ASYMPTOTIC PROPERTIES OF LEAST-SQUARES ESTIMATES OF PARAMETERS OF THE SPECTRUM OF A STATIONARY NON-DETERMINISTIC TIME-SERIES

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1. Introduction

Let $\{x_t\}$ $(t = 0, \pm 1, \pm 2 \cdots)$ be a stationary non-deterministic time series with $E(x_t^2) < \infty$, $E(x_t) = 0$, and let its spectrum be continuous (strictly absolutely continuous) so that the spectral distribution function $F(\omega) = \int_{-\pi}^{\omega} f(\lambda) d\lambda$ $(-\pi \leqq \omega \leqq \pi)$, where $f(\omega)$ is the spectral density function. It is well known that $\{x_t\}$ then has a unique one-sided moving-average representation

$$x_t = \sum_{u=0}^{\infty} g_u \varepsilon_{t-u} \tag{1}$$

with $g_0 = 1$,

$$\sum_{u=0}^{\infty} g_u^2 < \infty,$$

and

$$E(\varepsilon_t) = 0,\ E(\varepsilon_t \varepsilon_u) = 0\ \ (t \neq u),\ E(\varepsilon_t^2) = \sigma^2 > 0$$

where

$$\sigma^2 = 2\pi \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\omega) d\omega\right\}. \tag{2}$$

(See, for example, Hannan, [7], pp. 21–2, Grenander and Rosenblatt, [6], pp. 67–76). The coefficients g_u are determined from $f(\omega)$ by

$$\sum_{u=0}^{\infty} g_u z^u = \exp\left(\sum_{u=1}^{\infty} a_u z^u\right) \tag{3}$$

where

$$a_u = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iu\omega} \log f(\omega) d\omega,$$

(3) defining a function of the complex variable z which is analytic and never zero in the interior of the unit circle $|z| < 1$. Conversely (1) gives

$$f(\omega) = (\sigma^2/2\pi)\left|\sum_{u=0}^{\infty} g_u e^{i\omega u}\right|^2. \tag{4}$$

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Suppose now that the spectral density is a specified function of a number of unknown parameters, and that we wish to obtain estimates of these from data consisting of part of a realisation of the series, which, as usual, will be assumed to be n consecutive observations, denoted by $x_1, x_2, \cdots, x_n$. We shall take the g_u to be functions of p independent parameters, $\theta_1, \theta_2, \cdots, \theta_p$, which may be called *structural* parameters, the set of parameters being completed by the addition of σ^2. Thus, writing $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)$, we have $|\sum_{u=0}^{\infty} g_u e^{i\omega u}|^2 = g(\omega, \boldsymbol{\theta})$, where g is a specified function, and then, from (4), the spectral density $f = (\sigma^2/2\pi)g(\omega, \boldsymbol{\theta})$. The term "structural parameter" is used because in most applications the series is *defined* by a linear model which can be put in the form (1), with ε_t representing the effect of a disturbance at time t, for example, a linear autoregressive process of order p given by the stationary solution of the set of difference equations $x_t + \sum_{u=1}^{p} \theta_u x_{t-u} = \varepsilon_t$, where the ε_t are independently and identically distributed and the roots of the equation $z^p + \sum_{u=1}^{p} z^{p-u}\theta_u = 0$ have moduli less than unity. The parameter σ^2, on the other hand, can be thought of as a scale factor, being proportional to the variance of x_t (that is, to the total power $F(\pi)$ in the whole frequency range of the spectrum); it is also equal to the minimum mean square error that can be achieved when x_{t+1} is predicted by a linear combination of observations x_u $(u \leqq t)$. Clearly if we wish to estimate the standardised spectral density $f(\omega)/\int_{-\pi}^{\pi} f(\lambda)\mathrm{d}\lambda$ giving the relative distribution of power over $[-\pi, \pi]$, we require only estimates of the structural parameters.

Let us first assume that $\{x_t\}$ is a normal process, namely that the joint distribution of any finite set of the x_t is normal; this is equivalent to taking the ε_t to be independently and identically distributed as $N(0, \sigma^2)$. The logarithm of the likelihood of the data $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)'$ can then be written

$$(5) \qquad L_n(\boldsymbol{\theta}, \sigma^2) = -\tfrac{1}{2}\log|V_n(\boldsymbol{\theta})| - \tfrac{1}{2}n\log 2\pi\sigma^2 - Q_n(\boldsymbol{x}, \boldsymbol{\theta})/2\sigma^2$$

where, if

$$v_{rs}(\boldsymbol{\theta}) = E(x_r x_s)/\sigma^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(r-s)} g(\omega, \boldsymbol{\theta})\mathrm{d}\omega,$$

$$V_n(\boldsymbol{\theta}) = (v_{rs}(\boldsymbol{\theta})) \qquad (1 \leqq r,\ s \leqq n), \qquad \text{and}$$

$$Q_n(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}'\{V_n(\boldsymbol{\theta})\}^{-1}\boldsymbol{x}.$$

It would be natural to consider estimation of $\boldsymbol{\theta}$ and σ^2 by the method of maximum likelihood, but this is awkward because of the term $-\frac{1}{2}\log|V_n(\boldsymbol{\theta})|$. However, Whittle ([12], Chapter 7; [13]) showed that $\lim_{n\to\infty}|V_n(\boldsymbol{\theta})| = 1$ (at least if $\sum_{u=0}^{\infty} g_u(\boldsymbol{\theta})z^u$ is analytic and never zero in $|z| < 1+\delta$, $\delta > 0$), so that when n is fairly large it should make little difference if instead the expression $-\frac{1}{2}n\log 2\pi\sigma^2 - Q_n(\boldsymbol{x}, \boldsymbol{\theta})/2\sigma^2$ is maximised.

Doing this gives us estimates $\hat{\boldsymbol{\theta}}_n = (\hat{\theta}_{n,1}, \hat{\theta}_{n,2}, \cdots \hat{\theta}_{n,p})$ which minimise $Q_n(\boldsymbol{x}, \boldsymbol{\theta})$ and therefore may be called least-squares estimates, Q_n being equal to a sum of squares of uncorrelated random variables, and then $\hat{\sigma}_n^2 = Q_n(\boldsymbol{x}, \hat{\boldsymbol{\theta}}_n)/n$. In most problems there will remain the difficulty of finding explicit expressions for the elements of $\{V_n(\boldsymbol{\theta})\}^{-1}$, but Whittle showed that this could also be avoided when n is large by replacing Q_n by

$$(6)\qquad U_n(\boldsymbol{x}, \boldsymbol{\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{t=1}^{n} x_t e^{i\omega t}\right|^2 \{g(\omega, \boldsymbol{\theta})\}^{-1} d\omega = n \sum_{s=-(n-1)}^{n-1} \alpha_s(\boldsymbol{\theta}) C_s$$

where $C_s = \sum_{t=1}^{n-|s|} x_t x_{t+|s|}/n$ is the sample serial covariance (with divisor n) for lag $|s|$, and

$$(7)\qquad \alpha_s(\boldsymbol{\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega s}\{g(\omega, \boldsymbol{\theta})\}^{-1} d\omega$$

since clearly

$$\left|\sum_{t=1}^{n} x_t e^{i\omega t}\right|^2 = n \sum_{s=-(n-1)}^{n-1} e^{i\omega s} C_s.$$

Whittle went on to investigate the asymptotic properties ($n \to \infty$) of the estimates obtained in this way, and established that under suitable regularity conditions on $g(\omega, \boldsymbol{\theta})$, these were what would be expected by analogy with the classical asymptotic theory of maximum likelihood estimation for data consisting of independent identically distributed observations. Thus

(i) $\hat{\boldsymbol{\theta}}_n$ is a (weakly) consistent estimate of $\boldsymbol{\theta}$, namely $p \lim_{n\to\infty} \hat{\boldsymbol{\theta}}_n = \boldsymbol{\theta}$;

(ii) $n^{\frac{1}{2}}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ has, as $n \to \infty$, a limiting distribution which is $N(\boldsymbol{0}, W^{-1})$, namely multinormal with mean $\boldsymbol{0}$ and covariance matrix equal to the inverse of the matrix $W = (w_{ij})$ $(1 \leqq i, j \leqq p)$, where

$$w_{ij} = \lim_{n\to\infty} n^{-1} E\left(-\frac{\partial^2 L_n^*}{\partial\theta_i \partial\theta_j}\right) = \lim_{n\to\infty} n^{-1} E\left(\frac{\partial L_n^*}{\partial\theta_i}\frac{\partial L_n^*}{\partial\theta_j}\right),$$

$L_n^* = -\frac{1}{2}n \log 2\pi\sigma^2 - U_n(\boldsymbol{x}, \boldsymbol{\theta})/2\sigma^2$ being the modified log-likelihood which is maximised, whence it follows that

$$(8)\qquad w_{ij} = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\partial \log g(\omega, \boldsymbol{\theta})}{\partial\theta_i}\,\frac{\partial \log g(\omega, \boldsymbol{\theta})}{\partial\theta_j}\, d\omega$$

and

(i′) $\displaystyle p \lim_{n\to\infty} \hat{\sigma}_n^2 = \sigma^2$;

(ii′) $n^{\frac{1}{2}}(\hat{\sigma}_n^2 - \sigma^2)$ has, as $n \to \infty$, a limiting distribution which is $N(0, 2\sigma^4)$, and in the limit $n^{\frac{1}{2}}(\hat{\sigma}_n^2 - \sigma^2)$ and $n^{\frac{1}{2}}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})$ are distributed independently.

(See Whittle, [14]; [15], pp. 211—5, Hannan, [7], pp. 46—7).

