is said to be an *fsg* group if it admits a global *fsg* type, and a *dfg* group if it admits a global *dfg* type.

In [11], Hrushovski, Peterzil, and Pillay proved that *fsg* groups coincide with definably compact groups in o -minimal expansions of real closed field. Moreover, in [23], Onshuus and Pillay proved that it is also true for groups definable in \mathbb{Q}_p . The above results say that the model-theoretic concept of *fsg* is able to describe the topological concept of “compactness” in o -minimal and p -adic context.

On the other hand, putting various results together, we know that for definably connected groups definable in o -minimal expansions of *RCF*, torsion-free is equivalent to being *dfg*. Here are some details. The left implies right direction is precisely Proposition 4.7 in [6]. For the right implies left direction: The proof of Lemma 1.15 in [28] shows that if G is *dfg*, then there is a G^0 -invariant definable type, so by connectedness there is a G -invariant definable type p . Now we know G to be definably amenable, so by Propositions 2.6 and 4.6 of [6], there is a normal definable torsion-free (solvable) subgroup H of G such that G/H is definably compact (and definably connected). We claim that G/H is trivial, which will show that $G = H$ is torsion-free. Assuming G/H nontrivial, p gives rise to a G/H -invariant type p/H of G/H , which implies that $(G/H)^{00} = G/H$, a contradiction to definable compactness of G/H (for example, to the truth of Pillay’s conjecture).

Since torsion-free means “totally non-definably compact” for groups definable in o -minimal expansions of *RCF* this translates into *dfg* meaning “totally non *fsg*” in the o -minimal context. This observation is also witnessed by the following:

FACT 1.1 [34]. $\text{Th}(\mathbb{Q}_p)$ and o -minimal theories are distal.

Note that distality of a theory implies that no global types can be both definable and finitely satisfiable (in a small model), and in fact such types are weakly orthogonal.

Anyway it is natural to ask what happens in the p -adic context. We will be working with $\text{Th}(\mathbb{Q}_p)$ in the ring language and we will actually be restricting ourselves to groups definable in or over the standard model \mathbb{Q}_p .

QUESTION 1. *Are dfg groups exactly “totally noncompact” groups in this p -adic context?*

The aim of the paper is to answer this question. Our first result is a structure theorem for dfg groups over \mathbb{Q}_p , which gives a positive answer to Question 1.

THEOREM 1. *Let G be a group definable over \mathbb{Q}_p with dfg, then there is a normal sequence*

$$G_0 \triangleleft \cdots \triangleleft G_i \triangleleft G_{i+1} \triangleleft \cdots \triangleleft G_n,$$

where the G_i are definable over \mathbb{Q}_p , and such that G_0 is finite, G_n is a finite index subgroup of G , and each G_{i+1}/G_i is definably (over \mathbb{Q}_p) isomorphic to either the additive group \mathbb{G}_a , or a finite index subgroup of the multiplicative group \mathbb{G}_m .

Recall that if G is a definable group in a structure M , then by definition a type in $S_G(M)$ is almost periodic if the closure of its G -orbit is a minimal G -invariant closed subset of $S_G(M)$. In the following, the ambient model M is assumed to be saturated. The following conjecture was raised in [28].

CONJECTURE 1. Let G be a dfg group definable in an NIP structure. Then any global f -generic type is almost periodic.

The following theorem gives a positive answer to Conjecture 1 in the p -adic case. The proof makes use of a recent result in [37].

THEOREM 2. *Suppose that G is a dfg group definable over \mathbb{Q}_p , then every global f -generic type of G is almost periodic.*

The paper is organized as follows. In the rest of this introduction we will recall precise notation, definitions, and results from earlier papers, relevant to our results. In Section 2.1, we will prove some general results for dfg and fsg groups. In Section 2.2, we will characterise the one-dimensional dfg groups definable over \mathbb{Q}_p . Section 2.3 contains the key lemmas of the paper, where we proved that every algebraic group containing an open dfg subgroup is trigonalizable. In Section 2.4, we prove a structure theorem for an arbitrary dfg group definable over \mathbb{Q}_p , showing that every dfg group is virtually trigonalizable over \mathbb{Q}_p , and concluding that every global f -generic type of a dfg group is almost periodic.

1.1. Notations. We will assume a basic knowledge of model theory. Good references are [19] and [31]. Let T be a complete theory in a language L and let \mathbb{M} be the monster model, in which every type over a small subset $A \subset \mathbb{M}$ is realized, where “small” means $|A| < |\mathbb{M}|$. We will assume T to be 1-sorted. By x, y, z we mean finite tuples of variables, and $a, b, c \in \mathbb{M}$ finite tuples from \mathbb{M} . For a subset A of \mathbb{M} , L_A is the language obtained from L by adjoining constants for elements of A .

As a rule, by a formula we will mean an $L_{\mathbb{M}}$ -formula (unless otherwise stated). For an arbitrary elementary submodel M of \mathbb{M} and an L_M -formula $\phi(x)$, $\phi(M)$ denotes the definable subset of $M^{|x|}$ defined by ϕ , and a set $X \subseteq M^n$ is definable if there is an L_M -formula $\phi(x)$ such that $X = \phi(M)$. If $\bar{X} \subseteq \mathbb{M}^n$ is definable, defined with parameters from M , then $\bar{X}(M)$ will denote $\bar{X} \cap M^n$, the realizations from M , which is clearly a definable subset of M^n . Suppose that $X \subseteq \mathbb{M}^n$ is a definable set, defined with parameters from M , then we write $S_X(M)$ for the space of complete types concentrating on X . We use freely basic notions of model theory such as definable type, heir, coheir, The book [31] is a possible source.

We will be distinguishing in the current paper, between definable and interpretable. So by a definable set in the saturated structure \mathbb{M} we mean a subset of \mathbb{M}^n defined by a formula with parameters.

1.2. Topological dynamics. Our reference for (abstract) topological dynamics is [1].

Given a (Hausdorff) topological group G , by a G -flow we mean a continuous action $G \times X \rightarrow X$ of G on a compact (Hausdorff) topological space X . We sometimes write the flow as (X, G) . Often it is assumed that there is a dense orbit, and a G -flow (X, G) with a distinguished point $x \in X$ whose orbit is dense is called a G -ambit (although we will not make much use of this notation).

In spite of p -adic algebraic groups being nondiscrete topological groups, we will be treating them as discrete groups so as to have their actions on type spaces being continuous. So in this background section we may assume G to be a discrete group, in which case a G -flow is simply an action of G by homeomorphisms on a compact space X .

By a subflow of (X, G) we mean a nonempty closed G -invariant nonempty subspace Y of X (together with the action of G on Y). (X, G) will always have minimal subflows. A point $x \in X$ is almost periodic if the closure of its orbit is a minimal subflow.

Let (X, G) and (Y, G) be flows (with the same acting group). A homomorphism from X to Y is a continuous map $f : X \rightarrow Y$ such that $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

FACT 1.2. *Let (X, G) and (Y, G) be flows, $f : X \rightarrow Y$ a homomorphism. Then:*

- (i) *If X_0 is a minimal subflow of X , then $Y_0 = f(X_0)$ is a minimal subflow of Y .*
- (ii) *If Y is a minimal flow, then f is onto.*

1.3. Definably amenable groups. Let T be any first-order theory, let $\mathbb{M} \models T$ be a very saturated model, and let G be a group \emptyset -definable in \mathbb{M} defined by the formula $G(x)$. For any $M \prec \mathbb{M}$, $G(M) = \{g \in M^n \mid g \in G\}$ is a subgroup of G . By $S_G(M)$ we mean the space of all complete types over M concentrating on $G(x)$. It is easy to see that $S_G(M)$ together with the type of the identity element is a $G(M)$ -ambit, taking the dense orbit to be $\{\text{tp}(g/M) \mid g \in G(M)\}$. From now on, we will, throughout this paper, assume that every formula $\phi(x) \in L_{\mathbb{M}}$, with parameters in \mathbb{M} , is contained in $G(x)$, namely, the subset $\phi(\mathbb{M})$ defined by ϕ is contained in G . Suppose that $\phi \in L_M$ and $g \in G(M)$, then the left translate $g\phi(x)$ is defined to be $\phi(g^{-1}x)$. It is easy to check that $(g\phi)(M) = gX$ with $X = \phi(M)$.

DEFINITION 1.3. Let the notation be as above.

