Proceedings of the Royal Society of Edinburgh, page 1 of 24
DOI:10.1017/prm.2024.28

Optimal inverse problems of potentials for two given eigenvalues of Sturm–Liouville problems

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(Received 7 September 2023; accepted 11 February 2024)

The present paper is concerned with the infimum of the norm of potentials for Sturm–Liouville eigenvalue problems with Dirichlet boundary condition such that the first two eigenvalues are known. The explicit quantity of the infimum is given by the two eigenvalues.

Keywords: optimal inverse problem; Lyapunov inequality; min-max principle; Sturm–Liouville problem; eigenvalue

2020 *Mathematics Subject Classification:* Primary: 34B20
Secondary: 34L99, 34L05

1. Introduction

The present paper is concerned with the Sturm–Liouville eigenvalue problem subject to Dirichlet boundary condition:

$$-y'' + qy = \lambda y,\ y = y(x),\ x \in [0, 1],\ y(0) = 0 = y(1), \tag{1.1}$$

where the potential $q \in L^1([0, 1],\ \mathbb{R})$ and $L^1([0, 1],\ \mathbb{R})$ is the space of Lebesgue integrable real-valued functions on $[0, 1]$. From the spectral theory of differential equations, it is known that (1.1) has a countable number of eigenvalues, which are algebraically simple, bounded below, and tend to ∞. Let $\lambda_n(q)$ be the n-th eigenvalue of (1.1). Then

$$-\infty < \lambda_1(q) < \lambda_2(q) < \cdots < \lambda_n(q) \to \infty,\ n \to \infty.$$

For example, if $q \equiv 0$, then $\lambda_n(0) = n^2\pi^2,\ n \in \mathbb{N}$. The set of all the eigenvalues of (1.1) is the spectral set, denoted by $\sigma(q)$.

The main objective in spectral theory of differential equations is relations between *geometric data:* coefficients of the equation, shape of the boundary, etc. and *spectral data:* eigenvalues and eigenfunctions of the differential equation. The direct problem is to determine spectral data from geometric data. The spectral theory for the direct problem is well developed for regular as well as singular problems, and the reader can refer to the books [**2**, **23**, **24**]. The inverse problem is to recover the geometric data (or part of it) from some spectral dada. Compared with the direct problem, the inverse problem is hard to solve and the corresponding spectral theory needs to be developed further. One of the reason is that the spectral set is determined uniquely by $q \in L^1[0,1]$, but the converse is not true. It was proved in [**13**] that for a given spectral set $\sigma(q_0)$ with $q_0 \in L^2[0,1]$, the isospectral set $M(q_0)$:

$$M(q_0) = \big\{q \in L^2[0,1] : \ \sigma(q) = \sigma(q_0)\big\} \tag{1.2}$$

is an infinite dimensional real analytic submanifold of $L^2[0,1]$ [**13**, p. 68], and only the even potentials in an isospectral set can be determined uniquely with the smallest L^2-norm, see [**13**, Corollary 1, p. 77]. In fact, earlier in 1946, Borg [**1**] gave the fundamental theorem that two sets of eigenvalues uniquely determine the potential. Since then, many scholars have carried out in-depth research and generalization of Borg's results, see [**10**, **11**]. Hochstadt and Lieberman [**7**] showed that the potential which is known on half of the interval can be recovered from a set of eigenvalues and it is also the start of recovering potentials from partial spectral data [**3**, **4**, **21**, **22**].

In this paper, we attempt to study the optimal inverse problem of the potentials with fixed finite eigenvalues $\lambda_j \in \mathbb{R}, j = 1,\dots,m$ and $\lambda_1 < \cdots < \lambda_m$, $m \in \mathbb{N}$. That is, we will estimate the infimum of the norm $\|q\|_1$ of q in $\Omega(\lambda_1,\dots,\lambda_m) \subset L^1[0,1]$, where

$$\Omega(\lambda_1,\dots,\lambda_m) = \big\{q : \ q \in L^1([0,1], \ \mathbb{R}), \ \lambda_j \in \sigma(q), \ j = 1,\dots,m\big\}.$$