Now suppose that the normality assumption does not hold. The above method of estimation, although no longer effectively that of maximum likelihood when n is large, is still intuitively reasonable. $Q_n(x, \theta)$ remains a sum of squares of uncorrelated random variables which in fact is approximately equal to the sum of squares of "residuals (see Whittle, [15], p. 207; the argument given there is heuristic, but can certainly be shown to be valid when $\sum_{u=0}^{\infty} g_u(\theta)z^u$ is analytic and never zero in $|z| < 1+\delta$, the representation (1) then being invertible to give $\varepsilon_t = \sum_{u=0}^{\infty} h_u x_{t-u}$ with the coefficients h_u tending to zero like some power of e^{-u} when $u \to \infty$). We might therefore expect that the estimates will still possess most of the asymptotic properties (i), (ii), (i′), (ii′) under wide conditions. Whittle found that this is so, the assumption that the ε_t are independently and identically distributed with $E(\varepsilon_t^4) < \infty$ being sufficient to ensure that all these asymptotic properties hold, with the exception of the formula for the asymptotic variance of $\hat{\sigma}_n^2$ which will depend on the value of $E(\varepsilon_t^4)$ (see Whittle, [15], p. 215). Indeed it is enough for $\{x_t\}$ to be a *linear* process with finite fourth moment, namely that

$$x_t = \sum_{u=0}^{\infty} l_u \eta_{t-u} \tag{9}$$

where the η_t are independently and identically distributed with $E(\eta_t) = 0$, $0 < E(\eta_t^4) < \infty$, and $\sum_{u=0}^{\infty} l_u^2 < \infty$ (compare Bartlett, [2], p. 146). (It should be noted that taking $l_0 = 1$ does not necessarily make (9) identical with the representation (1) because the function $\sum_{u=0}^{\infty} l_u z^u$ might have zeros in $|z| < 1$; for example this would happen when $x_t = \eta_t + 2\eta_{t-1}$).

The arguments presented by Whittle in obtaining these results are on the whole heuristic, although extremely interesting and ingenious (compare Hannan, [7], p. 46, footnote). In this paper we provide rigorous proofs under fairly general conditions. Whittle ([15], p. 213) does in fact indicate that methods similar to those used here will yield rigorous proofs, but he gives no details. As we shall see, there are many points in the arguments that require careful treatment, although the main ideas are quite simple.

The following assumptions will be made throughout.

(1) $\{x_t\}$ is a linear process with finite fourth moment defined by (9), and $E(\eta_t^2) = 1$ (which clearly involves no loss of generality).

(2) The estimates $\hat{\theta}_n$, $\hat{\sigma}_n^2$ are such that

$L_n^*(\theta, \sigma^2) = -\frac{1}{2}n \log 2\pi\sigma^2 - U_n(x, \theta)/2\sigma^2$ is an absolute maximum when $\theta = \hat{\theta}_n$, $\sigma^2 = \hat{\sigma}_n^2$. (We do not exclude the possibility that there is more than one set of estimates maximising L_n^*, although this will be extremely unlikely).

(3) The true values $\boldsymbol{\theta}$, σ^2 of the parameters lie in a region defined by $0 < \sigma^2 < \infty$, $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is a bounded closed set contained in an open set S in p-dimensional Euclidean space.

(4) If $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$ are any two points of $\boldsymbol{\Theta}$, $g(\omega, \boldsymbol{\theta}_1)$ and $g(\omega, \boldsymbol{\theta}_2)$ are not equal almost everywhere (ω). (This ensures that two different sets of parameter values cannot give the same spectral distribution).

(5) If $\boldsymbol{\theta} \in S$, $g(\omega, \boldsymbol{\theta})$ and $\{g(\omega, \boldsymbol{\theta})\}^{-1}$ are continuous functions of ω for $-\pi \leqq \omega \leqq \pi$.

2. Consistency of $\hat{\boldsymbol{\theta}}_n$

We first establish the following lemma.

LEMMA 1. *Let $\boldsymbol{\theta}_0$ be the true value of $\boldsymbol{\theta}$ and let $\boldsymbol{\theta}^*$ be any other point of $\boldsymbol{\Theta}$. Then there is a positive constant $K(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)$ such that*

$$\lim_{n\to\infty} p\{n^{-1}[U_n(\boldsymbol{\theta}_0)-U_n(\boldsymbol{\theta}^*)] < -K(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)\} = 1 \tag{10}$$

[From now on we write $U_n(\boldsymbol{\theta})$ for $U_n(\boldsymbol{x}, \boldsymbol{\theta})$].

PROOF. Write

$$Y_n = n^{-1}[U_n(\boldsymbol{\theta}_0)-U_n(\boldsymbol{\theta}^*)] = \sum_{|s| \leqq n-1} C_s\{\alpha_s(\boldsymbol{\theta}_0)-\alpha_s(\boldsymbol{\theta}^*)\},$$

from (6).

Then

$$E(Y_n) = \sum_{|s| \leqq n-1} \gamma_s^{(0)} \left(1-\frac{|s|}{n}\right) \{\alpha_s(\boldsymbol{\theta}_0)-\alpha_s(\boldsymbol{\theta}^*)\},$$

where $\gamma_s^{(0)} = E(x_t x_{t+s} | \boldsymbol{\theta} = \boldsymbol{\theta}_0, \sigma^2 = \sigma_0^2)$, σ_0^2 denoting the true value of σ^2. Now by Parseval's formula we have

$$\sum_{s=-\infty}^{\infty} \gamma_s^{(0)}\alpha_s(\boldsymbol{\theta}_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (2\pi f_0/g_0)\mathrm{d}\omega = \sigma_0^2$$

and

$$\sum_{s=-\infty}^{\infty} \gamma_s^{(0)}\alpha_s(\boldsymbol{\theta}^*) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (2\pi f_0/g^*)\mathrm{d}\omega = \frac{\sigma_0^2}{2\pi}\int_{-\pi}^{\pi} (g_0/g^*)\mathrm{d}\omega,$$

writing f_0, g_0, g^* respectively for $f(\omega, \boldsymbol{\theta}_0)$, $g(\omega, \boldsymbol{\theta}_0)$, $g(\omega, \boldsymbol{\theta}^*)$.

Hence

$$\begin{aligned}\lim_{n\to\infty} E(Y_n) &= \sum_{s=-\infty}^{\infty} \gamma_s^{(0)}\{\alpha_s(\boldsymbol{\theta}_0)-\alpha_s(\boldsymbol{\theta}^*)\} \\ &= \sigma_0^2\left\{1-\frac{1}{2\pi}\int_{-\pi}^{\pi}(g_0/g^*)\mathrm{d}\omega\right\}.\end{aligned} \tag{11}$$

Now equation (2) may be rewritten

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$$\log(\sigma^2/2\pi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\{\sigma^2 g(\omega)/2\pi\}d\omega$$

i.e.

(2a)
$$\int_{-\pi}^{\pi} \log g(\omega)d\omega = 0.$$

Therefore

(12)
$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log(g_0/g^*)d\omega < \log\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}(g_0/g^*)d\omega\right\},$$

since the logarithmic function is concave, and by assumption (4), g_0, g^* are not equal almost everywhere. From (11) and (12),

$$\lim_{n\to\infty} E(Y_n) = -\mu(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) \text{ say, where } \mu(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*) > 0.$$

Also, using the alternative formula for $U_n(\boldsymbol{\theta})$ given in (6), we have

(13)
$$\begin{aligned}\operatorname{var} Y_n &= \operatorname{var}\left\{\frac{1}{2\pi n}\int_{-\pi}^{\pi}\left|\sum_{t=1}^{n} x_t e^{i\omega t}\right|^2 (g_0^{-1}-(g^*)^{-1})d\omega\right\}\\ &= \operatorname{var}\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} I_n(\omega)(g_0^{-1}-(g^*)^{-1})\,d\omega\right\},\end{aligned}$$

where $I_n(\omega) = (2/n)\,|\sum_{t=1}^{n} x_t e^{i\omega t}|^2$ denotes the usual periodogram intensity for frequency ω. Now there is a theorem due to Grenander and Rosenblatt ([5], p. 541; [6], p. 137) which states that when $\{x_t\}$ satisfies assumption (1), i.e. is a linear process with finite fourth moment, and has a continuous spectral density $f(\omega)$, then

(14)
$$\begin{aligned}&\lim_{n\to\infty} n \operatorname{var}\left\{\int_{-\pi}^{\pi} I_n(\omega)W(\omega)d\omega\right\}\\ &\qquad = 16\pi^2\left[4\pi\int_{-\pi}^{\pi} f^2(\omega)W^2(\omega)d\omega + \kappa_4\left\{\int_{-\pi}^{\pi} f(\omega)W(\omega)d\omega\right\}^2\right]\end{aligned}$$

where $W(\omega)$ is any bounded even function of ω with at most a finite number of discontinuities, κ_4 denoting the fourth cumulant of η_t. Applying this theorem to (13) we see that $\lim_{n\to\infty} \operatorname{var} Y_n = 0$. The result (10) then follows by a simple application of Chebyshev's inequality; $K(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)$ can be any constant less than $\mu(\boldsymbol{\theta}_0, \boldsymbol{\theta}^*)$.

From Lemma 1 it is easily seen that $\hat{\boldsymbol{\theta}}_n$ is consistent when $\boldsymbol{\Theta}$ is a finite set. To obtain consistency when $\boldsymbol{\Theta}$ is an arbitrary bounded closed set we require $U_n(\boldsymbol{\theta})$ to satisfy a suitable continuity condition.

LEMMA 2. *Let*

(15)
$$|n^{-1}[U_n(\boldsymbol{\theta}_2) - U_n(\boldsymbol{\theta}_1)]| < H_{\delta,n}(\boldsymbol{x}, \boldsymbol{\theta}_1)$$

for all $\theta_1 \in \Theta$, $\theta_2 \in S$ *such that* $|\theta_2 - \theta_1| < \delta$ (δ *possibly depending on* θ_1), *where*

(16) $$\lim_{\delta \to 0} E(H_{\delta,n}) = 0 \text{ uniformly in } n,$$

and

(17) $$\lim_{n\to\infty} \operatorname{var}(H_{\delta,n}) = 0 \text{ for each } \delta.$$

Then $p \lim_{n\to\infty} \hat{\theta}_n = \theta_0$, *the true value of* θ.