- A definable subset $X \subseteq G$ is generic if finitely many left translates of X cover G . Namely, there are $g_1, \dots, g_n \in G$ such that $G = \cup_{1 \leq i \leq n} g_i X$.
- A definable subset $X \subseteq G$ is weakly generic if there is a non-generic definable subset $Y \subseteq G$ such that $X \cup Y$ is generic.
- A definable subset $X \subseteq G$ is f -generic if for some/any model M over which X is defined and any $g \in G$, gX does not divide over M . Namely, for any M -indiscernible sequence $(g_i : i < \omega)$, with $g = g_0$, $\{g_i X : i < \omega\}$ is consistent.
- A formula $\phi(x)$ is generic if the definable set $\phi(\mathbb{M})$ is generic. Similarly for weakly generic and f -generic formulas.
- A type $p \in S_G(M)$ is generic if every formula $\phi(x) \in p$ is generic. Similarly for weakly generic and f -generic types.
- A type $p \in S_G(M)$ is almost periodic if p is an almost periodic point of the $G(M)$ -flow $S_G(M)$.
- A global type $p \in S_G(\mathbb{M})$ is strongly f -generic over a small model M if every left $G(\mathbb{M})$ -translate of p does not fork over M . A global type $p \in S_G(\mathbb{M})$ is strongly f -generic if it is strongly f -generic over some small model.

FACT 1.4 [22]. Let $AP \subseteq S_G(M)$ be the space of almost periodic types, and $WG \subseteq S_G(M)$ the space of weakly generic types. Then $WG = \text{cl}(AP)$.

Recall that a type definable over A subgroup $H \leq G$ has bounded index if $|G/H| < 2^{|T|+|A|}$. For groups definable in NIP structures, the smallest type-definable subgroup G^{00} exists (see [11]). Namely, the intersection of all type-definable subgroups of bounded index still has bounded index. We call G^{00} the type-definable connected component of G . Another model theoretic invariant is G^0 , called the definable-connected component of G , which is the intersection of all definable subgroups of G of finite index. Clearly, $G^{00} \leq G^0$.

Recall also that a Keisler measure over M on X , with X a definable subset of M^n , is a finitely additive probability measure on the Boolean algebra of M -definable subsets of X . When we take the monster model, i.e., $M = \mathbb{M}$, we call it a global Keisler measure. A definable group G is said to be definably amenable if it admits a global (left) G -invariant Keisler measure.

The following facts will be used later.

FACT 1.5 [4]. Let G be a group definable in a saturated NIP structure \mathbb{M} . Then:

- (i) G is definably amenable if and only if it admits a global type $p \in S_G(\mathbb{M})$ with bounded G -orbit.
- (ii) G is definably amenable if and only if it admits a strongly f -generic type.

Moreover,

FACT 1.6 [4]. Let G be a definably amenable group definable in a saturated NIP structure \mathbb{M} . Then:

- (i) Weakly generic definable subsets, formulas, and types coincide with f -generic definable subsets, formulas, and types, respectively.
- (ii) $p \in S_G(\mathbb{M})$ is f -generic if and only if it has bounded G -orbit.
- (iii) $p \in S_G(\mathbb{M})$ is f -generic if and only if it is G^{00} -invariant.

- (iv) *The type-definable subgroup H fixing a global f -generic type is exactly G^{00} .*
- (v) *G/G^{00} is isomorphic to the Ellis group of $S_G(M)$ for any $M \prec \mathbb{M}$.*

Note that in the context of Fact 1.6, if p is global f -generic type which is either definable over, or finitely satisfiable in, a small model, then p will be strongly f -generic (because of parts (iii) or (iv)).

REMARK 1.7. Fact 1.6(v) gives a positive answer to the so-called Ellis group conjecture. The equality between G/G^{00} and the Ellis group, under possibly additional assumptions, was suggested first by Newelski [22] in Question 5.5 and Problem 5.6. A precise formulation under an assumption of NIP and definable amenability was made by Pillay [27]. This makes it reasonable to consider definably amenable groups as the “stable-like” groups in NIP environment.

1.4. Groups definable in $(\mathbb{Q}_p, +, \times, 0, 1)$. We will be referring a lot to the comprehensive survey in [2] for the basic model theory of the p -adic field \mathbb{Q}_p . A key point is Macintyre’s theorem [18] that $\text{Th}(\mathbb{Q}_p, +, \times, 0, 1)$ has quantifier elimination in the language where we add predicates $P_n(x)$ for the n th powers for each $n \in \mathbb{N}^+$. The valuation is quantifier-free definable in this expanded language, in particular is definable in the language of rings (see Section 3.2 of [2]). Let us give some notation we will be using. M denotes the structure $(\mathbb{Q}_p, +, \times, 0, 1)$, $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ is the multiplicative group and \mathbb{Z} is the ordered additive group of integers, the value group of \mathbb{Q}_p . The group homomorphism $v : \mathbb{Q}_p^* \rightarrow \mathbb{Z}$ is the valuation map. The valuation map v induces an absolute valuation $|\cdot|$ on \mathbb{Q}_p : for each $x \in \mathbb{Q}_p$, $|x| = p^{-v(x)}$ if $x \neq 0$ and $|x| = 0$ otherwise. The absolute valuation makes the p -adic field \mathbb{Q}_p into a locally compact topological field, with basis given by the sets $v(x - a) \geq n$ for $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. The ring of p -adic integers (valuation ring)

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid x = 0 \vee v(x) \geq 0\}$$

is of course a definable subset of \mathbb{Q}_p . We can add new predicates $\mathcal{O}(x)$ for the valuation ring and $\mathcal{O}^\times(x)$ for the elements with valuation 0. So $\mathcal{O}(M) = \mathcal{O}(\mathbb{Q}_p) = \mathbb{Z}_p$ and $\mathcal{O}^\times(M) = \mathcal{O}^\times(\mathbb{Q}_p) = \{x \in \mathbb{Q}_p \mid v(x) = 0\}$.

It is convenient to refer to Section 3 of [25] for various notions such as definable p -adic analytic manifold and definable p -adic analytic group. (One can also consult [23].) In the background are the notions of dimension of definable (with parameters) sets in the structure $M = (\mathbb{Q}_p, +, \times, 0, 1)$ as well as theorems on p -adic cell decomposition and definable functions, which are due to van den Dries and Scowcroft [9]. But we mention here a few details: For X a definable set in M , say $X \subseteq \mathbb{Q}_p^n$, $\dim(X)$ is the greatest $k \leq n$ such that some projection on \mathbb{Q}_p^k contains an open set. An n -dimensional definable p -adic analytic manifold is a topological space with a covering by finitely many open sets each homeomorphic to an open definable (in M) subset of \mathbb{Q}_p^n such that the transition maps are definable and analytic. See [32] for analytic functions over complete fields. A definable p -adic analytic group is a definable p -adic analytic manifold equipped with a group structure which is definable and analytic when read in the appropriate charts. Such a definable p -adic analytic group is a definable group in M and Lemma 3.8 of [25] says that conversely any group G definable in M can be definably equipped with the structure of a definable p -adic analytic group.

In the current paper we will really be concerned with groups G definable in the structure $M = (\mathbb{Q}_p, +, \times, 0, 1)$. However, the various notions introduced in the previous section depend on an ambient saturated model. So we let \mathbb{M} denote a very saturated elementary extension $(\mathbb{K}, +, \times, 0, 1)$ of M . We also let $N = (K, +, \times, 0) \prec \mathbb{M}$ be an elementary extension of M . So K and \mathbb{K} are p -adically closed fields. The dimension of definable sets is defined as before, but can also be defined in terms of the underlying algebraic closure relation, which coincides with field-theoretic algebraic closure: assuming X is definable over a finite set A , then $\dim(X)$ can be described as the maximum of $\dim(\bar{a}/A)$ as \bar{a} ranges over points of X . We have the notion of a “definable C^k -manifold in N (or over K),” a topological space with a finite covering by open sets each homeomorphic to an open definable subset of K^n (fixed n) with transition maps definable and C^k . Either adapting the methods of [25] or by transfer from the case $K = \mathbb{Q}_p$, one sees that for any group G definable in N and for any $k < \omega$, G can be definably equipped with the structure of a definable C^k -manifold in N with respect to which the group structure is C^k . In contrast with the analogous situation for real closed fields, it seems not to be known whether we can also do it for $k = \infty$. On the other hand if G is defined with parameters from \mathbb{Q}_p , then $G(\mathbb{Q}_p)$ is as in the previous paragraph, and so the definable analytic manifold and group structure on $G(\mathbb{Q}_p)$ DOES give rise to a definable C^∞ -structure on G with respect to which the group operation is C^∞ .

