SOME ELEMENTARY PROPERTIES OF BILINEAR FORMS

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The purpose of the present note is to help popularize a section of Artin's "Geometric Algebra" (chapter I, 4; Interscience, New York (1957)) by elaborating on its contents. The author will have succeeded when the reader discovers that his results are either presented more simply in Artin's book or that they are trivial corollaries of its theorems, in particular of theorem 1.11.

1. Let V be a finite dimensional vector space over an arbitrary field F. The letters V_0, V_1, ... denote subspaces of V. The dimension of V_0 is denoted by $\dim V_0$. The codimension of V_0 is defined through

$$\operatorname{codim} V_0 = \dim V - \dim V_0.$$

Obviously

(1) $$V_0 = V_1 \leftrightarrow V_0 \subset V_1 \text{ and } \dim V_0 = \dim V_1.$$

The set $V_0 \cap V_1$ of all the vectors which lie in both V_0 and V_1 is a subspace. The sum $V_0 + V_1$ of V_0 and V_1 is the smallest subspace of V which contains both V_0 and V_1 It consists of all the vectors

$$v_2 = v_0 + v_1 \quad \text{where} \quad v_0 \in V_0,\ v_1 \in V_1.$$

If $V_0 \cap V_1 = 0$, this decomposition of v_2 is unique and the sum $V_0 + V_1$ is said to be direct. We then write $V_0 + V_1$.

It is well known that

(2) $$\dim (V_0 + V_1) + \dim (V_0 \cap V_1) = \dim V_0 + \dim V_1.$$

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Hence

(3) $\operatorname{codim}(V_0 + V_1) + \operatorname{codim}(V_0 \cap V_1) = \operatorname{codim} V_0 + \operatorname{codim} V_1$.

2. Let W be a second finite dimensional vector space over F with the subspaces $W_0, W_1, \ldots$. The bilinear form

$$f: \qquad v, w \rightarrow (v, w)$$

maps all the pairs of vectors $v \in V$, $w \in W$ onto elements of F. If v or w is kept fixed, this mapping is required to be linear in w respectively v. Thus e.g.

$$(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 (v, w_1) + \lambda_2 (v, w_2)$$

for all $v \in V$, $w_1 \in W$, $w_2 \in W$, $\lambda_1 \in F$, $\lambda_2 \in F$. The vector spaces V and W are said to be paired.

Every subspace V_0 of V now determines a new subspace $V_0^* \subset W$ through

$$V_0^* = \{w \mid (v_0, w) = 0 \text{ for all } v_0 \in V_0\}.$$

Similarly define ${}^*W_0 \subset V$ through

$${}^*W_0 = \{v \mid (v, w_0) = 0 \text{ for all } w_0 \in W_0).$$

We call

$$V^* = \{w \mid (v, w) = 0 \text{ for all } v \in V\}$$

the right kernel and

$${}^*W = \{v \mid (v, w) = 0 \text{ for all } w \in W\}$$

the left kernel of f.

Obviously

(4) $$\begin{cases} V_0 \subset V_1 & \text{implies} \quad V_1^* \subset V_0^* \\ W_0 \subset W_1 & \text{implies} \quad {}^*W_1 \subset {}^*W_0 \end{cases}$$

In particular

(5) $$V^* \subset V_0^* \quad \text{and} \quad {}^*W \subset {}^*W_0 \quad \text{for all } V_0, W_0.$$

If $v \in V$, then $(v, v^*) = 0$ for all $v^* \in V^*$. Thus $v \in {}^*(V^*)$. Hence $V \subset {}^*(V^*)$. Trivially ${}^*(V^*) \subset V$. Hence

(6) $$V = {}^*(V^*), \quad W = ({}^*W)^*.$$

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3. We wish to verify

(7) $(V_0 + V_1)^* = V_0^* \cap V_1^*, \quad {}^*(W_0 + W_1) = {}^*W_0 \cap {}^*W_1.$

It suffices to discuss the first formula.

Since $V_0 \subset V_0 + V_1$ and $V_1 \subset V_0 + V_1$, (4) implies

$$(V_0 + V_1)^* \subset V_0^* \quad \text{and} \quad (V_0 + V_1)^* \subset V_1^*.$$

Thus

$$(V_0 + V_1)^* \subset V_0^* \cap V_1^*.$$

Conversely let $w \in V_0^* \cap V_1^*$, $v \in V_0 + V_1$. Then there are vectors $v_0 \in V_0$, $v_1 \in V_1$ such that $v = v_0 + v_1$. Since $w \in V_0^*$, we have $(v_0, w) = 0$; also $w \in V_1^*$ implies $(v_1, w) = 0$. Hence

$$(v, w) = (v_0 + v_1, w) = (v_0, w) + (v_1, w) = 0 + 0 = 0.$$

This remains valid for every choice of v. Hence $w \in (V_0 + V_1)^*$ or

$$V_0^* \cap V_1^* \subset (V_0 + V_1)^*.$$

This yields (7).