The form of this condition was suggested by the work of Wald ([10]) and Wolfowitz([16]) on consistency of maximum likelihood estimates when the observations are independently and identically distributed (compare also Silvey, [8], pp. 446—7).

PROOF. Suppose $\theta_1 \neq \theta_0$. Then by Lemma 1,

(18) $$\lim_{n\to\infty} p\,\{n^{-1}[U_n(\theta_0) - U_n(\theta_1)] < -K(\theta_0, \theta_1)\} = 1.$$

Also by (16) and (17) we have

(19) $$\lim_{n\to\infty} p\,\{H_{\delta,n}(x, \theta_1) < K(\theta_0, \theta_1)\} = 1,$$

since we can first choose δ so that $E\{H_{\delta,n}(x, \theta_1)\} \leqq \frac{1}{2}K(\theta_0, \theta_1)$ and then let $n \to \infty$, using Chebyshev's inequality. Hence, since simultaneous occurrence of the events in (18) and (19) implies

$$n^{-1}[U_n(\theta_0) - U_n(\theta_2)] < 0 \text{ for } |\theta_2 - \theta_1| < \delta_1 \text{ (say)},$$

we have

(20) $$\lim_{n\to\infty} p\{\sup_{\theta_2 \in N(\theta_1)} [U_n(\theta_0) - U_n(\theta_2)] < 0\} = 1,$$

where $N(\theta_1)$ denotes the set $\{\theta : |\theta - \theta_1| < \delta_1\}$.

Now the collection of sets $\{N(\theta_1) : \theta_1 \neq \theta_0\}$, obtained by letting θ_1 run through all points of Θ except θ, and the set $N(\theta_0) = \{\theta : |\theta - \theta_0| < \delta_0\}$, where δ_0 is arbitrary, together constitute an open covering of Θ. Since Θ is a compact set, this contains a finite open covering, $\{N(\theta_j);\ 0 \leqq j \leqq m\}$, say. (22) then gives

$$\lim_{n\to\infty} p\,\{\sup_{\theta_2 \in \bigcup_{j=1}^{m} N(\theta_j)} [U_n(\theta_0) - U_n(\theta_2)] < 0\} = 1,$$

or

$$\lim_{n\to\infty} p\,\{\inf_{\theta \in \Theta} U_n(\theta) = \inf_{\theta \in N(\theta_0)} U_n(\theta)\} = 1,$$

so that

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(21) $$\lim_{n\to\infty} p\,\{|\hat{\boldsymbol{\theta}}_n-\boldsymbol{\theta}_0| < \delta_0\} = 1.$$

Thus since δ_0 can be arbitrarily small, $p \lim_{n\to\infty} \hat{\boldsymbol{\theta}}_n = \boldsymbol{\theta}_0$.

We now show that continuity of the partial derivatives of $h(\omega, \boldsymbol{\theta}) = [g(\omega, \boldsymbol{\theta})]^{-1}$ with respect to the components of $\boldsymbol{\theta}$ implies the above continuity condition and so is sufficient for $\hat{\boldsymbol{\theta}}_n$ to be consistent.

THEOREM 1. *Let* $h^{(i)}(\omega, \boldsymbol{\theta}) = \big(\partial h(\omega, \boldsymbol{\theta})/\partial\theta_i\big)$ $(1 \leqq i \leqq p)$ *be continuous functions of* $(\omega, \boldsymbol{\theta})$ *for* $-\pi \leqq \omega \leqq \pi$, $\boldsymbol{\theta} \in S$. *Then* $p \lim_{n\to\infty} \hat{\boldsymbol{\theta}}_n = \boldsymbol{\theta}_0$.

PROOF. We have

$$n^{-1}[U_n(\boldsymbol{\theta}_2)-U_n(\boldsymbol{\theta}_1)] = \frac{1}{4\pi}\int_{-\pi}^{\pi} I_n(\omega)\{h(\omega, \boldsymbol{\theta}_2)-h(\omega, \boldsymbol{\theta}_1)\}\mathrm{d}\omega,$$

and by the mean value theorem,

$$h(\omega, \boldsymbol{\theta}_2)-h(\omega, \boldsymbol{\theta}_1) = \sum_{i=1}^{p} (\theta_{2,i}-\theta_{1,i})h^{(i)}(\omega, \lambda\boldsymbol{\theta}_1+(1-\lambda)\boldsymbol{\theta}_2) \quad (0 < \lambda < 1)$$

so that if $|\boldsymbol{\theta}_2-\boldsymbol{\theta}_1| < \delta$,

$$|h(\omega, \boldsymbol{\theta}_2)-h(\omega, \boldsymbol{\theta}_1)| \leqq \delta \sum_{i=1}^{p} |h^{(i)}(\omega, \lambda\boldsymbol{\theta}_1+(1-\lambda)\boldsymbol{\theta}_2)|.$$

Hence

(22) $$|n^{-1}[U_n(\boldsymbol{\theta}_2)-U_n(\boldsymbol{\theta}_1)]| \leqq (\delta/4\pi) \sum_{i=1}^{p} M_i(\boldsymbol{\theta}_1) \int_{-\pi}^{\pi} I_n(\omega)\mathrm{d}\omega,$$

where $M_i(\boldsymbol{\theta}_1) < \infty$ is the supremum of $|h^{(i)}(\boldsymbol{\theta})|$ over $-\pi \leqq \omega \leqq \pi$, $|\boldsymbol{\theta}-\boldsymbol{\theta}_1| \leqq \delta(\boldsymbol{\theta}_1)$, $\delta(\boldsymbol{\theta}_1) > \delta$ being chosen so that the set $\{\boldsymbol{\theta} : |\boldsymbol{\theta}-\boldsymbol{\theta}_1| \leqq \delta(\boldsymbol{\theta}_1)\}$ is contained in S.

Now

$$\int_{-\pi}^{\pi} I_n(\omega)\mathrm{d}\omega = \int_{-\pi}^{\pi} 2 \sum_{|s| \leqq n-1} C_s e^{i\omega s}\mathrm{d}\omega = 4\pi C_0,$$

$E(C_0) = E(x_t^2) = \gamma_0^{(0)}$, and

$$\lim_{n\to\infty} n \operatorname{var} C_0 = 2 \sum_{v=-\infty}^{\infty} \{\gamma_v^{(0)}\}^2+\kappa_4^{(0)}\{\gamma_0^{(0)}\}^2,$$

$\kappa_4^{(0)}$ denoting the true value of the fourth cumulant of η_t (see, for example, Hannan, [7], p. 40).

Hence if the right-hand side of (22) is defined to be $H_{\delta,n}(\boldsymbol{x}, \boldsymbol{\theta}_1)$, this will satisfy the conditions (16) and (17). The result then follows from Lemma 2.

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3. Asymptotic normality of $\hat{\theta}_n$

Having established the consistency of $\hat{\theta}_n$, we can go on to obtain the limiting distribution of $n^{\frac{1}{2}}(\hat{\theta}_n - \theta_0)$ in the usual way by applying the mean value theorem to $U_n^{(i)}(\hat{\theta}_n) - U_n^{(i)}(\theta_0)$ $(1 \leqq i \leqq p)$, where $U_n^{(i)}$ denotes the partial derivative $\partial U_n / \partial \theta_i$ and θ_0, as in § 2, is the true value of θ. Here further conditions must be imposed on $h(\omega, \theta)$ to ensure that the second order partial derivatives $\partial^2 U_n / \partial \theta_i \partial \theta_j$ satisfy a suitable continuity condition and that a central limit theorem can be applied to give the limiting joint distribution of $n^{-\frac{1}{2}} U_n^{(i)}(\theta_0)$ $(1 \leqq i \leqq p)$.

THEOREM 2. *Let* $h^{(i)}(\omega, \theta)$ $(1 \leqq i \leqq p)$ *be continuous in* (ω, θ) *for* $-\pi \leqq \omega \leqq \pi$, $\theta \in S$, and $h^{(ij)}(\omega, \theta) = \partial^2 h / \partial \theta_i \partial \theta_j$, $h^{(ijk)}(\omega, \theta) = \partial^3 h / \partial \theta_i \partial \theta_j \partial \theta_k$ $(1 \leqq i, j, k \leqq p)$ be continuous in (ω, θ) *for* $-\pi \leqq \omega \leqq \pi$, $\theta \in N_\delta(\theta_0)$, *where* $N_\delta(\theta_0) = \{\theta : |\theta - \theta_0| < \delta\}$ *is some neighbourhood of* θ_0. *Also let*

$$\sum_{u=0}^{\infty} u |l_u^{(0)}| < \infty \tag{23}$$

where the $l_u^{(0)}$ *are the values of the coefficients in the linear process* (9) *when* $\theta = \theta_0$, *and let the matrix* $W_0 = (w_{ij}^{(0)})$ $(1 \leqq i, j \leqq p)$ *with*

$$w_{ij}^{(0)} = \frac{1}{4\pi} \int_{-\pi}^{\pi} \{h^{(i)}(\omega, \theta_0) h^{(j)}(\omega, \theta_0) / h^2(\omega, \theta_0)\} d\omega$$
$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\frac{\partial \log g}{\partial \theta_i}\right)_0 \left(\frac{\partial \log g}{\partial \theta_j}\right)_0 d\omega$$

(using an obvious notation), be non-singular.

Then the limiting distribution of $n^{\frac{1}{2}}(\hat{\theta}_n - \theta_0)$ *when* $n \to \infty$ *is* $N(\mathbf{0}, W_0^{-1})$.