Recall from [23] that a definable manifold X over K is definably compact, if for any definable continuous function $f : \mathcal{O}(K) \setminus \{0\} \rightarrow X$, the limit $\lim_{x \rightarrow 0} f(x)$ exists in X . Definable compactness agrees with compactness if K is \mathbb{Q}_p . A definable subset of K^n with the induced topology is definably compact if and only if it is closed and bounded.

Let G be a group definable in N with parameters from \mathbb{Q}_p . As remarked above $G(\mathbb{Q}_p)$ has (definably in M) the structure of a p -adic analytic (Lie) group, and as such has an open compact subgroup, which must be definable in M . It follows that G has a \mathbb{Q}_p -definable subgroup C with $\dim(C) = \dim(G)$ and $C(\mathbb{Q}_p)$ is compact (and C definably compact).

Let C be a group definable in N over \mathbb{Q}_p which is (with its definable manifold topology) definably compact. Then $C(\mathbb{Q}_p)$ is compact. We have the standard part map $st : C \rightarrow C(\mathbb{Q}_p)$. The kernel of st is precisely the group of infinitesimals of C and coincides with the intersection of all \mathbb{Q}_p -definable subgroups of C of finite index. This is elaborated on in the following fact.

FACT 1.8. *Let C be a group definable in \mathbb{M} over \mathbb{Q}_p .*

- (i) *If C definably compact, then $C^{00} = C^0$ coincides with $\ker(st)$, and st induces a homeomorphism between C/C^0 (with its logic topology) and the p -adic Lie group $C(\mathbb{Q}_p)$.*
- (ii) *C is definably compact iff C has fsg.*

PROOF. For (i) see Corollary 2.4 of [23].

The left implies right direction of (ii) is Corollary 2.3(iv) of [23]. The right to left direction (easy direction) does not appear explicitly in [23], but appears in somewhat greater generality in Proposition 3.1 of [16] –1

FACT 1.9 [29]. *If G is a one-dimensional group definable over \mathbb{Q}_p , then G is abelian-by-finite, hence definably amenable.*

1.5. Definable groups and quotient groups. We will try to give some coherent notation regarding definable groups and algebraic groups. Our underlying first-order theory is $\text{Th}(\mathbb{Q}_p)$ in the ring language (or in the Macintyre language, a definitional expansion). As in Section 1.4, M denotes the standard model, \mathbb{M} a “monster model,” and $N \prec \mathbb{M}$ a not necessarily saturated elementary extension of M . When we speak of a group definable in M , N , or \mathbb{M} , we mean the obvious thing. When we speak of a group G definable over \mathbb{Q}_p (for example) we typically mean a group definable in \mathbb{M} defined with parameters from \mathbb{Q}_p . In this case $G(\mathbb{Q}_p)$ denotes the group definable in \mathbb{Q}_p by the same formulas defining G in \mathbb{M} . On the other hand if for example G is a group definable in \mathbb{Q}_p then we can consider the groups definable in N or \mathbb{M} by the same formulas, which we may refer to as $G(K)$, $G(\mathbb{K})$, or $G(N)$, $G(\mathbb{M})$. In general, when we speak of a definable object (set, or group) we mean a definable object in the monster model \mathbb{M} of $\text{Th}(\mathbb{Q}_p)$. As mentioned in Section 1.4 we are mainly interested in groups definable in the standard model M , but we will need information about their interpretations in \mathbb{M} . Let us again emphasize that we distinguish between definable and interpretable (definable in \mathbb{K}^{eq}) and our groups will be definable.

The usual mathematical notation for algebraic groups is to consider an algebraic group G over a field K as something like a functor which takes a field L containing K to the group $G(L)$. Alternatively, one can identify G with $G(L)$ for a given algebraically closed field containing K , where the variety structure as well as the group structure are given by data (polynomial equations, transition maps, morphisms) over K . (See [26] for more details on this and the later discussion of algebraic varieties and groups over algebraically closed fields.) With this notation the algebraic groups we will consider will be over \mathbb{Q}_p and we will consider their groups $G(\mathbb{Q}_p)$ and $G(\mathbb{K})$ of \mathbb{Q}_p -points, and \mathbb{K} -points. Of course the latter will be also definable groups in the structures M , \mathbb{M} , respectively, in the earlier sense, but essentially just quantifier-free definable in the ring language.

So in this paper we will slightly modify notation, by defining a *p -adic algebraic group* to be the group $G(\mathbb{Q}_p)$ of \mathbb{Q}_p -points of an algebraic group G over \mathbb{Q}_p . When G is a linear algebraic group over \mathbb{Q}_p , then $G(\mathbb{Q}_p)$ will simply be a subgroup of some $GL_n(\mathbb{Q}_p)$ defined by a finite system of polynomial equations over \mathbb{Q}_p . So a p -adic algebraic group is a special case of a group definable in M . As above we denote by $G(\mathbb{K})$ the group of \mathbb{K} -points of G , which is consistent with our earlier notation regarding definable groups.

We now just want to observe that the quotient of a p -adic algebraic group by a p -adic algebraic subgroup can be seen as a definable (rather than just interpretable) set in M : So we are given algebraic groups G, H over \mathbb{Q}_p such that $H(\mathbb{Q}_p)$ is a subgroup of $G(\mathbb{Q}_p)$. We assume G to be connected as an algebraic group. Let L be an algebraically closed field containing \mathbb{Q}_p such as $\mathbb{Q}_p^{\text{alg}}$. Then $H(L)$ is a subgroup of $G(L)$. But then by elimination of imaginaries in ACF together with a theorem of Weil, $G(L)/H(L)$ is, in the structure $(L, +, \times)$, definably isomorphic over \mathbb{Q}_p to an algebraic variety $X(L)$ over \mathbb{Q}_p , which is a homogeneous space for $G(L)$ all defined over \mathbb{Q}_p . Namely we have a surjective morphism $f : G(L) \rightarrow X(L)$ defined over \mathbb{Q}_p , constant on cosets of $H(L)$ in $G(L)$ and inducing a bijection between

$G(L)/H(L)$ and $X(L)$. Restricting f to $G(\mathbb{Q}_p)$ gives a definable (in M) bijection between $G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ and a subset Y of $X(\mathbb{Q}_p)$ which is definable in M and can be seen, by dimension reasons, to be an open subset of the p -adic manifold $X(\mathbb{Q}_p)$, so a p -adic manifold itself. In the case where $H(\mathbb{Q}_p)$ is normal in $G(\mathbb{Q}_p)$, then Y becomes an open subgroup of the p -adic algebraic group $X(\mathbb{Q}_p)$.

§2. Main results.

2.1. fsg groups and dfg groups over \mathbb{Q}_p .

FACT 2.1. *Let G be a group definable in \mathbb{M} over \mathbb{Q}_p and suppose G to be definably amenable. Then there is an algebraic group H over \mathbb{Q}_p and a finite-to-one \mathbb{Q}_p -definable group homomorphism from G^{00} to $H(\mathbb{K})$.*

Explanation. This follows from Theorem 2.19 of [21], making use of the proof of Corollary 2.22 of [21].

LEMMA 2.2. *Let G be a definably amenable group over \mathbb{Q}_p such that $G^{00} = G^0$. Then there is a \mathbb{Q}_p -definable subgroup $A \leq G(\mathbb{Q}_p)$ of finite index and a finite normal subgroup $A_0 \subseteq A$ such that A/A_0 is \mathbb{Q}_p -definably isomorphic to an open subgroup of a p -adic algebraic group.*

PROOF. By Fact 2.1, there is an algebraic group H over \mathbb{Q}_p and a definable over \mathbb{Q}_p finite-to-one group homomorphism from G^{00} to $H(\mathbb{K})$. Since $G^{00} = G^0$, by compactness, there is a \mathbb{Q}_p -definable finite index subgroup $A \leq G$ and a \mathbb{Q}_p -definable finite-to-one group homomorphism f from $A(\mathbb{Q}_p)$ to $H(\mathbb{Q}_p)$. Let $H_0(\mathbb{Q}_p)$ be the Zariski closure of $\text{im}(f)$ in $H(\mathbb{Q}_p)$, then $H_0(\mathbb{Q}_p)$ is a p -adic algebraic group. By Remark 2.13 of [12], $\dim(\text{im}(f)) = \dim(H_0(\mathbb{Q}_p))$. By the topological definition of dimension $\text{im}(f)$ has interior in $H_0(\mathbb{Q}_p)$, hence $\text{im}(f)$, being a subgroup, is open in $H_0(\mathbb{Q}_p)$. ⊣

FACT 2.3 [28]. *Assuming NIP. If a definable group G has dfg, then $G^{00} = G^0$.*

COROLLARY 2.4. *Let G be a group definable over \mathbb{Q}_p which is either fsg or dfg. Then there is a finite index \mathbb{Q}_p -definable subgroup $A \leq G(\mathbb{Q}_p)$ and a finite normal subgroup $A_0 \subseteq A$ such that A/A_0 is \mathbb{Q}_p -definably isomorphic to an open subgroup of a p -adic algebraic group.*

PROOF. By Facts 1.8 and 2.3 and Lemma 2.2. ⊣

2.2. One-dimensional dfg groups over \mathbb{Q}_p . Recall that the dp-rank of a partial type $\pi(x)$ over A is $\geq \kappa$ if there are $a \models \pi(x)$ and κ sequences $(I_i, i < \kappa)$ mutually indiscernible over A and none of which is indiscernible over Aa . Dp-minimal theories are theories in which all nonalgebraic 1-types have dp-rank 1. Every dp-minimal theory has NIP.