Since $\Omega(\lambda)$ is an infinite-dimensional submanifold in $L^2[0,1]$ with codimension one and

$$\Omega(\lambda_1,\lambda_2,\dots,\lambda_m) = \cap_{k=1}^m \Omega(\lambda_k),$$

the set $\Omega(\lambda_1,\lambda_2,\dots,\lambda_m)$ must be an infinite set in $L^2[0,1]$, and hence in $L^1[0,1]$ due to $L^2[0,1] \subset L^1[0,1]$. So that, the uniqueness of potential for the recovery problem does not hold. Thus, we recover the optimal potential under the condition that the L^1-norm of the potential is the infimum in $\Omega(\lambda_1,\lambda_2,\dots,\lambda_m)$. For the optimal recovery problem, we also study the existence—whether the infimum is attained in $\Omega(\lambda_1,\lambda_2,\dots,\lambda_m)$. Furthermore, we will present the quantitative representation of the optimal potential. The optimal recovery problem, or the optimal inverse problem, is very related to the extremal problem of eigenvalues, particularly, the two problems are equivalent to each other provided that $m = 1$ [**14**].

For the case $m = 1$, the optimal inverse problem has been solved in [**14**] by using the generalized Lyapunov-type inequality together with Rayleigh–Ritz principle, and this method has been used successfully to solve the norm estimating of the optimal potentials for Sturm–Liouville problem with general separated boundary conditions (see [**5**]) and with Dirac distribution weights (see [**6**]). However, the above technique is hardly applicable to solve the problem directly for the cases $m \geqslant 2$ since the (generalized) Lyapunov-type inequality involves only one eigenvalue.

Another efficient method to solve the optimal inverse problem is the critical equations in $L^p[0,1]$ for $p > 1$, which were early used by the authors in [**20**, **25**] for Sturm–Liouville operators in studying the extremal problems of eigenvalues, in which the similar results for the optimal potentials with Dirichlet and Neumann boundary conditions have been obtained based on the critical equations. Recently, the critical equations are also constructed in [**18**] for elliptic operators with the case $m = 1$, then the authors in [**8**, **19**] extended the case to any finite m. Such method is also used in [**16**] to obtain the optimal weight of vibrating string equations for the case $m = 1$.

For $m \geqslant 2$, the optimal inverse problem can be turned into finding the solution of a boundary value problem for a system of nonlinear differential equations. For example, the authors in [**19**] used this method to find the optimal potential $q \in L^2$ which is the nearest to given q_0 with prescribed partial trace. And the authors in [**8**] extended to estimate the infimum of the norm $\|q - q_0\|_2$ for fixed finite eigenvalues of q. In both of the above papers, the existence of the optimal potentials is proved and the expressions of such potentials are given by the solutions of boundary value problems of nonlinear differential equations.

In the present paper, the similar problem for $m = 2$ is considered in the space $L^1[0, 1]$ with $q_0 = 0$. Since the norm in $L^1[0, 1]$ is not differentiate, the critical equation method could not be applied directly. Besides, when $m \geqslant 2$, it is difficult to obtain the explicit form of the optimal potential by the critical equation method due to the nonlinearity of the corresponding problems.

Similar to the present problem, the inverse optimal problem of weights for the problem

$$-y'' = \lambda w y,\ y(0) = 0 = y(1)$$

has been investigated in [**15**] when the first two eigenvalues are known. To date, few results of estimating the extremal norm of potentials for the cases $m \geqslant 2$ are available.

In this paper, we will study the case for $m = 2$. For fixed $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2$, the paper considers the infimum:

$$E(\lambda_1, \lambda_2) = \inf \{\|q\|_1 :\ q \in \Omega(\lambda_1, \lambda_2)\} \tag{1.3}$$

of the L^1-norm of q in the set:

$$\Omega(\lambda_1, \lambda_2) = \left\{q :\ q \in L^1[0, 1], \lambda_1(q) = \lambda_1, \lambda_2(q) = \lambda_2\right\}. \tag{1.4}$$

For the purpose of the clear statement of the methods, we consider only the special case, where

$$\begin{cases} \lambda_1 \in (-\infty, \pi^2),\ \lambda_2 \in (\lambda_1, 4\pi^2),\ \rho_j = \sqrt{\lambda_j},\ j = 1, 2, \\ \rho_1 \left(\cot \dfrac{\rho_1}{4} - \tan \dfrac{\rho_1}{4}\right) \geqslant 2\rho_2 \cot \dfrac{\rho_2}{4}. \end{cases} \tag{1.5}$$