PROOF. Since from (7), $\alpha_s(\theta) = (1/2\pi) \int_{-\pi}^{\pi} e^{i\omega s} h(\omega, \theta) d\omega$, the partial derivative $\partial \alpha_s / \partial \theta_i = \alpha_s^{(i)}(\theta) = (1/2\pi) \int_{-\pi}^{\pi} e^{i\omega s} h^{(i)}(\omega, \theta) d\omega$ exists, the validity of differentiation under the integral sign being guaranteed by the fact that $h^{(i)}(\omega, \theta')$ is bounded for $-\pi \leqq \omega \leqq \pi$, $\theta' \in N(\theta)$, some neighbourhood of θ (compare Cramer, [3], p. 68). Similarly

$$\partial^2 \alpha_s / \partial \theta_i \partial \theta_j = \alpha_s^{(ij)}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega s} h^{(ij)}(\omega, \theta) d\omega$$

and

$$\partial^3 \alpha_s / \partial \theta_i \partial \theta_j \partial \theta_k = \alpha_s^{(ijk)}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega s} h^{(ijk)}(\omega, \theta) d\omega$$

exist when $\theta \in N_\delta(\theta_0)$. Hence all the corresponding partial derivatives $U_n^{(i)}(\theta)$, $U_n^{(ij)}(\theta)$, $U_n^{(ijk)}(\theta)$ of $U_n(\theta) = n \sum_{|s| \leqq n-1} \alpha_s(\theta) C_s$ exist when $\theta \in N_\delta(\theta_0)$.

Since $\hat{\theta}_n$ is consistent, by Theorem 1, we can obtain the limiting distribution on the assumption that $\hat{\boldsymbol{\theta}} \in N_\delta(\boldsymbol{\theta}_0)$. (For if we denote the event $\hat{\boldsymbol{\theta}}_n \in N_\delta(\boldsymbol{\theta})$ by B_n we have $\lim_{n\to\infty} p(B_n) = 1$, and it is easily seen that $\lim_{n\to\infty} p(A_n|B_n) = l$ for some sequence of events $\{A_n\}$ then implies that $\lim_{n\to\infty} p(A_n) = l$.) Clearly we must have $U_n^{(j)}(\hat{\boldsymbol{\theta}}_n) = 0$ $(1 \leqq j \leqq p)$, and so by the mean value theorem,

$$0 = U_n^{(j)}(\boldsymbol{\theta}_0) + \sum_{i=1}^{p} (\hat{\theta}_{n,i} - \theta_{0,i}) U_n^{(ij)}(\boldsymbol{\theta}_n^*),$$

where $\boldsymbol{\theta}_n^* = \lambda\hat{\boldsymbol{\theta}}_n + (1-\lambda)\boldsymbol{\theta}_0 \in N_\delta(\boldsymbol{\theta}_0)$. [Strictly we should write $\boldsymbol{\theta}_n^{*(j)}$ but the omission of the superfix will cause no ambiguity]. Hence

$$\sum_{i=1}^{p} \{-n^{1} U_n^{(ij)}(\hat{\boldsymbol{\theta}}_n^*)\} n^{-\frac{1}{2}} \{\hat{\theta}_{n,i} - \theta_{0,i}\} = n^{-\frac{1}{2}} U_n^{(j)}(\boldsymbol{\theta}_0). \tag{24}$$ [1]

We shall now show

(a) that $p \lim_{n\to\infty} \{n^{-1} U_n^{(ij)}(\boldsymbol{\theta}_n^*)\} = \lim_{n\to\infty} n^{-1} E\{U_n^{(ij)}(\boldsymbol{\theta}_0)\} = 2\sigma_0^2 w_{ij}^{(0)}$, and

(b) that the limiting distribution of $n^{-\frac{1}{2}} U_n^{(j)}(\boldsymbol{\theta}_0)$ $(1 \leqq j \leqq p)$ is $N(\mathbf{0}, 4\sigma_0^4 W_0)$.

The conclusion of the theorem will then follow at once from (24).

Proof of (a). We first show that

$$p \lim_{n\to\infty} n^{-1}[U_n^{(ij)}(\boldsymbol{\theta}_n^*) - U_n^{(ij)}(\boldsymbol{\theta}_0)] = 0 \tag{25}$$

For, proceeding as in the proof of Theorem 1, we have

$$\begin{aligned} |n^{-1}[U_n^{(ij)}(\boldsymbol{\theta}_n^*) - U_n^{(ij)}(\boldsymbol{\theta}_0)]| &= \left| \frac{1}{4\pi} \int_{-\pi}^{\pi} I_n(\omega) \{h^{(ij)}(\omega, \boldsymbol{\theta}_n^*) - h^{(ij)}(\omega, \boldsymbol{\theta}_0)\} d\omega \right| \\ &\leqq \{|\boldsymbol{\theta}_n^* - \boldsymbol{\theta}_0| M^{(ij)}(\boldsymbol{\theta}_0)/4\pi\} \int_{-\pi}^{\pi} I_n(\omega) d\omega \\ &= |\boldsymbol{\theta}_n^* - \boldsymbol{\theta}_0| M^{(ij)}(\boldsymbol{\theta}_0) C_0, \end{aligned} \tag{26}$$

where

$$M^{(ij)}(\boldsymbol{\theta}_0) = \sum_{k=1}^{p} \sup_{|\omega| \leqq \pi, \boldsymbol{\theta} \in N_\delta(\boldsymbol{\theta}_0)} |h^{(ijk)}(\omega, \boldsymbol{\theta})|,$$

and (26) clearly converges to zero in probability.

Next we have

$$p \lim_{n\to\infty} n^{-1}[U_n^{(ij)}(\boldsymbol{\theta}_0) - E\{U_n^{(ij)}(\boldsymbol{\theta}_0)\}] = 0, \tag{27}$$

since

$$\operatorname{var} \{n^{-1} U_n^{(ij)}(\boldsymbol{\theta}_0)\} = \operatorname{var} \left\{ \frac{1}{4\pi} \int_{-\pi}^{\pi} I_n(\omega) h^{(ij)}(\omega, \boldsymbol{\theta}_0) d\omega \right\}$$

tends to zero as $n \to \infty$ by Grenander and Rosenblatt's theorem (equation (14)).

[1] Expression (24) should read after correction

$$\sum_{i=1}^{p} \{-n^{-1} U_n^{(ij)}(\hat{\boldsymbol{\theta}}_n^*)\} n^{\frac{1}{2}} \{\hat{\theta}_{n,i} - \theta_{0,i}\} = n^{-\frac{1}{2}} U_n^{(j)}(\boldsymbol{\theta}_0).$$

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Finally

$$(28)\qquad \lim_{n\to\infty} n^{-1}E\{U_n^{(ij)}(\theta_0)\} = \lim_{n\to\infty} \sum_{|s|\leq n-1}\left(1-\frac{|s|}{n}\right)\gamma_s^{(0)}\alpha_s^{(ij)}(\theta_0) = \sum_{s=-\infty}^{\infty}\gamma_s^{(0)}\alpha_s^{(ij)}(\theta_0)$$

provided that this series converges. The $\alpha_s^{(ij)}(\theta_0)$ are Fourier coefficients of the function $h^{(ij)}(\omega, \theta_0)$ whose square is certainly integrable over $-\pi \leqq \omega \leqq \pi$, and Parseval's formula can therefore be applied, showing that (28) is equal to

$$(29)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} h^{(ij)}(\omega, \theta_0)2\pi f(\omega, \theta_0)\mathrm{d}\omega = \frac{\sigma_0^2}{2\pi}\int_{-\pi}^{\pi}(h_0^{(ij)}/h_0)\mathrm{d}\omega$$

(writing $h_0 = h(\omega, \theta_0)$, $h_0^{(ij)} = h^{(ij)}(\omega, \theta_0)$).

Now from equation (2a) we have

$$\int_{-\pi}^{\pi}\log h(\omega, \theta)\mathrm{d}\omega = 0,$$

and for $\theta \in N_\delta(\theta_0)$ this can be differentiated under the integral sign with respect to θ_j and then θ_i to give

$$(30)\qquad \int_{-\pi}^{\pi} h^{(j)}(\omega, \theta)g(\omega, \theta)\mathrm{d}\omega = 0,$$

and

$$\int_{-\pi}^{\pi}\{h^{(ij)}(\omega, \theta)g(\omega, \theta)+h^{(j)}(\omega, \theta)g^{(i)}(\omega, \theta)\}\mathrm{d}\omega = 0$$

or

$$(31)\qquad \int_{-\pi}^{\pi}\{(h^{(ij)}/h)-(h^{(i)}h^{(j)}/h^2)\}\mathrm{d}\omega = 0.$$

(29) is therefore equal to

$$\frac{\sigma_0^2}{2\pi}\int_{-\pi}^{\pi}(h_0^{(i)}h_0^{(j)}/h_0^2)\mathrm{d}\omega = 2\sigma_0^2\omega_{ij}^{(0)},$$

and the result follows from (25) and (27).