FACT 2.5 [7]. *$\text{Th}(\mathbb{Q}_p)$ is dp-minimal.*

FACT 2.6 [35]. *Assume that T has NIP. Let $N \models T$ be a small model. If p is an N -invariant global type of dp-rank 1, then p is either finitely satisfiable in N or definable over N .*

FACT 2.7 [35]. *Assume that T is dp -minimal, and acl satisfies exchange. Then dp -rank coincides with the acl -dimension.*

FACT 2.8 [13]. *If T has NIP then a global complete type does not fork over a small submodel M iff it is M -invariant.*

LEMMA 2.9. *Let G be a \mathbb{Q}_p -definable group of dimension 1. Then G is either dfg or fsg .*

PROOF. By Fact 1.9, G is definably amenable. By Fact 1.6, G admits a strongly f -generic type. Let $p \in S_G(\mathbb{K})$ be a global strongly f -generic type. There is a small submodel M such that every left translate of p does not fork over M , and hence M -invariant by Fact 2.8. By Facts 2.5 and 2.6, p is either definable over M or finitely satisfiable in M . So G is either dfg or fsg . \dashv

Recall the notion of generically stable type from [13]: a global type p is generically stable if it is both definable and finitely satisfiable in some small model.

FACT 2.10 [34]. *Let $p \in S(\mathbb{K})$ be a non-algebraic global type. Then p is not generically stable.*

COROLLARY 2.11. *If G is an infinite \mathbb{Q}_p -definable group which has dfg then G is NOT definably compact.*

PROOF. Suppose for a contradiction (using Fact 1.8) that G were both dfg and fsg . Let p be a global definable (strongly) f -generic type. By [14] p is generic, namely every formula in p is generic. By Proposition 4.2 of [11], p (and all its translates) are finitely satisfiable in a small model. So p is generically stable, which contradicts Fact 2.10. This proves the corollary. \dashv

The following is from [24]. Part (a) is Lemma 2.1 there, part (b)(i) is contained in Proposition 2.3 there, and part (b)(ii) in Proposition 2.4. Let us emphasize that the data in items (ii) and (iii) of part (a) determine complete types over M as is proved in [24].

FACT 2.12. (a) *The complete 1-types over M (or \mathbb{Q}_p) are precisely the following:*

- (i) *The realized types $\text{tp}(a/M)$ for each $a \in \mathbb{Q}_p$.*
- (ii) *For each $a \in \mathbb{Q}_p$ and coset C of $(\mathbb{K}^*)^0$, the type $p_{a,C}$ saying that x is infinitesimally close to a (i.e., $v(x - a) > n$ for each $n \in \mathbb{N}$), and $(x - a) \in C$.*
- (iii) *For each coset C as above the type $p_{\infty,C}$ saying that $x \in C$ and $v(x) < n$ for all $n \in \mathbb{Z}$.*

(b)

- (i) *A global type of $\mathbb{G}_a(\mathbb{K})$ is (strongly) f -generic iff it is an heir of some $p_{\infty,C}$.*
- (ii) *A global type of $\mathbb{G}_m(\mathbb{K})$ is (strongly) f -generic iff it is an heir of some $p_{\infty,C}$ or $p_{0,C}$.*

Part (ii) of the next remark is precisely Remark 2.2(ii) from [24].

REMARK 2.13. (i) Let $q(x) \in S_1(\mathbb{K})$ be a \emptyset -definable 1-type. By Fact 2.12, we see that if q is NOT infinitesimally close to any point of \mathbb{Q}_p over \mathbb{Q}_p , then q is a global heir of some $p_{\infty,C}$.

(ii) Conversely, any global complete 1-type extending the partial type $v(x) < \Gamma$ is an heir of some $p_{\infty,C}$, and hence definable over \mathbb{Q}_p .

LEMMA 2.14. *Suppose that G is a one-dimensional group definable over \mathbb{Q}_p . Let $f : \mathbb{Z}_p \setminus \{0\} \rightarrow G(\mathbb{Q}_p)$ be a definable continuous function such that $\lim_{x \rightarrow 0} f(x) \notin G(\mathbb{Q}_p)$, and $\alpha \in \bar{\mathbb{K}} \succ \mathbb{K}$ such that $\alpha \neq 0$ and $v(\alpha) > \Gamma$. Then $\text{tp}(f(\alpha)/\mathbb{K})$ is a global definable f -generic type of G .*

PROOF. First let us check that the notation in the statement of the lemma makes sense. First the continuity of f is meant with respect to the p -adic topology on \mathbb{Z}_p and the definable p -adic analytic group topology on $G(\mathbb{Q}_p)$ from Section 1.4. For any elementary extension $N = (K, +, \times, 0)$ of $M = (\mathbb{Q}_p, +, \times, 0)$, $f(K)$ is a continuous definable over \mathbb{Q}_p function from $\mathcal{O} \setminus \{0\}$ to $G(K)$, which we may also write as just f . So for $\alpha \in \bar{\mathbb{K}} \succ \mathbb{K}$, nonzero such that $v(\alpha) > \Gamma$, $f(\alpha) \in G(\bar{\mathbb{K}})$.

Note that $\text{tp}(\alpha/\mathbb{K})$ is \emptyset -definable. Since $\lim_{x \rightarrow 0} f(x) \notin G(\mathbb{Q}_p)$, $f(\alpha)$ is not infinitesimally close (over \mathbb{Q}_p) to any point of $G(\mathbb{Q}_p)$. Moreover, since $G(\mathbb{Q}_p)$ has a definable manifold structure, for any $g \in G(\mathbb{Q}_p)$, $gf(\alpha)$ is not infinitesimally close to any point of $G(\mathbb{Q}_p)$ over \mathbb{Q}_p .

By Lemma 2.2, let $H(\mathbb{Q}_p)$ be a one-dimensional algebraic group, $U(\mathbb{Q}_p) \leq G(\mathbb{Q}_p)$ a \mathbb{Q}_p -definable subgroup of finite index, and $\pi : U(\mathbb{Q}_p) \rightarrow H(\mathbb{Q}_p)$ a definable finite-to-one homomorphism. Let $V(\mathbb{Q}_p) = \pi(U(\mathbb{Q}_p))$. Since $U(\mathbb{Q}_p)$ is generic, we may assume that $f(\alpha) \in U(\bar{\mathbb{K}})$. Let $\beta = \pi(f(\alpha))$. Then, for any $g \in V(\mathbb{Q}_p)$, $g\beta$ is not infinitesimally close to any point of $V(\mathbb{Q}_p)$ over \mathbb{Q}_p .

CLAIM. β is not infinitesimally close to any point of $H(\mathbb{Q}_p)$.

PROOF. As any open subgroup is clopen, we see that $V(\mathbb{Q}_p)$ is a clopen subgroup of $H(\mathbb{Q}_p)$. Now suppose for a contradiction that β is infinitesimally close to $h \in H(\mathbb{Q}_p)$ over \mathbb{Q}_p , then $h \in \text{cl}(V(\mathbb{Q}_p)) = V(\mathbb{Q}_p)$. A contradiction. \dashv

Now $H(\mathbb{Q}_p)$ is a p -adic algebraic group (as defined in Section 1.5). So $H(\mathbb{Q}_p)$ is definably isomorphic to $(\mathbb{Q}_p, +)$, or (\mathbb{Q}_p^*, \times) , or a definably compact group $C(\mathbb{Q}_p)$ (which is either an anisotropic group or an elliptic curve). But any point in $C(\bar{\mathbb{K}})$ is infinitesimally close to its standard part in $C(\mathbb{Q}_p)$ over \mathbb{Q}_p . So $H(\mathbb{Q}_p)$ is isomorphic to $(\mathbb{Q}_p, +)$, or (\mathbb{Q}_p^*, \times) .