Note that the functions $\sqrt{\lambda}\cot(\sqrt{\lambda}/4)$ and $\sqrt{\lambda}\tan\left(\sqrt{\lambda}/4\right)$ are analytic in the half-plane $\operatorname{Re}\lambda < 4\pi^2$ except for the only removable singularity $\lambda = 0$, and hence for $\sqrt{\lambda}\cot(\sqrt{\lambda}/4)$, we use $\sqrt{\lambda}\cot(\sqrt{\lambda}/4)|_{\lambda=0} = 4$ and for $\lambda < 0$:

$$\sqrt{\lambda}\cot\sqrt{\lambda} = \sqrt{|\lambda|}\coth\sqrt{|\lambda|}, \quad \sqrt{\lambda}\tan\sqrt{\lambda} = -\sqrt{|\lambda|}\tanh\sqrt{|\lambda|}.$$

The admissible set of the above restrictions on λ_1 and λ_2 in (1.5) is not empty, see remark 2 in §3. The reason why we consider condition (1.5) is that for general

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situations, it follows from the existing results, see [**14**, p. 11], the optimal potential is either the Dirac-delta function or the bathtub function. We will prove that condition (1.5) guarantees that the optimal potential is a Dirac-delta function.

Since the Dirac-delta function is not Lebesgue integral, then we introduce the measure space $\mathcal{M}_0$ [**12**, **26**] with the norm $\|\mu\|_V$ for $\mu \in \mathcal{M}_0$, see the definitions in §2. Then, we consider the eigenvalue problem of the second-order measure differential equations with the given measure $\mu \in \mathcal{M}_0$:

$$\begin{cases} -\mathrm{d}\dot{y}(x) + y(x)\,\mathrm{d}\mu(x) = \lambda y\,\mathrm{d}x, \;\; x \in [0,1], \\ y(0) = 0 = y(1), \end{cases} \tag{1.6}$$

where $\dot{y}$ expresses the generalized derivative of the solution $y(x)$ and (1.6) has a countable number of eigenvalues, which are algebraically simple, bounded below, and tend to ∞ [**12**].

Therefore, we can introduce the similar optimal recovery problem as (1.3) and (1.4) in $\mathcal{M}_0$. Let $\lambda_1(\mu),\ \lambda_2(\mu)$ be the first and second eigenvalues of (1.6) and define:

$$E_0(\lambda_1,\lambda_2) = \inf\left\{\|\mu\|_V : \ \mu \in \Omega_0(\lambda_1,\lambda_2)\right\}, \tag{1.7}$$

$$\Omega_0(\lambda_1,\lambda_2) = \left\{\mu : \ \mu \in \mathcal{M}_0,\ \lambda_1(\mu) = \lambda_1,\ \lambda_2(\mu) = \lambda_2\right\}. \tag{1.8}$$

Applying the spectral shifting lemma, see lemma 2.4, we prove in §2 that

THEOREM 1.1. *$E_0(\lambda_1,\lambda_2)$ is accessible and $E(\lambda_1,\lambda_2) = E_0(\lambda_1,\lambda_2)$.*

Theorem 1.1 indicates that the above optimal recovery problems are well posed and they are equivalent to each other. This enables us to look for the optimal solution $\mu_0 \in \Omega_0(\lambda_1,\lambda_2)$ such that $\|\mu_0\|_V = E_0(\lambda_1,\lambda_2)$ in $\mathcal{M}_0$. The following theorem is the main result of this paper.

THEOREM 1.2. *Consider the eigenvalue problem (1.1) with symmetric potential q. Let $E(\lambda_1,\lambda_2)$ and $\Omega(\lambda_1,\lambda_2)$ be defined in (1.3) and (1.4), respectively. If λ_1 and λ_2 satisfy (1.5), then*

$$E(\lambda_1,\lambda_2) = 2r,\;\; r = \rho_1\left[\cot(\rho_1 a) - \tan\rho_1(1/2 - a)\right] > 0,$$

where $a \in [1/4, 1/2)$ is the unique root of the equation:

$$\rho_1\left[\cot(\rho_1 a) - \tan\rho_1(1/2 - a)\right] = \rho_2\left[\cot(\rho_2 a) + \cot\rho_2(1/2 - a)\right].$$

Furthermore, $E(\lambda_1,\lambda_2)$ is attained in $\mathcal{M}_0$ by $\mu(x) = -r(H_a(x) + H_{1-a}(x))$ in (1.6), where $H_a(x)$ is Heaviside function, see definition (2.2) in §2.