Proof of (b). Consider first the behaviour of a single component $n^{-\frac{1}{2}}U_n^{(j)}(\theta_0)$. We have

$$(32)\qquad E\{n^{-\frac{1}{2}}U_n^{(j)}(\theta_0)\} = n^{\frac{1}{2}}E\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}I_n(\omega)h_0^{(j)}\mathrm{d}\omega\right\}$$

Now Grenander and Rosenblatt ([5], p. 543) have also shown that in the notation of equation (14),

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$$E\left\{\int_{-\pi}^{\pi} I_n(\omega)W(\omega)\mathrm{d}\omega\right\} = 4\pi\int_{-\pi}^{\pi} f(\omega)W(\omega)\mathrm{d}\omega + O(\log n/n) \tag{33}$$

provided that the spectral density $f(\omega)$ has a derivative which is bounded for $-\pi \leqq \omega \leqq \pi$. Since

$$f(\omega, \boldsymbol{\theta}_0) = \frac{1}{2\pi}\sum_{s=-\infty}^{\infty} \gamma_s^{(0)} e^{i\omega s}, \tag{34}$$

with

$$\gamma_s^{(0)} = \sum_{u=0}^{\infty} l_u^{(0)} l_{u+s}^{(0)} \qquad (s \geqq 0)$$

(the convergence of $\sum_{s=-\infty}^{\infty} |\gamma_s^{(0)}|$ certainly ensuring the validity of the inversion formula (34)), and

$$\begin{aligned}\sum_{s=-\infty}^{\infty} |s\gamma_s^{(0)}| &= 2\sum_{s=1}^{\infty} s\left|\sum_{u=0}^{\infty} l_u^{(0)} l_{u+s}^{(0)}\right| \\ &\leqq 2\sum_{u=0}^{\infty} |l_u^{(0)}| \sum_{s=1}^{\infty} s\,|l_s^{(0)}| < \infty \qquad \text{by (23)},\end{aligned}$$

(34) can be differentiated with respect to ω under the summation sign, so that $\partial f(\omega, \boldsymbol{\theta}_0)/\partial\omega$ is bounded for $-\pi \leqq \omega \leqq \pi$. We can therefore use (33) in (32) to obtain

$$\lim_{n\to\infty} E\{n^{-}\ U_n^{(j)}(\boldsymbol{\theta}_0)\} = \lim_{n\to\infty} n^{\frac{1}{2}}(\sigma_0^2/2\pi)\int_{-\pi}^{\pi} g_0 h_0^{(j)}\mathrm{d}\omega = 0 \qquad \text{by (30)}.$$

Hence the limiting distribution of $n^{-\frac{1}{2}}U_n^{(j)}(\theta_0)$ is the same as that of

$$n^{-\frac{1}{2}}[U_n^{(j)}(\boldsymbol{\theta}_0) - E\{U_n^{(j)}(\boldsymbol{\theta}_0)\}] = n^{\frac{1}{2}}\sum_{|s|\leqq n-1} \alpha_s^{(j)}(\boldsymbol{\theta}_0)\{C_s - E(C_s)\}. \tag{35}$$

Now let

$$\begin{aligned}Z_{m,n} &= n^{\frac{1}{2}}\sum_{|s|\leqq m} \alpha_s^{(j)}(\boldsymbol{\theta}_0)\{C_s - E(C_s)\}, \\ R_{m,n} &= n^{\frac{1}{2}}\sum_{m<|s|\leqq n-1} \alpha_s^{(j)}(\boldsymbol{\theta}_0)\{C_s - E(C_s)\} \qquad (m < n-1),\end{aligned}$$

and let $n \to \infty$, m remaining fixed. The condition $\sum_{u=0}^{\infty} u|l_u^{(0)}| < \infty$ (equation (23)) certainly ensures that the limiting joint distribution of $n^{\frac{1}{2}}\{C_r - E(C_r)\}$ $(0 \leqq r \leqq m)$ is normal with mean $\mathbf{0}$ and covariance matrix $\Lambda^{(0)} = (\lambda_{rs}^{(0)})$, where

$$\begin{aligned}\lambda_{rs}^{(0)} &= \lim_{n\to\infty} n \operatorname{cov}(C_r, C_s) \\ &= \sum_{v=-\infty}^{\infty} \{\gamma_v^{(0)}\gamma_{v+r-s}^{(0)} + \gamma_v^{(0)}\gamma_{v+r+s}^{(0)}\} + \kappa_4^{(0)}\gamma_r^{(0)}\gamma_s^{(0)} \\ &= \lambda_{r-s}^{(0)} + \lambda_{r+s}^{(0)} + \kappa_4^{(0)}\gamma_r^{(0)}\gamma_s^{(0)},\end{aligned}$$

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$\sum_{v=-\infty}^{\infty} \gamma_v^{(0)}\gamma_{v+r}^{(0)}$ being denoted by $\lambda_r^{(0)}$ (see Walker [11], Hannan, [7], pp. 40—1).

Hence $Z_{m,n}$ has the limiting distribution $N(0, \sigma_{m,z}^2)$, where

$$\sigma_{m,z}^2 = \sum_{|r|,|s| \leq m} \alpha_r^{(j)}(\theta_0)\alpha_s^{(j)}(\theta_0)\{\lambda_{r-s}^{(0)}+\lambda_{r+s}^{(0)}+\kappa_4^{(0)}\gamma_r^{(0)}\gamma_s^{(0)}\} \tag{36}$$

(note that $\alpha_s^{(j)}(\theta_0) = \alpha_{-s}^{(j)}(\theta_0)$).

Now the double series

$$\sum_{r,s=-\infty}^{\infty} \alpha_r^{(j)}(\theta_0)\alpha_s^{(j)}(\theta_0)\lambda_{r-s}^{(0)} = \sum_{r,s=-\infty}^{\infty} \alpha_r^{(j)}(\theta_0)\alpha_s^{(j)}(\theta_0)\lambda_{r+s}^{(0)} \tag{37}$$

is absolutely convergent, since

$$\begin{aligned} \sum_{r,s=-\infty}^{\infty} |\alpha_r^{(j)}(\theta_0)\,\alpha_s^{(j)}(\theta_0)|\;|\lambda_{r-s}^{(0)}| &= \sum_{t=-\infty}^{\infty} |\lambda_t^{(0)}| \sum_{r=-\infty}^{\infty} |\alpha_r^{(j)}(\theta_0)\,a_{r+t}^{(j)}(\theta_0)| \\ &\leq \sum_{t=-\infty}^{\infty} |\lambda_t^{(0)}| \sum_{r=-\infty}^{\infty} |\alpha_r^{(j)}(\theta_0)|^2, \end{aligned}$$

and

$$\sum_{t=-\infty}^{\infty} |\lambda_t^{(0)}| \leq \Big(\sum_{v=-\infty}^{\infty} |\gamma_v|\Big)^2 < \infty,$$

$$\sum_{r=-\infty}^{\infty} |\alpha_r^{(j)}(\theta_0)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{h_0^{(j)}\}^2 \mathrm{d}\omega < \infty.$$

Hence (37) is equal to

$$\lim_{m\to\infty} \sum_{|r|\leq m} \alpha_r^{(j)}(\theta_0) \lim_{n\to\infty} \sum_{|s|\leq n} \alpha_s^{(j)}(\theta_0)\lambda_{r-s}^{(0)} = 2\pi\int_{-\pi}^{\pi} \{h_0^{(j)}\}^2 f_0^2 \mathrm{d}\omega,$$

by Parseval's formula (used three times). From (36) we therefore obtain

$$\lim_{m\to\infty} \sigma_{m,z}^2 = 4\pi\int_{-\pi}^{\pi} \{h_0^{(j)}\}^2 f_0^2 \mathrm{d}\omega + \kappa_4^{(0)} \left[\int_{-\pi}^{\pi} h_0^{(j)} f_0 \mathrm{d}\omega\right]^2,$$

the second term on the right-hand side resulting from yet another application of Parseval's formula. By (30) this term is zero, and so

$$\lim_{m\to\infty} \sigma_{m,z}^2 = (\sigma_0^4/\pi)\int_{-\pi}^{\pi} \{h_0^{(j)}/h_0\}^2 \mathrm{d}\omega = 4\sigma_0^4 w_{jj}^{(0)}. \tag{38}$$

Also, using the exact expression for cov (C_r, C_s) $(r, s \geqq 0)$, which is of the form

$$\begin{aligned} \operatorname{cov}(C_r, C_s) = n^{-1} \sum_{v=-n+s+1}^{n-r-1} &\left\{1 - \frac{\zeta(v, r, s)}{n}\right\} \\ &\cdot \left\{\gamma_v^{(0)}\gamma_{v+r-s}^{(0)} + \gamma_{v+r}^{(0)}\gamma_{v-s}^{(0)} + \kappa_4^{(0)} \sum_{u=0}^{\infty} l_u^{(0)} l_{u+v}^{(0)} l_{u+s}^{(0)} l_{u+v+r}^{(0)}\right\}, \end{aligned} \tag{39}$$

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where $|\zeta(v, r, s)| < n$ (compare Bartlett, [2], p. 255, Hannan, [7], p. 39), we have

$$
\begin{aligned}
nE\,|R_{m,n}|^2 \leq & \sum_{|r|,|s|=m+1}^{n-1} |\alpha_r^{(j)}(\theta_0)\alpha_s^{(j)}(\theta_0)| \\
& \sum_{v=-\infty}^{\infty}\left\{|\gamma_v^{(0)}\gamma_{v+r-s}^{(0)}|+|\gamma_{v+r}^{(0)}\gamma_{v-s}^{(0)}|+|\kappa_4^{(0)}|\sum_{u=0}^{\infty}|l_u^{(0)}l_{u+v}^{(0)}l_{u+s}^{(0)}l_{u+v+r}^{(0)}|\right\} \\
\leq & \; 2\sum_{|r|,|s|=m+1}^{n-1}|\alpha_r^{(j)}(\theta_0)\alpha^{(j)}(\theta_0)|\sum_{v=-\infty}^{\infty}|\gamma_v^{(0)}\gamma_{v+r-s}^{(0)}| \\
& +|\kappa_4^{(0)}|\left\{\sup_{|r|>m}|\alpha_r^{(j)}(\theta_0)|\right\}^2\left\{\sum_{u=0}^{\infty}|l_u^{(0)}|\right\}^4 .
\end{aligned} \tag{40}
$$

The first term in (40) is certainly not greater than

$$
\begin{aligned}
& \sum_{t=-2(n-1)}^{2(n-1)}\sum_{v=-\infty}^{\infty}|\gamma_v^{(0)}\gamma_{v+t}^{(0)}|\sum_{|r|=m}^{\infty}|\alpha_r^{(j)}(\theta_0)\alpha_{r+t}^{(j)}(\theta_0)| \\
& \qquad \leq \left\{\sum_{v=-\infty}^{\infty}|\gamma_v^{(0)}|\right\}^2\left\{\sum_{|r|=m}^{\infty}|\alpha_r^{(j)}(\theta_0)|^2\sum_{r=-\infty}^{\infty}|\alpha_r^{(j)}(\theta_0)|^2\right\}^{\frac{1}{2}}.
\end{aligned}
$$

It follows that

$$
E(R_{m,n}^2) \leq K_m \text{ (say) for all } n, \text{ where } \lim_{m\to\infty} K_m = 0. \tag{41}
$$

From (38) and (41) we deduce that $Z_{m,n}+R_{m,n}$ has the limiting distribution $N(0, 4\sigma_0^4 w_{jj}^{(0)})$. using a form of a limit theorem of P. H. Diananda given by T. W. Anderson ([1], p. 687, Theorem 4.5).[1] A similar argument can be used to show that the limiting distribution of $n^{-\frac{1}{2}}\sum_{j=1}^{p} k_j[U_n^{(j)}(\theta_0)-E\{U_n^{(j)}(\theta_0)\}]$, where the k_j are arbitrary constants, is $N(0, 4\sigma_0^4\sum_{i,j=1}^{p}k_i k_j w_{ij}^{(0)})$. It follows, by using the continuity theorem for characteristic functions (see, for example, Walker, [11], p. 64), that the limiting joint distribution of $n^{-\frac{1}{2}}[U_n^{(i)}(\theta_0)-E\{U_n^{(i)}(\theta_0)\}]$ $(1 \leq i \leq p)$ is $N(\mathbf{0}, 4\sigma_0^4 W_0)$, which establishes the required result.