As β is not infinitesimally close to any point of $H(\mathbb{Q}_p)$, and $\text{tp}(\beta/\mathbb{K})$ is definable over \emptyset , we conclude by Remark 2.13 and Fact 2.12 that $\text{tp}(\beta/\mathbb{K})$ is the heir of some $p_{\infty, C}$ in the case $H = (\mathbb{Q}_p, +)$, and the heir of some $p_{\infty, C}$ or some $p_{0, C}$ in the case that $H = (\mathbb{Q}_p^*, \times)$. Either way $\text{tp}(\beta/\mathbb{K})$ is a global definable (strong) f -generic of H . So the $V(\bar{\mathbb{K}})$ -orbit of $\text{tp}(\beta/\mathbb{K})$ is bounded. Since $\pi : U(\bar{\mathbb{K}}) \rightarrow V(\bar{\mathbb{K}})$ is a finite-to-one homomorphism, the $U(\bar{\mathbb{K}})$ -orbit of $\text{tp}(f(\alpha)/\mathbb{K})$ is bounded. This implies that $\text{tp}(f(\alpha)/\mathbb{K})$ is a global f -generic type of G . \dashv

PROPOSITION 2.15. *Let G be a one-dimensional \mathbb{Q}_p -definable group. If $H \subseteq G$ is an open dfg subgroup of G definable over \mathbb{Q}_p , then H has finite index in G .*

PROOF. By Corollary 2.11, there is a definable continuous function $f : \mathbb{Z}_p \setminus \{0\} \rightarrow H(\mathbb{Q}_p)$ such that $\lim_{x \rightarrow 0} f(x) \notin H(\mathbb{Q}_p)$. Moreover, we claim the following.

CLAIM. *There is a definable continuous function $f : \mathbb{Z}_p \setminus \{0\} \rightarrow H(\mathbb{Q}_p)$ such that $\lim_{x \rightarrow 0} f(x) \notin G(\mathbb{Q}_p)$.*

PROOF. Suppose NOT. Then $H(\mathbb{Q}_p) = \text{cl}(H(\mathbb{Q}_p))$ is definably compact. This contradicts Corollary 2.11. \dashv

Let f be a definable continuous function $f : \mathbb{Z}_p \setminus \{0\} \rightarrow H(\mathbb{Q}_p)$ such that $\lim_{x \rightarrow 0} f(x) \notin G(\mathbb{Q}_p)$, and $\alpha \in \bar{\mathbb{K}} \succ \mathbb{K}$ such that $v(\alpha) > \Gamma$. By Lemma 2.14, $\text{tp}(f(\alpha)/\mathbb{K})$ is a global definable f -generic type of both G and H . Now $\text{tp}(f(\alpha)/\mathbb{K}) \in S_H(\mathbb{K})$ is G^{00} -invariant. So $G^{00} \subseteq H$. By compactness, H has finite index. \dashv

COROLLARY 2.16. *Let G be a one-dimensional dfg group definable over \mathbb{Q}_p . Then $G^{00} = \bigcap_{n \in \mathbb{N}^+} nG$.*

PROOF. By Fact 1.9, we may assume that G is commutative. Consider the map $f_n : G \rightarrow nG$ given by $x \mapsto nx$. Let $\text{tp}(\alpha/\mathbb{K}) \in S_G(\mathbb{K})$ be a definable f -generic type. Then there is a small submodel $M \prec \mathbb{K}$ such that every left nG -translate of $\text{tp}(n\alpha/\mathbb{K})$ is definable over M . So nG is a dfg group. By Proposition 2.15, nG has finite index for each $n \in \mathbb{N}^+$. On the other hand, if $H \leq G$ is a definable subgroup of index n with $n \in \mathbb{N}^+$, then $nG \leq H$. So $G^0 = \bigcap_{n \in \mathbb{N}^+} nG$. By Fact 2.3, we have $G^{00} = \bigcap_{n \in \mathbb{N}^+} nG$ as required. \dashv

Recall that a definable group is definably connected if it has no proper finite index definable subgroups. We will call a definable group G definably totally disconnected if every finite index definable subgroup of G is NOT definably connected. Namely, a definably totally disconnected group has no minimal finite index definable subgroups.

Note that $G_a(\mathbb{K})$ is definably connected since $G_a(\mathbb{K})^0 = G_a(\mathbb{K})$, while $G_m(\mathbb{K})$ is definably totally disconnected since $G_m(\mathbb{K})^0 = \bigcap_{n \in \mathbb{N}^+} P_n(\mathbb{K}^*)$ (see [24]).

PROPOSITION 2.17. *Let G be a one-dimensional dfg group definable over \mathbb{Q}_p .*

1. *If $G(\mathbb{Q}_p)$ is definably connected, then there is a finite normal subgroup A_0 such that $G(\mathbb{Q}_p)/A_0$ is definably isomorphic to $(\mathbb{Q}_p, +)$.*
2. *If $G(\mathbb{Q}_p)$ is definably totally disconnected, then there is a definable subgroup $A \leq G$ of finite index, and a finite normal subgroup $A_0 \leq A(\mathbb{Q}_p)$ such that $A(\mathbb{Q}_p)/A_0$ is definably isomorphic to $(P_n(\mathbb{Q}_p^*), \times)$.*

PROOF. By Lemma 2.2, there are a one-dimensional algebraic group H and a definable finite-to-one group homomorphism f from $A(\mathbb{Q}_p)$ to $H(\mathbb{Q}_p)$, where $A \leq G$ is a finite index subgroup definable over \mathbb{Q}_p . Clearly, A has dfg. Let $\text{tp}(\alpha/\mathbb{K})$ be a global definable f -generic type of A . Then $\text{tp}(f(\alpha)/\mathbb{K})$ is also a global definable f -generic type of $\text{im}(f)$. As $\text{im}(f)$ has dfg, $\text{im}(f) \leq H$ has finite index by Proposition 2.15. So $H(\mathbb{Q}_p)$ is dfg, and hence is either $(\mathbb{Q}_p, +)$ or (\mathbb{Q}_p^*, \times) .

If $G(\mathbb{Q}_p)$ is definably connected. Then $\text{im}(f)$ is definably connected. So $H(\mathbb{Q}_p)$ has to be $(\mathbb{Q}_p, +)$, and hence is $\text{im}(f)$.

If $G(\mathbb{Q}_p)$ is totally disconnected, then $\text{im}(f)$ is also totally disconnected. So $H(\mathbb{Q}_p)$ has to be (\mathbb{Q}_p^*, \times) , and hence $\text{im}(f)$ contains $(P_n(\mathbb{Q}_p^*), \times)$ for some n . Replace $A(\mathbb{Q}_p)$ by $f^{-1}(P_n(\mathbb{Q}_p^*))$ if necessary. \dashv

2.3. Algebraic groups with open dfg subgroups. The following will be used later:

FACT 2.18 [23]. *Let $X \subseteq \mathbb{K}^n$ be open, definable and defined over \mathbb{Q}_p . Let $X = Y_1 \cup Y_2$, where the Y_i are definable in \mathbb{K} . Then one of the Y_i contains an open \mathbb{Q}_p -definable subset.*

LEMMA 2.19. *Let H be an algebraic group over \mathbb{Q}_p , and G an open \mathbb{Q}_p -definable dfg subgroup of $H(\mathbb{K})$. Suppose A is an algebraic subgroup of H , also defined over \mathbb{Q}_p . Let $G/A(\mathbb{K})$ be the set of cosets $g/A(\mathbb{K})$ for $g \in G$. Then $G/A(\mathbb{K})$ infinite implies $G/A(\mathbb{K})$ is not definably compact (equivalently $G(\mathbb{Q}_p)/A(\mathbb{Q}_p)$ is not compact.)*

PROOF. We start with some explanations, partly suggested by the referee. First the quotient space $H(\mathbb{Q}_p)/A(\mathbb{Q}_p)$ (a homogeneous space for $H(\mathbb{Q}_p)$) can be viewed as a definable (rather than interpretable) set in M , and moreover a definable p -adic manifold. This is by the last paragraph of Section 1.5. So $G(\mathbb{Q}_p)/A(\mathbb{Q}_p)$, as defined in the statement of the lemma will be a definable submanifold of $H(\mathbb{Q}_p)/A(\mathbb{Q}_p)$, and a homogeneous space for $G(\mathbb{Q}_p)$, and we call it $C(\mathbb{Q}_p)$. So $C = C(\mathbb{K})$ is a definable (over \mathbb{Q}_p) manifold in the sense of Section 1.4, and also a homogeneous space for $G = G(\mathbb{K})$. Note also that the lemma we are proving is a natural generalization of Corollary 2.11 to homogeneous spaces (transitive flows).