If we specialize in (7) $V_1 = {}^*W$, we obtain on account of (6)

$$(V_0 + {}^*W)^* = V_0^* \cap ({}^*W)^* = V_0^* \cap W$$

or

(8) $(V_0 + {}^*W)^* = V_0^*$; symmetrically ${}^*(W_0 + V^*) = {}^*W_0$.

In the next three sections we determine $\dim {}^*W_0$.

4. If Y_0 is a proper subspace of the arbitrary finite dimensional vector space Y, then there exists a linear form in Y which vanishes identically in Y_0 but not in Y. This can

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be restated as follows: Let Y_0 be a subspace of Y. If every linear form which vanishes identically in Y_0 also vanishes identically in Y, then $Y_0 = Y$.

Apply this observation to the dual vector space X' of the vector space X and to a subspace X'_0 of X' (thus X' consists of the linear forms in X). Since the space of all the linear forms in X' may be identified with X, we obtain:

Let X'_0 be a subspace of the dual space X' of the vector space X. Suppose to each $x \in X$, $x \neq 0$ there exists an element of X'_0 which does not annihilate x. Then $X'_0 = X'$.

5. We now return to our bilinear form f. Let

$$W_0 \cap V^* = 0.$$

Map each vector $v \in V$ onto the linear form (v, w_0) in W_0. Thus V is mapped homomorphically into the vector space W'_0 of all the linear forms in W_0. By our assumption, there exists to each w_0 a v such that $(v, w_0) \neq 0$. Hence by 4. the image of our homomorphism is the whole of W'_0.

The image of the vector v, i.e. the linear form (v, w_0) vanishes identically in W_0 if and only if $v \in {}^*W_0$. Thus ${}^*W_0$ is the kernel of this homomorphism and $V/{}^*W_0$ is isomorphic to W'_0. In particular

$$\text{codim } {}^*W_0 = \dim V/{}^*W_0 = \dim W'_0.$$

Since a vector space and its dual have the same dimension, we therefore have

(9) $$\text{codim } {}^*W_0 = \dim W_0 \quad \text{if } W_0 \cap V^* = 0.$$

6. If $V^* \subset W_0$, then there is a W_1 such that $W_0 = W_1 \dotplus V^*$; cf. 1. By (8)

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$$*W_0 = *(W_1 + V^*) = *W_1.$$

Hence by (9)

$$\operatorname{codim} *W_0 = \operatorname{codim} *W_1 = \dim W_1$$

or

(10) $\operatorname{codim} *W_0 = \dim W_0 - \dim V^*$ if $V^* \subset W_0$.

Symmetrically

(10') $\operatorname{codim} V_0^* = \dim V_0 - \dim *W$ if $*W \subset V_0$.

Finally let W_0 be any subspace of W. Consider the restriction of the form f to the pair of subspaces V, W_0. If $w_0 \in W_0$ is given, then $(v, w_0) = 0$ for all $v \in V$ if and only if $w_0 \in V^* \cap W_0$. Hence (10) implies

(11) $\operatorname{codim} *W_0 = \dim W_0 - \dim (V^* \cap W_0)$.

This formula contains (9) and (10).

The case $W_0 = W$ of (11) yields

(12) $\operatorname{codim} *W = \dim W - \dim V^* = \operatorname{codim} V^*$.

This number is called the <u>rank</u> of f.

7. If $w_0 \in W_0$, then $(*w_0, w_0) = 0$ for all $*w_0 \in *W_0$. Hence $w_0 \in (*W_0)^*$ and therefore $W_0 \subset (*W_0)^*$. By (5), $V^* \subset (*W_0)^*$. This yields

$$W_0 + V^* \subset (*W_0)^*.$$

On the other hand, $*W \subset *W_0$. Hence by (10'), (12), and (11)

$$\operatorname{codim} (*W_0)^* = \dim *W_0 - \dim *W$$

$$= \operatorname{codim} *W - \operatorname{codim} *W_0$$

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$$= \operatorname{codim} V^* - \dim W_0 + \dim (V^* \cap W_0)$$

$$= \operatorname{codim} V^* + \operatorname{codim} W_0 - \operatorname{codim} (V^* \cap W_0)$$

$$= \operatorname{codim} (V^* + W_0).$$

The principle (1) therefore implies

(13) $({}^*W_0)^* = W_0 + V^*$, symmetrically ${}^*(V_0^*) = V_0 + {}^*W$.