The arrangement of this paper is as follows. Section 2.1 provides some preliminary knowledge about the real measure space on $[0,1]$. In §2.2, we introduce the inverse spectral theory in $L^2[0,1]$, and then provide the proof of theorem 1.1. We will introduce the min-max principle of quadratic form in §2.3 and provide the generalized Lyapunov inequality with modification in §2.4. The proof of theorem 1.2 is given in §3. In the Appendix, we will supplement some concepts of quadratic form and add the proof of the correspondence between the quadratic form and the problem proposed in §2.3 and 2.4.

2. Preliminaries

2.1. The measure space

This subsection introduces some basic knowledge of measure space, see [12, 26] for more details.

For a real function $\mu: [0,1] \to \mathbb{R}$, the total variation of μ on $[0,1]$ is defined as:

$$\|\mu\|_V = \sup \left\{ \sum_{i=0}^{n-1} |\mu(x_{i+1}) - \mu(x_i)| : \ 0 = x_0 < \cdots < x_n = 1, n \in \mathbb{N} \right\}.$$

The space of measures on $[0,1]$ is defined as:

$$\mathcal{M}_0 = \{\mu: [0,1] \to \mathbb{R}: \ \mu(0+) = 0, \ \mu(x+) = \mu(x), \ \forall x \in (0,1), \ \|\mu\|_V < \infty\},$$

where $\mu(x+) := \lim_{s \downarrow x} \mu(s), \ x \in [0,1)$ is the right-limit. $\mathrm{d}\mu$ can be represented as $\rho = \mathrm{d}\mu$ and call ρ as 'density function'. For example, $f \in L^1[0,1]$ is the density function of the absolutely continuous measure:

$$\mu_f(x) := \int_{[0,x]} f(s)\,\mathrm{d}s, \ x \in [0,1]. \tag{2.1}$$

And the Dirac-delta function $\delta(x-a)$ at point $a \in (0,1)$ is the density function of the Heaviside function $H_a(x)$, where

$$H_a(x) = \begin{cases} 0, & x \in [0,a), \\ 1, & x \in [a,1]. \end{cases} \tag{2.2}$$

In the measures space $\mathcal{M}_0$, besides the usual topology induced by the norm $\|\cdot\|_V$, there also is the following weak* topology, denoted by w^*.

Definition 2.1. *Let $\mu_n, \ \mu_0 \in \mathcal{M}_0, \ n \in \mathbb{N}$. μ_n is said to converge weakly* to μ_0, denoted as $\mu_n \xrightarrow{w^*} \mu_0$ in $(\mathcal{M}_0, w^*)$, if*

$$\lim_{n\to\infty} \int_{[0,1]} u(t)\,\mathrm{d}\mu_n(t) = \int_{[0,1]} u(t)\,\mathrm{d}\mu_0(t), \ \forall u \in C[0,1].$$

Remark 2.2. From the properties of bounded variation functions and the density of the space of absolutely continuous functions in $(L^1[0,1], \|\cdot\|_1)$, it follows that $L^1[0,1]$ is dense in $(\mathcal{M}_0, w^*)$, that is for $\forall \mu \in \mathcal{M}_0$, there exists $f_n \in L^1[0,1]$ such that $f_n \xrightarrow{w^*} \mathrm{d}\mu$.

Theorem 1.1 of [12] shows the continuity of the solutions of the initial value problem with respect to $\mu \in (\mathcal{M}_0, w^*)$. Applying Montel theorem and the similar method, we can prove the more general uniformly convergent properties of the solutions.

Lemma 2.3. *Let $\mu_n,\ \mu_0 \in \mathcal{M}_0$, and $y_n(\lambda),\ y_0(\lambda),\ z_n(\lambda),\ z_0(\lambda)$ be functions of λ on a bounded domain $D \subset \mathbb{C}$. Let $y(x,\mu_n,\lambda)$ and $y(x,\mu_0,\lambda)$ respectively be the solutions of the problem:*

$$\begin{cases} -\mathrm{d}\dot{y}(x) + y(x)\,\mathrm{d}\mu_n(x) = \lambda y\,\mathrm{d}x, & x \in [0,1], \\ y(0) = y_n(\lambda), & \dot{y}(0) = z_n(\lambda) \end{cases} \tag{2.3}$$

and

$$\begin{cases} -\mathrm{d}\dot{y}(x) + y(x)\,\mathrm{d}\mu_0(x) = \lambda y\,\mathrm{d}x, & x \in [0,1], \\ y(0) = y_0(\lambda), & \dot{y}(0) = z_0(\lambda). \end{cases} \tag{2.4}$$

If $\mu_n \xrightarrow{w^*} \mu_0$ and $y_n \to y_0,\ z_n \to z_0,\ n \to \infty$ uniformly in D, then $y(x,\mu_n,\lambda) \to y(x,\mu_0,\lambda),\ n \to \infty$ uniformly for $(x,\lambda) \in [0,1] \times D$.