4. Consistency and asymptotic normality of $\hat{\sigma}_n^2$

THEOREM 3. *Under the conditions of Theorem* 1, $p\lim_{n\to\infty}\hat{\sigma}_n^2 = \sigma_0^2$.

PROOF. We have $\hat{\sigma}_n^2 = n^{-1}U_n(\hat{\theta}_n)$, and since

$$
\begin{aligned}
|n^{-1}[U_n(\hat{\theta}_n)-U_n(\theta_0)]| &= \left|\frac{1}{4\pi}\int_{-\pi}^{\pi}\{h(\omega, \hat{\theta}_n)-h(\omega, \theta_0)\}I_n(\omega)\mathrm{d}\omega\right| \\
&\leq \{|\hat{\theta}_n-\theta_0|/4\pi\}\sum_{i=1}^{p}\sup_{|\omega|\leq\pi,\,\theta\in N_\delta(\theta_0)}|h^{(i)}(\omega, \theta_0)|\int_{-\pi}^{\pi}I_n(\omega)\mathrm{d}\omega
\end{aligned}
$$

when $\hat{\theta}_n \in N_\delta(\theta_0)$,

[1] As stated, this theorem requires $Z_{m,n}$, $R_{m,n}$ to be defined for all positive integers m, n; when $n-1 \leq m$ we can define $Z_{m,n}$ to be equal to (35) and $R_{m,n}$ to be zero.

$$p\lim_{n\to\infty} n^{-1}U_n(\hat{\boldsymbol{\theta}}_n) = p\lim_{n\to\infty} n^{-1}U_n(\boldsymbol{\theta}_0).$$

Also

$$\lim_{n\to\infty} E\{n^{-1}U_n(\boldsymbol{\theta}_0)\} = \lim_{n\to\infty} \sum_{s=-(n-1)}^{n-1} \gamma_s^{(0)} \left(1 - \frac{|s|}{n}\right) \alpha_s(\boldsymbol{\theta}_0)$$
$$= \sum_{s=-\infty}^{\infty} \gamma_s^{(0)} \alpha_s(\boldsymbol{\theta}_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi f_0 h_0 \mathrm{d}\omega = \sigma_0^2$$

by Parseval's formula, and var $\{n^{-1}U_n(\boldsymbol{\theta}_0)\}$ is of order n^{-1} by the theorem of Grenander and Rosenblatt used in § 2, so that $p\lim_{n\to\infty} n^{-1}U_n(\boldsymbol{\theta}_0) = \sigma_0^2$.

THEOREM 4. *Under the conditions of Theorem 2, the limiting distribution of* $n^{\frac{1}{2}}(\hat{\sigma}_n^2-\sigma_0^2)$ *when* $n\to\infty$ *is* $N(0, \sigma_0^4(2+\kappa_4^{(0)}))$, *and in the limit* $n^{\frac{1}{2}}(\hat{\sigma}_n^2-\sigma_0^2)$ *and* $n^{\frac{1}{2}}(\hat{\boldsymbol{\theta}}_n-\boldsymbol{\theta}_0)$ *are independent.*

PROOF. By Taylor's Theorem,

$$n^{-1}U_n(\boldsymbol{\theta}_0) = n^{-1}U_n(\hat{\boldsymbol{\theta}}_n)+\sum_{i,j=1}^{p} (\theta_{0,i}-\hat{\theta}_{n,i})(\theta_{0,j}-\hat{\theta}_{n,j})n^{-1}U_n^{(ij)}(\boldsymbol{\theta}_n^*) \tag{42}$$

where

$$\boldsymbol{\theta}_n^* = \lambda\boldsymbol{\theta}_0+(1-\lambda)\hat{\boldsymbol{\theta}}_n \qquad (0<\lambda<1).$$

From Theorem 2 the second term on the right-hand side of (42) is clearly of order $n^{-\frac{1}{2}}$ in probability, so that

$$n^{\frac{1}{2}}(\hat{\sigma}_n^2-\sigma_0^2) = n^{-\frac{1}{2}}U_n(\boldsymbol{\theta}_0)-n^{\frac{1}{2}}\sigma_0^2+O_p(n^{-\frac{1}{2}}).$$

Now by proceeding in the same way as in the proof of (b) in Theorem 2, it being in fact only necessary to replace $h_0^{(j)}$ by h_0 throughout the argument, we can show that

$$n^{-\frac{1}{2}}E\{U_n(\boldsymbol{\theta}_0)\} = (n^{\frac{1}{2}}\sigma_0^2/2\pi)\int_{-\pi}^{\pi} g_0 h_0 \mathrm{d}w+O(\log n/n^{\frac{1}{2}})$$
$$= n^{\frac{1}{2}}\sigma_0^2+o(1),$$

and that the limiting distribution of $n^{-\frac{1}{2}}[U_n(\boldsymbol{\theta}_0)-E\{U_n(\boldsymbol{\theta}_0)\}]$ is normal with mean 0 and variance equal to

$$4\pi\int_{-\pi}^{\pi} (h_0f_0)^2\mathrm{d}\omega+\kappa_4^{(0)}\left(\int_{-\pi}^{\pi} h_0f_0\mathrm{d}w\right)^2 = \sigma_0^4(2+\kappa_4^{(0)}).$$

Hence the limiting distribution of $n^{\frac{1}{2}}(\hat{\sigma}_n^2-\sigma_0^2)$ is $N(0, \sigma_0^4(2+\kappa_4^{(0)}))$.

Also a similar argument can be used to derive the limiting distribution of $n^{\frac{1}{2}}(\hat{\sigma}_n^2-\sigma_0^2)+n^{-\frac{1}{2}}\sum_{j=1}^{p} k_j[U_n^{(j)}(\boldsymbol{\theta}_0)-E_n^{(j)}\{U(\boldsymbol{\theta}_0)\}]$, where the k_j are arbitrary constants, this being normal with mean 0 and variance

$$4\pi\int_{-\pi}^{\pi}\left\{h_0+\sum_{j=1}^{p} k_jh_0^{(j)}\right\}^2 f_0^2\mathrm{d}\omega+\kappa_4^{(0)}\left[\int_{-\pi}^{\pi}\left\{h_0+\sum_{j=1}^{p} k_jh_0^{(j)}\right\} f_0\mathrm{d}\omega\right]^2$$

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which is easily seen to be equal to

$$4\sigma_0^4 \sum_{i,j=1}^{p} k_i k_j w_{ij}^{(0)} + \sigma_0^2(2+\kappa_4^{(0)})$$

It follows that in the limit, $n^{\frac{1}{2}}(\hat{\sigma}_n^2 - \sigma_0^2)$ is independent of

$$n^{-\frac{1}{2}}[U_n^{(j)}(\boldsymbol{\theta}_0) - E\{U_n^{(j)}(\boldsymbol{\theta}_0)\}]$$

and therefore, from (24), of $n^{\frac{1}{2}}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0)$.

5. Use of a truncated "sum of squares"

The coefficients α_s in the "sum of squares" $U_n(\boldsymbol{\theta})$ tend to zero as $|s| \to \infty$, at least as fast as $|s|^{-1}$ by the Riemann-Lebesgue Theorem (Titchmarsh, [9], p. 403), and typically like $e^{-k|s|}$ where $k > 0$. We should therefore expect the above results to remain true when a truncated "sum of squares" is used in L_n^*, that is, when the definition of U_n is altered to $U_n = n \sum_{|s| \leq \phi(n)} \alpha_s C_s$, where $\phi(n) < n-1$, but still $\to \infty$ with n, which requires less computation. In fact by imposing slightly stronger conditions we can then establish Theorems 1—4 in much the same way, the main changes in the argument arising from our no longer being able to use Grenander and Rosenblatt's theorems.

Thus in lemma 1 the proof of (11) will clearly not be affected by the altered definition of U_n. Also to show that $\lim_{n\to\infty} \operatorname{var} Y_n = 0$ we use the exact expression for cov (C_r, C_s) (equation (39)), but without requiring that $\sum_{v=-\infty}^{\infty} |\gamma_v^{(0)}|$ and $\sum_{u=0}^{\infty} |l_u^{(0)}|$ be finite, as in the derivation of (40). For, if we write for the moment $\beta_r = \alpha_r(\boldsymbol{\theta}_0) - \alpha_r(\boldsymbol{\theta}^*)$, we have

$$\begin{aligned}\operatorname{var} Y_n &\leqq n^{-1} \sum_{|r|,|s| \leqq \phi(n)}^{\infty} |\beta_r \beta_s| \\ &\quad \cdot \sum_{v=-\infty}^{\infty} \{|\gamma_v^{(0)} \gamma_{v+r-s}^{(0)}| + |\gamma_{v+r}^{(0)} \gamma_{v-s}^{(0)}| + |\kappa_4^{(0)}| \sum_{u=0}^{\infty} |l_u^{(0)} l_{u+s}^{(0)}|\, |l_{u+v}^{(0)} l_{u+v+r}^{(0)}|\} \\ &\leqq n^{-1} \left(\sum_{|r| \leqq \phi(n)} |\beta_r| \right)^2 \left\{ 2 \sum_{v=-\infty}^{\infty} (\gamma_v^{(0)})^2 + |\kappa_4^{(0)}| \left(\sum_{u=0}^{\infty} \{l_u^{(0)}\}^2 \right)^2 \right\}\end{aligned} \tag{43}$$

by Schwartz's inequality, and this tends to zero as $n \to \infty$, since β_r is $O(|r|^{-1})$ so that $\sum_{|r| \leqq \phi(n)} |\beta_r|$ is $O(\log \phi(n))$.