8. The equation

(14) $${}^*(W_0 \cap W_1) = {}^*W_0 + {}^*W_1$$

need not be true.

Since $W_0 \cap W_1 \subset W_0$ and $W_0 \cap W_1 \subset W_1$, we always have

$${}^*W_0 \subset {}^*(W_0 \cap W_1) \quad \text{and} \quad {}^*W_1 \subset {}^*(W_0 \cap W_1)$$

and hence

(15) $${}^*W_0 + {}^*W_1 \subset {}^*(W_0 \cap W_1).$$

Thus by (1), (14) is equivalent to

(16) $$\operatorname{codim} ({}^*W_0 + {}^*W_1) = \operatorname{codim} {}^*(W_0 \cap W_1).$$

By (3), (7), and (11)

$$\operatorname{codim} ({}^*W_0 + {}^*W_1) - \operatorname{codim} {}^*(W_0 \cap W_1)$$

$$= \operatorname{codim} {}^*W_0 + \operatorname{codim} {}^*W_1 - \operatorname{codim} ({}^*W_0 \cap {}^*W_1) - \operatorname{codim} {}^*(W_0 \cap W_1)$$

$$= \operatorname{codim} {}^*W_0 + \operatorname{codim} {}^*W_1 - \operatorname{codim} {}^*(W_0 + W_1) - \operatorname{codim} {}^*(W_0 \cap W_1)$$

$$= \{ \dim W_0 + \dim W_1 - \dim (W_0 + W_1) - \dim (W_0 \cap W_1)\}$$

$$- \{ \dim (V^* \cap W_0) + \dim (V^* \cap W_1) - \dim (V^* \cap (W_0 + W_1))$$

$$- \dim (V^* \cap W_0 \cap W_1)\}$$

$$= 0 - \{ \dim (V^* \cap W_0 + V^* \cap W_1) - \dim (V^* \cap (W_0 + W_1))\} .$$

Thus (16) is equivalent to

(17) $$\dim (V^* \cap W_0 + V^* \cap W_1) = \dim (V^* \cap (W_0 + W_1)).$$

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Obviously

$$V^*\cap W_0 + V^*\cap W_1 \subset V^*\cap(W_0 + W_1).$$

Hence (17) is equivalent to

(18) $$V^*\cap W_0 + V^*\cap W_1 = V^*\cap(W_0 + W_1).$$

This yields the result that (14) and (18) are equivalent.

If $V^* \subset W_1$, both sides of (18) are equal to V^*. Hence (14) then holds true.

The reader will verify that

$$^*(V_0^*\cap W_0) = V_0 + {}^*W_0$$

and prove that

(19) $$V_0 + {}^*W_0 = V \quad \leftrightarrow \quad V_0^*\cap W_0 \subset V^*.$$

9. We have $V^* = {}^*W = 0$ if and only if

$$\text{codim } {}^*W = \dim V, \qquad \text{codim } V^* = \dim W.$$

The form f is then said to be regular. Formula (12) then implies

$$\dim V = \dim W.$$

We call f regular in V_0, W_0 if the restriction of f to V_0, W_0 is regular. Since the restriction has the kernels ${}^*W_0\cap V_0$ and $V_0^*\cap W_0$, we have

THEOREM 9.1. f is regular in V_0, W_0 if and only if

$$V_0^*\cap W_0 = {}^*W_0\cap V_0 = 0.$$

This regularity implies

(20) $$\dim V_0 = \dim W_0.$$

We readily deduce by means of (19)

COROLLARY 9.2. f is regular in V_0, W_0 if and only if

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(21) $$V_0 \dot{+} {}^*W_0 = V, \qquad W_0 \dot{+} V_0^* = W.$$

Formulas (20) and (21) imply

COROLLARY 9.3. If f is regular in V_0, W_0, then

(22) $$\begin{aligned} \dim V_0 = \dim W_0 &= \operatorname{codim} {}^*W_0 = \operatorname{codim} V_0^* \\ &\leq \operatorname{codim} {}^*W = \operatorname{codim} V^* = \operatorname{rank} f. \end{aligned}$$

10. We call f <u>maximally regular</u> in V_0, W_0 if f is regular in V_0, W_0 and if equality holds in (22). From

$$\operatorname{codim} {}^*W_0 = \operatorname{codim} {}^*W, \qquad {}^*W \subset {}^*W_0$$

we then obtain

(23) $${}^*W_0 = {}^*W; \quad \text{symmetrically } V_0^* = V^*.$$

Hence by (21),

(24) $$V_0 \dot{+} {}^*W = V, \qquad W_0 \dot{+} V^* = W.$$

Conversely, (24) yields on account of (11) and (12) that

$$\begin{aligned} \operatorname{codim} {}^*W_0 &= \dim W_0 - \dim (V^* \cap W_0) \\ &= \dim W_0 = \operatorname{codim} V^* = \operatorname{codim} {}^*W. \end{aligned}$$

This implies (23) and (21). This proves

THEOREM 10.1. f is maximally regular in V_0, W_0 if and only if (24) holds true.