2.2. The optimal recovery problem in measure space

In this subsection, we will use the inverse spectral theory in $L^2[0,1]$ [**13**] to prove theorem 1.1. And hence, we need some knowledge of the spectrum for $q \in L^2[0,1]$.

The spectrum of (1.1) with $q \in L^2[0,1]$ belongs to the space S of all real, strictly increasing sequence $\sigma = (\sigma_1,\ \sigma_2,\ldots)$ of the form:

$$\sigma_n = n^2\pi^2 + s + \widetilde{\sigma},\ n \geqslant 1,$$

where $s = \int_0^1 q\,\mathrm{d}x \in \mathbb{R}$ and $\widetilde{\sigma} = (\widetilde{\sigma_1},\ \widetilde{\sigma_2},\ldots) \in l^2$, i.e. $\sum_{i=1}^{\infty} \widetilde{\sigma}_i^{\,2} < \infty$ [**13**, Theorem 2.4]. Let T be the spectral mapping such that

$$T : q \in L^2[0,1] \to \sigma(q) \in S.$$

Let E be the subspace of even functions in $L^2[0,1]$. It was proved in [**13**, Theorem 6.2] that T maps E onto S. The main tool in the proof of this subsection is the spectral 'shifting' theorem, see [**13**, Theorem 6.1]. In order to make the statement clearly, we need some more notations in the following.

For $p \in E$, let $\lambda_k(p)$ $(1 \leqslant k \in \mathbb{N})$ be the k-th eigenvalue of

$$-y'' + py = \lambda y,\ y = y(x),\ x \in [0,1] \tag{2.5}$$

associated with the Dirichlet condition $y(0) = 0 = y(1)$. Let $\varphi(x,\lambda,p),\ \phi(x,\lambda,p)$ respectively be the solutions of equation (2.5) with initial condition:

$$\varphi(0) = 1,\ \varphi'(0) = 0;\ \phi(0) = 0,\ \phi'(0) = 1.$$

Define

$$\begin{cases} Z(x,p) = \phi(x,\lambda_k(p),p),\ C(\lambda,p) = \dfrac{\varphi(1,\lambda_k(p),p) - \varphi(1,\lambda,p)}{\phi(1,\lambda,p)}; \\ W(x,\lambda,p) = \varphi(x,\lambda,p) + C(\lambda,p)\phi(x,\lambda,p); \\ w(x,\lambda,p) = W(x,\lambda,p)Z'(x,p) - W'(x,\lambda,p)Z(x,p). \end{cases} \tag{2.6}$$

Clearly, as a function of λ, $C(\lambda,p)$ is analytic for $\lambda \in (\lambda_{k-1}(p),\lambda_{k+1}(p))$ with the removable singularity at $\lambda = \lambda_k(p)$.

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LEMMA 2.4. *(cf. [**13**, Theorem 6.1]) If $p \in E$ and $\lambda_{k-1}(p) < \lambda_k(p) + t < \lambda_{k+1}(p)$, then the potential*

$$q = p - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log w(x, \lambda_k(p) + t, p)$$

satisfies that $q \in E$ and

$$\lambda_j(q) = \lambda_j(p),\ j \neq k,\ 1 \leqslant j \in \mathbb{N};\ \lambda_k(q) = \lambda_k(p) + t,\ j = k.$$

The above result indicates that one can shift one eigenvalue $\lambda_k(p)$ of p to the desired position $\lambda_k(q)$ as long as $\lambda_{k-1}(p) < \lambda_k(q) < \lambda_{k+1}(p)$ and the other eigenvalues without moving. So that, we call lemma 2.4 'the shifting lemma of spectral set', or simply 'the spectral shifting lemma'.

The proof of theorem 1.1. The accessibility of $E_0(\lambda_1, \lambda_2)$ is clearly from the weakly* closeness of the set $\Omega_0(\lambda_1, \lambda_2)$. So, we need only to prove $E_0(\lambda_1, \lambda_2) = E(\lambda_1, \lambda_2)$. Let $q \in \Omega(\lambda_1, \lambda_2)$. It follows from (2.1) that there exist $\mu_q \in \Omega_0(\lambda_1, \lambda_2)$ and $\|q\|_1 = \|\mu_q\|_V$. Then, $E_0(\lambda_1, \lambda_2) \leqslant E(\lambda_1, \lambda_2)$. It remains to prove that $E_0(\lambda_1, \lambda_2) \geqslant E(\lambda_1, \lambda_2)$.