Let $\pi : G \rightarrow C$ be the natural \mathbb{Q}_p -definable projection map.

So we have the action of $G = G(\mathbb{M})$ on $C = C(\mathbb{M})$ and hence also an action on the type space $S_C(\mathbb{M})$ of global types concentrating on C . We will call a definable subset X of C *generic* if finitely many G -translates of X cover C , and likewise a type $p \in S_C(\mathbb{M})$ *generic* if all formulas in p are generic.

We first show that for any \mathbb{Q}_p -definable open subset $O(\mathbb{M})$ of $G(\mathbb{M})$, $\pi(O(\mathbb{M}))$ contains a \mathbb{Q}_p -definable open subset of $C(\mathbb{M})$. Take $g \in O(\mathbb{M})$ such that $\dim(g/\mathbb{Q}_p) = \dim(G)$, then we have $g \in \pi^{-1}(\pi(g))$ and

$$\dim(\pi^{-1}(\pi(g))) = \dim(gA) = \dim(A).$$

So $\dim(g/\mathbb{Q}_p, \pi(g)) \leq \dim(A)$, and thus

$$\dim(\pi(g)/\mathbb{Q}_p) = \dim(g, \pi(g)/\mathbb{Q}_p) - \dim(g/\mathbb{Q}_p, \pi(g)) \geq \dim(G) - \dim(A).$$

As $\pi(g) \in \pi(O(\mathbb{M}))$, we see that

$$\dim(O(\mathbb{M})) \geq \dim(\pi(g)/\mathbb{Q}_p) \geq \dim(G) - \dim(A) = \dim(H/A) \geq \dim(C).$$

So $\pi(O(\mathbb{M}))$ contains a \mathbb{Q}_p -definable open subset of $C(\mathbb{M})$ as required.

Now suppose for a contradiction that $C(\mathbb{Q}_p)$ is definably compact and infinite. Then $C(\mathbb{Q}_p)$ is compact as a topological space (as mentioned earlier, definable compactness and compactness agree for definable p -adic manifolds).

CLAIM. *$S_C(\mathbb{M})$ has a generic type which is moreover finitely satisfiable in \mathbb{Q}_p (i.e., in $C(\mathbb{Q}_p)$).*

PROOF. Let $p(x)$ be any type in $S_C(M)$ which contains only open definable sets, and let $p'(x) \in S_C(\mathbb{M})$ be a coheir of p . We will show that p' is a generic type of $S_C(\mathbb{M})$ as defined above. So let X be a definable set in p' . By definability of types over \mathbb{Q}_p , $X \cap C(\mathbb{Q}_p) = Y$ is a definable set in the structure M . Note that Y has interior in the sense of the space $C(\mathbb{Q}_p)$, because otherwise $Y \notin p$, so $X \wedge \neg(Y(\mathbb{M})) \in p'$,

but the latter definable set does not meet $C(\mathbb{Q}_p)$, contradicting p' being finitely satisfiable in M . We have shown that Y has interior in $C(\mathbb{Q}_p)$. It follows that $Y(\mathbb{M})$ contains a \mathbb{Q}_p -definable open set.

Now consider $Y(\mathbb{M}) = (X \cap Y(\mathbb{M})) \cup ((\neg X) \cap Y(\mathbb{M}))$. Note that $(\neg(X) \cap Y(\mathbb{M})) \cap C(\mathbb{Q}_p) = \emptyset$ so in particular does not contain any \mathbb{Q}_p -definable open set. It follows from Fact 2.18 that $X \cap Y(\mathbb{M})$ contains a \mathbb{Q}_p -definable open set, so in particular, X contains a \mathbb{Q}_p -definable open set Z . As $C(\mathbb{Q}_p)$ is compact, finitely many $G(\mathbb{Q}_p)$ translates of $Z(\mathbb{Q}_p)$ cover $C(\mathbb{Q}_p)$. So finitely many G -translates of Z cover C , so finitely many G -translates of X cover C . So X is generic, and as X was an arbitrary definable set in p' we have shown that p' is generic. This completes the proof of the claim. \dashv

Let us now complete the proof of the lemma. Note that we have a $G(\mathbb{M})$ -flow map from $(G(\mathbb{M}), S_G(\mathbb{M}))$ to $(G(\mathbb{M}), S_C(\mathbb{M}))$ induced by π . By our assumptions, let $r \in S_G(\mathbb{M})$ be a definable f -generic type. By Lemma 1.15 of [28] (and its proof) the orbit $(G(\mathbb{M}) \cdot r)$ is closed, hence a minimal subflow of $S_G(\mathbb{M})$. Then $\pi(G(\mathbb{M}) \cdot r)$ is a minimal subflow of $S_C(\mathbb{M})$. But $S_C(\mathbb{M})$ has generic types, by the claim, hence as is well-known (see [22]) has a unique minimal subflow. By the claim, let $p \in S_C(\mathbb{M})$ be generic and finitely satisfiable in $C(\mathbb{Q}_p)$. But then p is of the form $\pi(q)$ for some $q \in G(\mathbb{M}) \cdot r$. As q is definable, so is $\pi(q) = p$. Then p is both definable and finitely satisfiable, which contradicts distality. \dashv

REMARK 2.20. Note that if G and H are definable groups, and G has dfg, and $f : G \rightarrow H$ is a definable onto homomorphism, then H has dfg.

At this point we will need some of the basic theory of linear algebraic groups. The notation in the literature (such as [20]) for (linear) algebraic groups over a field k is consistent with our notation from Section 1.5 where $k = \mathbb{Q}_p$. See Chapter 14 of [20] for the notions of unipotent elements in a linear algebraic group G and unipotent algebraic subgroups G . An algebraic group G over \mathbb{Q}_p is *trigonalizable* over \mathbb{Q}_p if it is isomorphic over \mathbb{Q}_p to a group of upper triangular matrices in some GL_n . An algebraic torus over \mathbb{Q}_p is *split* over \mathbb{Q}_p if it is isomorphic over \mathbb{Q}_p to a product of copies of the multiplicative group. A connected linear solvable algebraic group G over \mathbb{Q}_p is *split* over \mathbb{Q}_p if G admits a subnormal series of algebraic subgroups over \mathbb{Q}_p whose quotients are isomorphic over \mathbb{Q}_p to the additive or multiplicative group. A connected semisimple algebraic group G over \mathbb{Q}_p is said to be *isotropic* if contains a nontrivial \mathbb{Q}_p -split algebraic torus defined over \mathbb{Q}_p . Let G be a reductive algebraic group over \mathbb{Q}_p . A parabolic subgroup is a connected subgroup P of G such that the quotient variety G/P is complete.

We summarise the various facts from the literature that we need (specialized to the case $k = \mathbb{Q}_p$).

- FACT 2.21. (i) (See Theorem 16.33 of [20]) Let G be a connected linear solvable algebraic group over \mathbb{Q}_p . Then G has a maximal unipotent algebraic subgroup, G_u which is normal and over \mathbb{Q}_p and G/G_u is an algebraic torus (over \mathbb{Q}_p).
- (ii) (See Proposition 16.52 of [20]) Any connected solvable linear algebraic group over \mathbb{Q}_p which is split over \mathbb{Q}_p is trigonalizable over \mathbb{Q}_p .

- (iii) (See Corollaire 4.17 of [3]) Let G be a (connected) semisimple algebraic group over \mathbb{Q}_p . Then G is \mathbb{Q}_p -isotropic if and only if G has a proper parabolic subgroup P defined over \mathbb{Q}_p .
- (iv) (Theorem 3.1 of [30]) Suppose G is a connected semisimple algebraic group over \mathbb{Q}_p which is not \mathbb{Q}_p -isotropic. Then $G(\mathbb{Q}_p)$ is compact.