COROLLARY 10.2. f is maximally regular in V_0, W_0 if and only if (21) and one of the equations (23) hold true.

By means of (24) we can readily construct pairs V_0, W_0 in which f is maximally regular. We only have to choose V_0, W_0 independently of one another such that

$$V_0 \cap {}^*W = 0, \qquad \dim V_0 = \operatorname{codim} {}^*W$$

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and

$$W_0 \cap V^* = 0, \qquad \dim W_0 = \operatorname{codim} V^*.$$

Obviously, V_0 and W_0 are determined uniquely (mod ${}^*W$) respectively (mod V^*).

11. We now assume that V and W are both equal to the same vector space E. We then obtain theorems on general bilinear forms f in E. On account of (12), the rank of f can be defined through

$$\operatorname{rank} f = \operatorname{codim} {}^*E = \operatorname{codim} E^*.$$

If $\operatorname{rank} f = \dim E$, f may be called regular in E. By corollary 9.2, the restriction of f to the subspace E_0 of E is regular if and only if

(25) $$E_0 \dotplus {}^*E_0 = E_0 \dotplus E_0^* = E.$$

Suppose e.g.

(26) $$E_0 \dotplus E_0^* = E.$$

Then $E_0 \cap E_0^* = 0$ and (19) implies $E_0 + {}^*E_0 = E$. By (9) we have $\operatorname{codim} {}^*E_0 = \dim E_0$. This yields $E_0 \dotplus {}^*E_0 = E$. Hence,

THEOREM 11.1. Formula (26) implies the regularity of the restriction of f to E_0.

We call f again maximally regular in E_0 if (25) holds true and if $\dim E_0 = \operatorname{rank} f$; cf. (22). By theorem 10.1, f is maximally regular in E_0 if and only if

(27) $$E_0 \dotplus {}^*E = E_0 \dotplus E^* = E.$$

This readily yields

THEOREM 11.2. f is maximally regular in E_0 if and only if

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$$E_0 \cap {}^*E = E_0 \cap E^* = 0$$

$$\dim E_0 = \text{rank } f.$$

Finally we show

THEOREM 11.3. There are subspaces E_0 of E in which f is maximally regular.

Since $\dim {}^*E = \dim E^*$, this theorem is an immediate corollary of the observation that two subspaces of the same dimension have a common complement. For the sake of completeness we include a proof.

Let $e_1, \ldots, e_k$ be a basis of ${}^*E \cap E^*$. By means of the vectors

$$^*e_1, \ldots, {}^*e_h \quad [e^*_1, \ldots, e^*_h]$$

we complete it to a basis of $^*E$ [of E^*]. Thus the vectors

(28) $$e_1, \ldots, e_k, {}^*e_1, \ldots, {}^*e_h, e^*_1, \ldots, e^*_h$$

form a basis of ${}^*E + E^*$. We complete it to a basis

(29) $$e_1, \ldots, e_k, {}^*e_1, \ldots, {}^*e_h, e^*_1, \ldots, e^*_h, e'_1, \ldots, e'_m$$

of E. We wish to show that the vectors

(30) $$^*e_1 + e^*_1, \ldots, {}^*e_h + e^*_h, e'_1, \ldots, e'_m$$

span a subspace E_0 satisfying (27).

Suppose

$$e = \Sigma \lambda^i({}^*e_i + e^*_i) + \Sigma \mu^j e'_j \in E_0 \cap E^*.$$

Then

$$\Sigma \mu^j e'_j = e - \Sigma \lambda^i({}^*e_i + e^*_i) \in {}^*E + E^*.$$

Hence this vector is a linear combination of the vectors (28). Since its representation as a linear combination of the vectors (29) is unique, the μ^j must vanish and we have

$$e = \Sigma \lambda^i({}^*e_i + e^*_i).$$

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This yields

$$\Sigma \lambda^{i*} e_i = e - \Sigma \lambda^i e_i^* \epsilon \; {}^*E \cap E^*.$$

Therefore $\Sigma \lambda^{i*} e_i = 0$ and hence

$$\lambda^1 = \ldots = \lambda^h = 0; \qquad e = 0.$$

Thus $E_0 \cap E^* = 0$ and the vectors (30) are linearly independent.

Symmetrically $E_0 \cap {}^*E = 0$. Finally

$$\dim E_0 + \dim {}^*E = (h + m) + (k + h) = k + 2h + m = \dim E.$$

This proves (27).

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