Firstly, we show that for every measure $\mu \in \Omega_0(\lambda_1, \lambda_2)$ and the even potential $q_0 = \mathrm{d}\mu$, there exists even q_n satisfies:

$$q_n \in \Omega(\lambda_1, \lambda_2),\ q_n \xrightarrow{w^*} q_0. \tag{2.7}$$

According to remark 2.2 and the symmetry of q_0, there exists $p_n \in E$ such that $p_n \xrightarrow{w^*} q_0$, for example, one can choose:

$$p_n = H_n(x - 1/2) * q_0(x),$$

where H_n is the standard Sobolev kernel function:

$$H_n(x) = cn \begin{cases} \exp\left(\dfrac{x^2}{x^2 - 1/(2n)^2}\right), & |x| < \dfrac{1}{2n}, \\ 0, & |x| \geqslant \dfrac{1}{2n}, \end{cases}$$

and c is a constant such that $\int_{\mathbb{R}} H_n(x)\,\mathrm{d}x = 1$. Write

$$\sigma(p_n) = (\lambda_1(p_n),\ \lambda_2(p_n),\ \lambda_3(p_n), \ldots) \in S,$$

$$\sigma_n = (\lambda_1,\ \lambda_2,\ \lambda_3(p_n), \ldots).$$

We first shift $\sigma(p_n)$ to $\sigma(\widehat{p}_n) = (\lambda_1,\ \lambda_2(n), \ldots, \lambda_j(n), \ldots)$ with $\widehat{p}_n = p_n + \lambda_1 - \lambda_1(p_n)$, where $\lambda_j(n) = \lambda_j(\widehat{p}_n)$ for $j \geqslant 2$. Clearly, $\widehat{p}_n \in E$. Since the eigenvalues of (1.6) are continuous in measure $\mu \in (\mathcal{M}_0, w^*)$, see [**12**, Theorem 1.3], it follows

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that:

$$\lambda_j(p_n) \to \lambda_j,\ n \to \infty,\ j = 1, 2.$$

Therefore, $\widehat{p}_n \xrightarrow{w^*} q_0$ and

$$\lambda_j(n) = \lambda_j(\widehat{p}_n) = \lambda_j(p_n) + \lambda_1 - \lambda_1(p_n) \to \lambda_j(q_0), j \geqslant 2$$

as $n \to \infty$. Replacing $\widehat{p}_n$ by p_n, we may assume that $\lambda_1(p_n) = \lambda_1$ and p_n can be selected to satisfy $\lambda_1(p_n) < \lambda_2 < \lambda_3(p_n),\ \forall 1 \leqslant n \in \mathbb{N}$.

Now, we can use the spectral shifting lemma, lemma 2.4 to find a potential $q_n \in E$ such that $\sigma(q_n) = \sigma_n$. In fact, if we take $t := t_n = \lambda_2 - \lambda_2(p_n)$ in lemma 2.4, then

$$q_n = p_n - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log w(x, \lambda_2(p_n) + t_n, p_n) = p_n - 2\left(\frac{w'(x, \lambda_2(p_n) + t_n, p_n)}{w(x, \lambda_2(p_n) + t_n, p_n)}\right)' \tag{2.8}$$

satisfies $\sigma(q_n) = \sigma_n$. Then, we need to prove $q_n \xrightarrow{w^*} q_0$ as $n \to \infty$. To this end, recall the definitions of w, W and Z in (2.6), one sees that $w(0) = 1$ and

$$Z(x, p_n) = \phi(x, \lambda_2(p_n), p_n), \tag{2.9}$$

$$W(x, \lambda_2, p_n) = \varphi(x, \lambda_2, p_n) + C_n(\lambda_2)\phi(x, \lambda_2, p_n), \tag{2.10}$$

respectively satisfy the equation:

$$-Z'' + p_n Z = \lambda_2(p_n) Z, \tag{2.11}$$

$$-W'' + p_n W = \lambda_2 W, \tag{2.12}$$

where

$$C_n(\lambda) = C(\lambda, p_n) = \frac{\varphi(1, \lambda_2(p_n), p_n) - \varphi(1, \lambda, p_n)}{\phi(1, \lambda, p_n)}.$$