PROPOSITION 2.22. *Let H be a connected (not necessarily linear) algebraic group over \mathbb{Q}_p . Suppose that $H(\mathbb{M})$ contains an open \mathbb{Q}_p -definable dfg subgroup G , then H is linear, solvable, and the algebraic torus H/H_u (from part (i) of the fact above) splits over \mathbb{Q}_p .*

PROOF. The algebraic group H admits a short exact sequence of connected algebraic groups over \mathbb{Q}_p

$$1 \rightarrow L \rightarrow H \rightarrow A \rightarrow 1$$

with L a linear algebraic, and A an abelian variety. Now $G(\mathbb{Q}_p)/L(\mathbb{Q}_p)$ definably over \mathbb{Q}_p embeds in $A(\mathbb{Q}_p)$ which is compact, so $G(\mathbb{Q}_p)/L(\mathbb{Q}_p)$ is compact so definably compact. If A is nontrivial, then $\dim(L) < \dim(H)$, so as $G(\mathbb{Q}_p)$ is open in $H(\mathbb{Q}_p)$, $G(\mathbb{Q}_p)/L(\mathbb{Q}_p)$ is infinite. By Lemma 2.19 this implies that $G(\mathbb{Q}_p)/L(\mathbb{Q}_p)$ is not definably compact, so not compact, a contradiction. So A is trivial and hence H is linear.

Let $R = \text{rad}(H)$ be the solvable radical of H , the maximal connected normal solvable algebraic subgroup of H , which is automatically defined over \mathbb{Q}_p . Let S be the quotient H/R a connected semisimple algebraic group over \mathbb{Q}_p . We want to show that S is trivial, so H will be solvable, as desired. Suppose for a contradiction that S is nontrivial.

As in the statement of Lemma 2.19, we define $G/R(\mathbb{K})$ to be the set of cosets $g/R(\mathbb{K})$ in $H(\mathbb{K})/R(\mathbb{K})$ for $g \in G$, which (as R is normal in H and G an open dfg subgroup of $H(\mathbb{K})$) is an open \mathbb{Q}_p -definable dfg subgroup of $S(\mathbb{K})$. As $G/R(\mathbb{K})$ is infinite, by Lemma 2.19 $G/R(\mathbb{K})$ is not definably compact, so $G(\mathbb{Q}_p)/R(\mathbb{Q}_p)$ is not compact. But the latter is a closed subgroup of $S(\mathbb{Q}_p)$, whereby $S(\mathbb{Q}_p)$ is not compact. By Fact 2.21(iv), S is \mathbb{Q}_p -isotropic. So by Fact 2.21(iii), S has a proper parabolic subgroup P defined over \mathbb{Q}_p . As S/P is a complete variety defined over \mathbb{Q}_p , $(S/P)(\mathbb{Q}_p)$ is compact, so $(S/P)(\mathbb{K})$ is definably compact. Let X be the image of G (or of $G/R(\mathbb{K})$) in $(S/P)(\mathbb{K})$. Then X is also definably compact, again contradicting Lemma 2.19. This contradiction proves that S was trivial, and so $H = R$ is solvable.

We finally have to prove that the toric part $T = H/H_u$ of H splits over \mathbb{Q}_p . T is an algebraic torus defined over \mathbb{Q}_p , and by page 53 of [30] for example, T is an almost direct product of tori T_1, T_2 defined over \mathbb{Q}_p where T_1 is \mathbb{Q}_p -split and T_2 has no nontrivial split subtorus over \mathbb{Q}_p . We have to show that T_2 is trivial. Otherwise T_2 is nontrivial, and by Theorem 3.1 of [30] $T_2(\mathbb{Q}_p)$ is compact so $T_2(\mathbb{K})$ is definably compact. Now H has a proper normal algebraic subgroup H_1 defined over \mathbb{Q}_p (namely the preimage of T_1 under the homomorphism $H \rightarrow H/H_u$) such that the quotient $H/H_1 = T_2$. But by Lemma 2.19, as usual the image of G in $(H/H_1)(\mathbb{K})$ is infinite and not definably compact, a contradiction. This completes the proof of the proposition. ◻

COROLLARY 2.23. *Let H be a connected algebraic group over \mathbb{Q}_p . If $H(\mathbb{K})$ contains an open \mathbb{Q}_p -definable dfg subgroup G say, then H is trigonalizable over \mathbb{Q}_p .*

PROOF. By the previous proposition H is linear, solvable, and H/H_u is a \mathbb{Q}_p -split torus. As we are in characteristic 0 the unipotent part H_u of H is split (i.e., admits a subnormal series of algebraic subgroups over \mathbb{Q}_p whose quotients are isomorphic over \mathbb{Q}_p to the additive group). Hence H itself is a \mathbb{Q}_p -split solvable group over \mathbb{Q}_p . By Fact 2.21(ii), H is trigonalizable over \mathbb{Q}_p . \dashv

2.4. dfg Groups and f -generic types. Let A, B, C be definable groups, defined over \mathbb{Q}_p , such that $A \leq B$ and $\pi : B \rightarrow C$ is \mathbb{Q}_p -definable, such that, moreover,

$$1 \rightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \rightarrow 1$$

is a short exact sequence, where $i : A \rightarrow B$ is the inclusion map. Since $\text{Th}(\mathbb{Q}_p)$ has definable Skolem functions (see [8]), let $f : C \rightarrow B$ be a \mathbb{Q}_p -definable section of π . Every element b of B can be written uniquely as $af(\pi(b))$ for some $a \in A$.

We will be freely using Facts 1.5 and 1.6 in the following.

LEMMA 2.24. *Fix notation as above. Suppose that B is definably amenable, then both A and C are definably amenable. Moreover if $b \in B(\bar{\mathbb{M}})$ is such that $\text{tp}(b/\bar{\mathbb{M}})$ is an f -generic type, then both $\text{tp}(\pi(b)/\bar{\mathbb{M}})$ and $\text{tp}(a/\bar{\mathbb{M}})$ are f -generic types of C and A , respectively.*

PROOF. Since $\text{tp}(b/\bar{\mathbb{M}})$ is B^{00} -invariant, we see that $\text{tp}(\pi(b)/\bar{\mathbb{M}})$ is $\pi(B^{00})$ -invariant, so C is definably amenable and $\text{tp}(\pi(b)/\bar{\mathbb{M}})$ is an f -generic type of C .

Let $\eta : B \rightarrow A$ be the function given by $\eta(x) = x \cdot f(\pi(x))^{-1}$, then $a = \eta(b)$. For each $a_0 \in A$, we have

$$a_0 \cdot a = a_0 \cdot b \cdot f(\pi(b))^{-1} = (a_0 \cdot b) \cdot f(\pi(a_0 \cdot b))^{-1} = \eta(a_0 \cdot b).$$

Since $\text{tp}(b/\bar{\mathbb{M}})$ is f -generic, we see that

$$A \cdot \text{tp}(b/\bar{\mathbb{M}}) = \{\text{tp}(a_0 \cdot b/\bar{\mathbb{M}}) \mid a_0 \in A\}$$

is bounded. So the A -orbit of $\text{tp}(a/\bar{\mathbb{M}})$

$$\{\text{tp}(a_0 \cdot a/\bar{\mathbb{M}}) \mid a_0 \in A\} = \{\text{tp}(\eta(a_0 \cdot b)/\bar{\mathbb{M}}) \mid a_0 \in A\}$$

is bounded. We conclude that A is definably amenable and $\text{tp}(a/\bar{\mathbb{M}})$ is an f -generic type of C . \dashv

LEMMA 2.25. *Let the notation be as above. Suppose that B has dfg. Then both A and C have dfg.*

PROOF. Let $\text{tp}(b/\bar{\mathbb{M}})$ be a global f -generic type definable over some small model M_0 . Then, by Lemma 2.24, we see that both $\text{tp}(\pi(b)/\bar{\mathbb{M}})$ and $\text{tp}(a/\bar{\mathbb{M}})$ are f -generic types of C and A , respectively.

Assume that M_0 contains the parameters of A, B, C and π , then $a, \pi(b) \in \text{dcl}(M_0, b)$, we see that $\text{tp}(\pi(b)/\bar{\mathbb{M}})$ and $\text{tp}(a/\bar{\mathbb{M}})$ are definable over M_0 , and hence dfg types of C and A , respectively. This concludes the proof that A and C have dfg. \dashv

REMARK 2.26. Let A, B be algebraic groups over \mathbb{Q}_p such that A is a subgroup of B , and $B(\mathbb{K})$ is dfg. Since the coset space $(B/A)(\mathbb{K})$ is definable, a similar argument to the proof of Lemma 2.25 shows that $A(\mathbb{K})$ is also dfg.