Next, in the proof of Theorem 1, we use the formula

$$n^{-1} U_n(\boldsymbol{\theta}) = \frac{1}{4\pi} \int_{-\pi}^{\pi} I_n(\omega) \sum_{|s| \leqq \phi(n)} \alpha_s(\boldsymbol{\theta}) e^{-i\omega s} d\omega$$

which gives

$$n^{-1}[U_n(\boldsymbol{\theta}_2)-U_n(\boldsymbol{\theta}_1)] = \frac{1}{4\pi}\int_{-\pi}^{\pi} I_n(\omega) \sum_{|s|\leqq\phi(n)} \sum_{j=1}^{p} (\theta_{2,j}-\theta_{1,j})\alpha_s^{(j)}(\boldsymbol{\theta}^*)\mathrm{d}\omega$$

where

$$\boldsymbol{\theta}^* = \lambda\boldsymbol{\theta}_1+(1-\lambda)\boldsymbol{\theta}_2,$$

when the derivatives $\alpha_s^{(j)}(\boldsymbol{\theta})$ exist and are continuous in $\boldsymbol{\theta}$ for $\boldsymbol{\theta}\in S$. Hence, when $|\boldsymbol{\theta}_2-\boldsymbol{\theta}_1|<\delta$,

$$(44) \qquad |n^{-1}[U_n(\boldsymbol{\theta}_2)-U_n(\boldsymbol{\theta}_1)]| < (\delta/4\pi) \sum_{|s|\leqq\phi(n)} \sum_{j=1}^{p} |\alpha_s^{(j)}(\boldsymbol{\theta}^*)| \int_{-\pi}^{\pi} I_n(\omega)\mathrm{d}\omega.$$

It follows that Theorem 1 will certainly be true when for $\boldsymbol{\theta}\in S$,

$$(45) \qquad \alpha_s^{(j)}(\boldsymbol{\theta}) \text{ is continuous in } \boldsymbol{\theta} \text{ and } \sum_{s=-\infty}^{\infty} |\alpha_s^{(j)}(\boldsymbol{\theta})| \text{ is bounded } (1\leqq j\leqq p).$$

In the same way, it is easily seen that the conditions (45) ensure the truth of Theorem 3.

Again in the proof of (a) in Theorem 2, we can obtain a bound for $|n^{-1}\{U_n^{(ij)}(\boldsymbol{\theta}^*)-U_n^{(ij)}(\boldsymbol{\theta}_0)\}|$ similar to (44), which will show that the condition:

$$(46) \qquad \begin{array}{l} \alpha_s^{(ijk)}(\boldsymbol{\theta}) \text{ is continuous in } \boldsymbol{\theta} \text{ and} \\ \sum_{s=-\infty}^{\infty} |\alpha_s^{(ijk)}(\boldsymbol{\theta})| \text{ is bounded } (1\leqq i,j,k\leqq p) \text{ for } \boldsymbol{\theta}\in N_\delta(\boldsymbol{\theta}_0), \end{array}$$

ensures convergence of this to zero in probability. Also the fact that the $\alpha_s^{(ij)}(\boldsymbol{\theta}_0)$ are Fourier coefficients of $h^{(ij)}(\omega, \boldsymbol{\theta}_0)$ gives us $\lim_{n\to\infty}\operatorname{var}\{n^{-1}U_n^{(ij)}(\boldsymbol{\theta}_0)\}=0$ by an argument using an inequality analogous to (43), and $\lim_{n\to\infty} E\{n^{-1}U_n^{(ij)}(\boldsymbol{\theta}_0)\}=0$ as before.

Finally, in the proof of (b) in Theorem 2, we must have

$$\lim_{n\to\infty} E\{n^{-\frac{1}{2}}U_n^{(j)}(\boldsymbol{\theta}_0)\} = \lim_{n\to\infty} n^{\frac{1}{2}} \sum_{|s|\leqq\phi(n)} \alpha_s^{(j)}(\boldsymbol{\theta}_0)\gamma_s^{(0)}\left(1-\frac{|s|}{n}\right) = 0,$$

or equivalently,

$$\lim_{n\to\infty} n^{\frac{1}{2}} \sum_{\phi(n)<|s|<n} \alpha_s^{(j)}(\boldsymbol{\theta}_0)\gamma_s^{(0)}\left(1-\frac{|s|}{n}\right) = 0.$$

Since we have $\sum_{s=-\infty}^{\infty}|s\gamma_s^{(0)}|<\infty$, this reduces to

$$(47) \qquad \lim_{n\to\infty} n^{\frac{1}{2}} \sum_{\phi(n)<|s|<n} \alpha_s^{(j)}(\boldsymbol{\theta}_0)\gamma_s^{(0)} = 0.$$

A sufficient condition for (47) is clearly

$$(48) \qquad \lim_{n\to\infty} n \sum_{s=\phi(n)+1}^{\infty} \{\gamma_s^{(0)}\}^2 = 0.$$

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Also the change of the upper limit for $|s|$ from $n-1$ to $\phi(n)$ in the definition of $R_{m,n}$ will clearly not affect the argument.

6. Effect of applying a mean correction

Suppose now that we consider the effect of replacing x_t by $x_t-\bar{x}$ (where $\bar{x} = \sum_{t=1}^{n} x_t/n$) in $U_n(x, \theta)$. If we write

$$U_n^*(\theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \left|\sum_{t=1}^{n} (x_t-\bar{x})e^{i\omega t}\right|^2 h(\omega, \theta)\,\mathrm{d}\omega,$$

we have, for $\theta \in S$,

$$E|U_n^*(\theta)-U_n(\theta)| = O(1). \tag{49}$$

For

$$4\pi[U_n^*(\theta)-U_n(\theta)] = \bar{x}^2\int_{-\pi}^{\pi} \left|\sum_{t=1}^{n} e^{i\omega t}\right|^2 h(\omega, \theta)\mathrm{d}\omega$$
$$-2\bar{x}\sum_{u=1}^{n} x_u \int_{-\pi}^{\pi} h(\omega, \theta)\sum_{t=1}^{n} e^{i\omega(t-u)}\mathrm{d}\omega$$

(h being an even function of ω)

$$= T_{1,n}(\theta)+T_{2,n}(\theta), \quad \text{say.}$$

Now

$$E|T_{1,n}(\theta)| \leqq E(\bar{x}^2)\sup_{|\omega|\leqq\pi} h(\omega, \theta)\int_{-\pi}^{\pi}\left|\sum_{t=1}^{n} e^{i\omega t}\right|^2 \mathrm{d}\omega \to 4\pi^2 f(0, \theta_0)\sup_{|\omega|\leqq\pi} h(\omega, \theta)$$

as $n \to \infty$, since

$$\lim_{n\to\infty} nE(\bar{x}^2) = \lim_{n\to\infty}\sum_{s=-(n-1)}^{n-1}\left(1-\frac{|s|}{n}\right)\gamma_s^{(0)} = \sum_{s=-\infty}^{\infty}\gamma_s^{(0)} = 2\pi f(0, \theta_0),$$

and therefore

$$E\,|T_{1,n}(\theta)| = O(1).$$

Also

$$\{E|T_{2,n}(\theta)|\}^2 \leqq 4E(\bar{x}^2)E\left(\sum_{u=1}^{n} x_u c_{n,u}\right)^2 \tag{50}$$

where

$$c_{n,u} = \int_{-\pi}^{\pi} h(\omega, \theta)\sum_{t=1}^{n} e^{i\omega(t-u)}\mathrm{d}\omega,$$

and

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$$E\left(\sum_{u=1}^{n} x_u c_{n,u}\right)^2 = \int_{-\pi}^{\pi} \sum_{u,v=1}^{n} f(\omega, \theta_0) e^{i\omega(u-v)} c_{n,u} c_{n,v} d\omega$$
$$\leqq \sup_{|\omega| \leqq \pi} f(\omega, \theta_0) \int_{-\pi}^{\pi} \left| \sum_{u=1}^{n} e^{i\omega u} c_{n,u} \right|^2 d\omega$$
$$= \sup_{|\omega| \leqq \pi} f(\omega, \theta_0) 2\pi \sum_{u=1}^{n} |c_{n,u}|^2,$$

while, since

$$\int_{-\pi}^{\pi} \left| h(\omega, \theta) \sum_{t=1}^{n} e^{i\omega t} - \sum_{u=1}^{n} c_{n,u} e^{i\omega u} \right|^2 d\omega = \int_{-\pi}^{\pi} \left| |h(\omega, \theta) \sum_{t=1}^{n} e^{i\omega t} \right|^2 d\omega - \sum_{u=1}^{n} |c_{n,u}|^2,$$
$$\sum_{u=1}^{n} |c_{n,u}|^2 \leqq 2\pi n \sup_{|\omega| \leqq \pi} |h(\omega, \theta)|^2.$$

It follows that $E\,|T_{2,n}(\theta)| = O(1)$, and (49) is therefore established.