And hence

$$w'(x, \lambda_2, p_n) = WZ'' - W''Z = t_n W(x, \lambda_2, p_n) Z(x, p_n). \tag{2.13}$$

Therefore,

$$p_n - q_n = 2t_n \left\{\frac{W'Z + WZ'}{w} - t_n \frac{(WZ)^2}{w^2}\right\}. \tag{2.14}$$

It follows from lemma 2.3 that, as $n \to \infty$:

$$\phi(\cdot, \lambda_2(p_n), p_n) \xrightarrow{\|\cdot\|_\infty} \phi(\cdot, \lambda_2, q_0), \quad \varphi(\cdot, \lambda_2, p_n) \xrightarrow{\|\cdot\|_\infty} \varphi(\cdot, \lambda_2, q_0),$$

$$\phi(\cdot, \lambda_2, p_n) \xrightarrow{\|\cdot\|_\infty} \phi(\cdot, \lambda_2, q_0), \quad \varphi(1, \lambda_2(p_n), p_n) \to \varphi(1, \lambda_2, q_0),$$

$$\varphi(1, \lambda, p_n) \to \varphi(1, \lambda, q_0), \quad \phi(1, \lambda, p_n) \to \phi(1, \lambda, q_0).$$

Hence, by definition (2.9), $Z(x, p_n)$ is uniformly bounded. Moreover, it holds that

$$C_n(\lambda) \to \frac{\varphi(1, \lambda_2, q_0) - \varphi(1, \lambda, q_0)}{\phi(1, \lambda, q_0)},\ n \to \infty \tag{2.15}$$

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uniformly on any compact subinterval of $(\lambda_1(q_0), \lambda_3(q_0))$. Then

$$\lim_{n\to\infty} C_n(\lambda_2) = \lim_{\lambda\to\lambda_2}\lim_{n\to\infty} C_n(\lambda) = -\frac{(\partial\varphi/\partial\lambda)(1,\lambda_2,q_0)}{(\partial\phi/\partial\lambda)(1,\lambda_2,q_0)}. \tag{2.16}$$

Hence, according to (2.10), $W(x,\lambda_2,p_n)$ is also uniformly bounded. It follows from (2.13) that

$$\begin{aligned} w(x,\lambda_2,p_n) &= w(0,\lambda_2,p_n) + \int_0^x w'(s,\lambda_2,p_n)\,\mathrm{d}s \\ &= 1 + t_n\int_0^x W(s,\lambda_2,p_n)Z(s,p_n)\,\mathrm{d}s. \end{aligned}$$

This together with $t_n \to 0,\ n\to\infty$ implies that w has a positive lower bound for sufficient large n. From (2.11) and (2.12), it follows that

$$\begin{cases} Z'(x,p_n) = 1 + \displaystyle\int_0^x (p_n - \lambda_2(p_n))Z(s,p_n)\,\mathrm{d}s, \\ W'(x,\lambda_2,p_n) = C_n(\lambda_2) + \displaystyle\int_0^x (p_n-\lambda_2)W(s,\lambda_2,p_n)\,\mathrm{d}s, \end{cases}$$

and according to $p_n \xrightarrow{w^*} q_0,\ \lambda_2(p_n)\to\lambda_2,\ n\to\infty$, the uniform boundedness of $Z(x,p_n)$ and $W(x,\lambda_2,p_n)$ and (2.16), Z' and W' are uniformly bounded. Hence (2.14) implies that there exists $M>0$ such that

$$\|p_n - q_n\|_\infty \leqslant M|t_n| \to 0,\ n\to\infty.$$

Therefore, $\forall f\in C[0,1]$:

$$\begin{aligned} \left|\int_0^1 (q_n-q_0)f\,\mathrm{d}t\right| &\leqslant \left|\int_0^1 (q_n-p_n)f\,\mathrm{d}t\right| + \left|\int_0^1 (p_n-q_0)f\,\mathrm{d}t\right| \\ &\leqslant \|f\|_\infty\|p_n-q_n\|_\infty + \left|\int_0^1 (p_n-q_0)f\,\mathrm{d}t\right| \to 0, \end{aligned}$$

i.e. $q_n \xrightarrow{w^*} q_0$ as $n\to\infty$. This proves (2.7).

Now, applying (2.7), we have

$$\|\mu\|_V = \int_0^1 |q_0| = \lim_{n\to\infty}\int_0^1 |q_n| \geqslant E(\lambda_1,\lambda_2).$$

Clearly, $E_0(\lambda_1,\lambda_2) \geqslant E(\lambda_1,\lambda_2)$ by the arbitrary of μ. This proves theorem 1.1. □

2.3. Quadratic form

Since we will use the min-max principle theory of quadratic form in the proof of the main result, in this section, we introduce some knowledge of the theory of quadratic form, see [**9**, **17**].