PROPOSITION 2.27. *Let H be an algebraic group over \mathbb{Q}_p and G an open dfg subgroup of $H(\mathbb{K})$ definable over \mathbb{Q}_p . Then G has finite index in $H(\mathbb{K})$.*

PROOF. The proof is by induction on $\dim(H)$. The statement holds for $\dim(H) = 1$ by Proposition 2.15.

Now assume that $\dim(H) > 1$.

By Corollary 2.23, H is trigonalizable over \mathbb{Q}_p , and hence there is a short exact sequence of algebraic groups over \mathbb{Q}_p

$$1 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} C \longrightarrow 1,$$

where A a one-dimensional algebraic group central in H , and C an algebraic group of dimension $\dim(H) - 1$. Then we have an induced short exact sequence:

$$1 \longrightarrow A(\mathbb{K}) \cap G \xrightarrow{i} G \xrightarrow{\pi} \pi(G) \longrightarrow 1.$$

By Lemma 2.25, we see that both $A(\mathbb{K}) \cap G$ and $\pi(G)$ are open dfg subgroups of $A(\mathbb{K})$ and $C(\mathbb{K})$, respectively. By induction hypothesis, both $A(\mathbb{K}) \cap G$ and $\pi(G)$ have finite index in $A(\mathbb{K})$ and $C(\mathbb{K})$, respectively. This concludes that G has finite index in $H(\mathbb{K})$. ⊢

COROLLARY 2.28. *Suppose that $G_1 \leq G_2$ are dfg groups definable over \mathbb{Q}_p . If $\dim(G_1) = \dim(G_2)$, then G_1 has finite index in G_2 .*

PROOF. By Corollary 2.4, there are a finite index subgroup A of G_2 and a finite normal subgroup A_0 of A , with both A and A_0 definable over \mathbb{Q}_p , and such that A/A_0 is \mathbb{Q}_p -definably isomorphic to an open subgroup of $H(\mathbb{K})$ for some algebraic group H defined over \mathbb{Q}_p . Let $f : A \rightarrow H(\mathbb{K})$ be the finite-to-one homomorphism with $\ker(f) = A_0$ as in Fact 2.1. By Proposition 2.27, $\text{im}(f)$ has finite index in $H(\mathbb{Q}_p)$. As $G_1 \cap A$ has finite index in G_1 , we see that $G_1 \cap A$ has dfg. So the image $f(G_1 \cap A)$ is an open dfg subgroup of $H(\mathbb{K})$, and hence has finite index in $H(\mathbb{K})$ by Proposition 2.27. It follows easily that G_1 has finite index in G_2 . ⊢

A generalization of Corollary 2.28 has been proved in [10, Corollary 4.4].

We now prove the following two structure theorems for dfg groups using Proposition 2.27:

THEOREM 2.29. *Let G be an open dfg \mathbb{Q}_p -definable subgroup of $H(\mathbb{K})$ where H is an algebraic group defined over \mathbb{Q}_p . Then there is a normal sequence*

$$\{\text{id}_G\} = G_0 \triangleleft \dots \triangleleft G_i \triangleleft G_{i+1} \triangleleft \dots \triangleleft G_n = G$$

such that each G_i is definable over \mathbb{Q}_p and each G_{i+1}/G_i is definably over \mathbb{Q}_p isomorphic to a one-dimensional dfg group.

PROOF. Induction on $\dim(G)$. Clearly, the statement is trivial if $\dim(G) = 1$. Now assume that $\dim(G) > 1$. As we showed in Proposition 2.27, there is a short exact sequence

$$1 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} C \longrightarrow 1$$

with A a one-dimensional algebraic group central in H , and C an algebraic group of dimension $\dim(H) - 1$, where everything is over \mathbb{Q}_p . Both $A(\mathbb{K}) \cap G$ and $\pi(G)$

are open dfg subgroups of $A(\mathbb{K})$ and $C(\mathbb{K})$, respectively. By induction hypothesis, there is a normal sequence

$$\{\text{id}_C\} = D_0 \triangleleft \cdots \triangleleft D_i \triangleleft D_{i+1} \triangleleft \cdots \triangleleft D_n = \pi(G)$$

such that each D_{i+1}/D_i is definably isomorphic to a one-dimensional dfg group over \mathbb{Q}_p . It is easy to see that the normal sequence

$$\{\text{id}_G\} \leq G_0 = G \cap A(\mathbb{K}) \triangleleft \cdots \triangleleft G_{i+1} = G \cap \pi^{-1}(D_i) \triangleleft \cdots \triangleleft G_{n+1} = G$$

meets our requirements. ⊢

THEOREM 2.30. *Let G be a group definable over \mathbb{Q}_p with dfg. Then there is a normal sequence*

$$G_0 \triangleleft \cdots \triangleleft G_i \triangleleft G_{i+1} \triangleleft \cdots \triangleleft G_n$$

of \mathbb{Q}_p -definable groups, such that G_0 is finite, G_n is a finite index subgroup of G , and G_{i+1}/G_i is isomorphic to a one-dimensional dfg group over \mathbb{Q}_p .

PROOF. By Corollary 2.4, there is a finite index subgroup $A_1 \leq G$ and a finite normal subgroup $A_0 \subseteq A$, all \mathbb{Q}_p -definable, such that A_1/A_0 is \mathbb{Q}_p -definably isomorphic to an open subgroup of $H(\mathbb{K})$ for H some (connected) algebraic group defined over \mathbb{Q}_p . Let $f : A_1 \rightarrow H(\mathbb{K})$ be the finite-to-one homomorphism with $\ker(f) = A_0$ as in Fact 2.1. By Proposition 2.27, $\text{im}(f)$ has finite index in $H(\mathbb{K})$. Moreover H is trigonalizable over \mathbb{Q}_p by Proposition 2.22.

Clearly, the statement is trivial if $\dim(G) = 1$. Now assume that $\dim(G) > 1$. As we showed in Proposition 2.27, there is a short exact sequence

$$1 \rightarrow A \xrightarrow{i} H \xrightarrow{\pi} C \rightarrow 1$$

with A a one-dimensional algebraic group central in H , and C an algebraic group of dimension $\dim(H) - 1$, all defined over \mathbb{Q}_p . Both $A(\mathbb{K}) \cap \text{im}(f)$ and $\pi(\text{im}(f))$ are open dfg subgroups of $A(\mathbb{K})$ and $C(\mathbb{K})$, respectively. By Theorem 2.29, there is a normal sequence

$$\{\text{id}_C\} = D_0 \triangleleft \cdots \triangleleft D_i \triangleleft D_{i+1} \triangleleft \cdots \triangleleft D_n = \pi(\text{im}(f))$$

such that each D_{i+1}/D_i is definably isomorphic to a one-dimensional dfg group over \mathbb{Q}_p . Let $E_i = \pi^{-1}(D_i) \cap \text{im}(f)$. It is easy to see that $D_i = \pi(E_i)$, thus $E_i/(A(\mathbb{K}) \cap \text{im}(f)) \cong D_i$. So

$$E_{i+1}/E_i \cong (E_{i+1}/(A(\mathbb{K}) \cap \text{im}(f)))/(E_i/(A(\mathbb{K}) \cap \text{im}(f))) \cong D_{i+1}/D_i$$

is definably isomorphic to a one-dimensional dfg group over \mathbb{Q}_p . We conclude that

$$E_0 \triangleleft \cdots \triangleleft E_i \triangleleft E_{i+1} \triangleleft \cdots \triangleleft E_n = \text{im}(f)$$

is a normal sequence such that $E_0 = A(\mathbb{K}) \cap \text{im}(f)$ and each E_{i+1}/E_i is definably isomorphic to a one-dimensional dfg group over \mathbb{Q}_p . Let $G_0 = A_0$ and $G_{i+1} = f^{-1}(E_i)$, we see that the normal sequence

$$G_0 \triangleleft \cdots \triangleleft G_i \triangleleft G_{i+1} \triangleleft \cdots \triangleleft G_{n+1} = A_1$$

satisfies our requirement. ⊢

Finally we answer the Conjecture 1 in the p -adic case, making use of the following.

FACT 2.31 [37]. *Let H be a trigonalizable algebraic group over \mathbb{Q}_p . Then every global complete f -generic type of $H(\mathbb{K})$ is almost periodic.*

By Proposition 2.27 and Fact 2.31, we easily conclude the following.

THEOREM 2.32. *Suppose that G is a dfg group over \mathbb{Q}_p . Then every global complete f -generic type of G is almost periodic.*

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