When $U_n(\theta)$ is replaced by $U_n^*(\theta)$, Lemma 1 will clearly still hold since an immediate consequence of (49) is $p \lim_{n\to\infty} \{n^{-1}U_n^*(\theta) - n^{-1}U_n(\theta)\} = 0$. Also, it is easily seen that Theorem 1 will remain true, by using the fact that $C_0^* = n^{-1} \sum_{t=1}^{n} (x_t - \bar{x})^2 = C_0 - \bar{x}^2$ has, asymptotically, the same mean and variance as C_0 (for the variance we need the result $E(\bar{x}^4) = O(n^{-2})$ which is easily obtained by direct calculation, using the condition $\sum_{u=0}^{\infty} |l_u^{(0)}| < \infty$). Again in the proof of (a) in Theorem 2 the argument leading to (25) will not be affected, since we have merely to substitute C_0^* for C_0 in (26), and the result $E\,|U_n^{*(ij)}(\theta_0) - U_n^{(ij)}(\theta_0)| = O(1)$, which can be proved in exactly the same way as (49), shows that the analogues of (27) and (28) will hold. Finally for the proof of (b) we have only to observe that we shall have

$$E|U_n^{*(j)}(\theta_0) - U_n^{(j)}(\theta_0)| = O(1), \text{ giving } p \lim_{n\to\infty} n^{-\frac{1}{2}}\{U_n^{*(j)}(\theta_0) - U_n^{(j)}(\theta_0)\} = 0.$$

We thus see that the use of the mean correction does not affect the validity of any of our results. Hence they are applicable when the series mean, μ say, is not known *a priori*, so that, denoting the observations now by $\{y_t\}$, we have $y_t = x_t + \mu$, μ being another parameter that must be estimated. The substitution of the estimate $\hat{\mu} = \sum_{t=1}^{n} y_t/n$ for μ in L_n^*, which now becomes $-\frac{1}{2}n \log 2\pi\sigma^2 - U_n(y - \boldsymbol{\mu}, \theta)/2\sigma^2$ $[\boldsymbol{\mu} = (\mu, \mu, \cdots, \mu)']$, is then an obvious procedure; moreover the asymptotic variance of $\hat{\mu}$ is the same as that of the estimate which would be obtained by minimising with respect to μ the "exact" sum of squares $Q_n(y - \boldsymbol{\mu}, \theta)$ occurring in L_n (equation (5)), assuming θ to be known, this being a special case of a result due to Grenander [4] (see also Hannan, [7], p. 127, Grenander and Rosenblatt [6], p. 246).

The same conclusion is reached with the "truncated" definition of U_n given in § 5. This depends on results of the form

(51) $$E\Big|\sum_{|s|\leq\phi(n)}\alpha_s(\theta)(C_s^*-C_s)\Big| = O(n^{-1}\log\phi(n)), \text{ for } \theta\in S,$$

where

$$C_s^* = \sum_{t=1}^{n-|s|}(x_t-\bar{x})(x_{t+|s|}-\bar{x})/n.$$

To prove (51) we write

$$\sum_{|s|\leq\phi(n)}\alpha_s(\theta)(C_s^*-C_s)$$
$$= -\alpha_0(\theta)\bar{x}^2+2n^{-1}\sum_{s=1}^{\phi(n)}\left\{(n-s)\bar{x}^2-\bar{x}\sum_{t=1}^{n-s}x_t-\bar{x}\sum_{t=s+1}^{n}x_t\right\}\alpha_s(\theta),$$

so that the left-hand side is not greater than

(52) $$E(\bar{x}^2)\left|2\sum_{s=1}^{\phi(n)}\left(1-\frac{s}{n}\right)\alpha_s(\theta)-\alpha_0(\theta)\right| +2n^{-1}\sum_{s=1}^{\phi(n)}|\alpha_s(\theta)|\left\{E\left|\bar{x}\sum_{t=1}^{n-s}x_t\right|+E\left|\bar{x}\sum_{t=s+1}^{n}x_t\right|\right\}.$$

The first term in (52) is $O(n^{-1})$ since the $(C, 1)$ sum of $\sum\alpha_s(\theta)$ converges, while the second and third terms are $O(n^{-1}\log\phi(n))$ because $\alpha_s(\theta)$ is certainly $O(s^{-1})$ so that $\sum_{s=1}^{\phi(n)}|\alpha_s(\theta)|$ is $O(\log\phi(n))$, and, for example,

$$E\left|\bar{x}\sum_{t=1}^{n-s}x_t\right| \leq \left[E(\bar{x}^2)E\left(\sum_{t=1}^{n-s}x_t\right)^2\right]^{\frac{1}{2}}, \text{ which is } O(1)$$

since

$$E\left\{\sum_{t=1}^{n-s}x_t\right\}^2 = (n-s)\sum_{u=-(n-s)}^{n-s}\left(1-\frac{|u|}{n-s}\right)\gamma_s^{(0)} \leq n\sum_{s=-\infty}^{\infty}|\gamma_s^{(0)}|.$$

(51) with $\phi(n) = n-1$ could incidentally have been used instead of (49) in the previous discussion. However (49) is a slightly stronger result, and is of interest in indicating that the effect of the mean correction for finite n may sometimes be less for $\phi(n) = n-1$ than for smaller values of $\phi(n)$, which is perhaps a little surprising.

7. Further remarks

The conditions that we have imposed on $h(\omega, \theta)$ in our theorems are certainly stronger than they need be. For example, in Theorem 1 the partial derivatives $h^{(i)}(\omega, \theta)$ could have discontinuities provided that they remain bounded in $|\omega| \leq \pi$, $\theta\in S$, and continuous in the argument θ alone, and in Theorem 2 the same is true for the second and third order partial derivatives $h^{(ij)}(\omega, \theta)$, $h^{(ijk)}(\omega, \theta)$. However in most problems these conditions, and also the conditions on the Fourier coefficients $\alpha_s(\theta)$ and

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their derivatives given in § 5, will be satisfied (typically $h(\omega, \boldsymbol{\theta})$ will be of the form $|(1+\sum_{k=1}^{r}\phi_k(\boldsymbol{\theta})e^{i\omega k})/(1+\sum_{k=1}^{s}\psi_k(\boldsymbol{\theta})e^{i\omega k})|^2$, where ϕ_k and ψ_k are functions of $\boldsymbol{\theta}$ having derivatives of arbitrarily high order), and weaker conditions will be much less easily verified.

The assumption that $\boldsymbol{\Theta}$ is a bounded closed set contained in S will sometimes be an unnatural restriction. For example, if $\{x_t\}$ is generated by a first-order autoregressive process, with

$$(53)\qquad x_t+\theta x_{t-1} = \eta_t \quad (|\theta| < 1),\ h(\omega, \theta) = 1+\theta^2+2\theta\cos\omega,$$

or by a two-term moving-average process, with

$$(54)\qquad x_t = \eta_t+\theta\eta_{t-1} \ (|\theta| < 1),\ h(\omega, \theta) = (1+\theta^2+2\theta\cos\omega)^{-1},$$

we would usually want to take $\boldsymbol{\Theta}$ to be the open interval $(-1, 1)$, that is, S itself, rather than a closed interval such as $[-\theta_1, \theta_1]$ $(0 < \theta_1 < 1)$. Again one might have a problem in which $\boldsymbol{\Theta}$ was unbounded, for example the whole real line. However it is not clear how one could establish consistency of $\hat{\boldsymbol{\theta}}_n$ under general conditions without this assumption; the difficulty is to show that

$$\lim_{n\to\infty} p\{\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}-\boldsymbol{\Theta}_b}[U_n(\boldsymbol{\theta}_0)-U_n(\boldsymbol{\theta})] < 0\} = 1,$$

$\boldsymbol{\Theta}-\boldsymbol{\Theta}_b$ being the part of $\boldsymbol{\Theta}$ not contained in some bounded closed set $\boldsymbol{\Theta}_b$.

In some cases the difficulty does not arise because it can be seen that the "least squares" equations $U_n^{(i)}(\boldsymbol{\theta}) = O(1 \leqq i \leqq p)$ have a unique solution. Provided that $\boldsymbol{\theta}_0$ lies in *some* bounded closed set $\boldsymbol{\Theta}_b \subset \boldsymbol{\Theta}$, Theorem 1 can be applied with $\boldsymbol{\Theta}_b$ in place of $\boldsymbol{\Theta}$ to show that there must be at least one consistent solution of the equations, which therefore can be identified as the unique $\hat{\boldsymbol{\theta}}_n$. For example, this clearly happens when $h(\omega, \boldsymbol{\theta})$ is given by (53), and more generally when $\{x_t\}$ is an autoregressive process of order p with coefficients $\theta_1, \theta_2, \cdots, \theta_p$, so that $h(\omega, \boldsymbol{\theta}) = |1+\sum_{r=1}^{p}\theta_r e^{i\omega r}|^2$ (almost all the results for this case were derived by Mann and Wald [17]). On the other hand, for the moving-average model (46) we have $\alpha_s = (-\theta)^s/(1-\theta^2)$, $U_n(\theta) = \{C_0+2\sum_{s=1}^{n}(-\theta)^s C_s\}/(1-\theta^2)$ and it is not clear that the equation $\partial U_n/\partial\theta = 0$ has (with probability tending to unity as $n \to \infty$) just one root in the interval $-1 < \theta < 1$, although intuitively one would feel that this must be so. When the solution of the equations is not unique we can of course define $\hat{\boldsymbol{\theta}}_n$ to be one of the consistent solutions, and then Theorems 2—4 will clearly still hold. There will normally be a unique consistent solution and indeed it is easy to see that two consistent solutions $\hat{\boldsymbol{\theta}}_n^{(1)}$, $\hat{\boldsymbol{\theta}}_n^{(2)}$ must be equal with probability tending to unity as $n \to \infty$ (by applying the mean value theorem to $U_n^{(i)}(\hat{\boldsymbol{\theta}}_n^{(1)})-U_n^{(i)}(\hat{\boldsymbol{\theta}}_n^{(2)})$). This definition of $\hat{\boldsymbol{\theta}}_n$ is in fact used by Whittle in his 1953 paper (compare also Hannan, [7], p. 46).

Note added in proof. The author apologises for overlooking a paper by P. Whittle, Gaussian Estimation in Stationary Time Series, Bull. I.S.I. 39 (1962), 105—129, which gives a rigorous treatment along somewhat different lines from those of the present paper.

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