Let $\mathcal{D}$ be a subspace of a Hilbert space H. A mapping $\mathrm{t}[u,v] : \mathcal{D}\times\mathcal{D}\to\mathbb{R}$ is called a sesquilinear form on H if it is linear in $u\in\mathcal{D}$ and semilinear in $v\in\mathcal{D}$. $\mathcal{D}$

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will be called the domain of t and is denoted by $\mathcal{D}(\mathrm{t})$. $\mathrm{t}[u]=\mathrm{t}[u,u]$ will be called the quadratic form or simply form associated with $\mathrm{t}[u,v]$. For the problems:

$$\begin{cases}-y''+qy=\lambda y,\ y=y(x),\ x\in[0,1/2],\\ y(0)=0=y'(1/2),\end{cases} \tag{2.17}$$

and

$$\begin{cases}-y''=\lambda y,\ y=y(x),\ x\neq a\in(0,1/2),\\ y(0)=0=y'(1/2),\ y'(a-0)-y'(a+0)=ry(a),\end{cases} \tag{2.18}$$

where $q\in L^1[0,1/2]$ and $a,\ r$ are defined in theorem 1.1, the associated forms are respectively as follows:

$$\mathrm{s}_1[u,v]=\int_0^{1/2}(u'\overline{v}'+qu\overline{v})\,\mathrm{d}x, \tag{2.19}$$

$$\mathrm{t}_1[u,v]=\int_0^{1/2}u'\overline{v}'\,\mathrm{d}x-ru(a)\overline{v}(a), \tag{2.20}$$

where $u,v\in\mathcal{D}(\mathrm{s}_1)=\mathcal{D}(\mathrm{t}_1)$ and

$$\mathcal{D}(\mathrm{t}_1):=\left\{u\in L^2[0,1/2]:\ u\in AC[0,1/2],\ u'\in L^2[0,1/2],\ u(0)=0\right\}.$$

The proof is given in the Appendix. Similarly, for the problems:

$$\begin{cases}-y''+qy=\lambda y,\ y=y(x),\ x\in[0,1/2],\\ y(0)=0=y(1/2),\end{cases} \tag{2.21}$$

and

$$\begin{cases}-y''=\lambda y,\ y=y(x),\ x\neq a\in(0,1/2),\\ y(0)=0=y(1/2),\ y'(a-0)-y'(a+0)=ry(a),\end{cases} \tag{2.22}$$

where $q\in L^1[0,1/2]$ and $a,\ r$ are defined in theorem 1.1, the associated forms are respectively given by

$$\mathrm{s}_2[u,v]=\int_0^{1/2}(u'\overline{v}'+qu\overline{v})\,\mathrm{d}x, \tag{2.23}$$

$$\mathrm{t}_2[u,v]=\int_0^{1/2}u'\overline{v}'\,\mathrm{d}x-ru(a)\overline{v(a)}, \tag{2.24}$$

where $u,v\in\mathcal{D}(\mathrm{s}_2)=\mathcal{D}(\mathrm{t}_2)$ and

$$\mathcal{D}(\mathrm{t}_2):=\left\{u\in L^2[0,1/2]:\ u\in AC[0,1/2],\ u'\in L^2[0,1/2],\ u(0)=0=u(1/2)\right\}.$$

According to the min-max principle of form [**17**, Theorem XIII.2], which yields the specific expression of eigenvalue by the associated form, the following result holds.

LEMMA 2.5. (*cf. [17, Theorem XIII.2]*) *Let λ_k be the k-th eigenvalue of the eigenvalue problem* (2.17) *and* $\mathrm{s}_1[u]$ *the associated form given by* (2.19). *Then*

$$\lambda_k = \sup_{E_{k-1}} \inf_{\phi \in E_{k-1}^{\perp}} \left\{ \mathrm{s}_1[\phi]: \ \phi \in \mathcal{D}(\mathrm{s}_1), \ \|\phi\|_2 = 1 \right\},$$

where $1 \leqslant k \in \mathbb{N}$, E_{k-1} *is any* $k-1$ *dimensional subspace of* $L^2[0,1/2]$, *and* $E_{k-1}^{\perp}$ *expresses the orthogonal complement space of* E_{k-1} *in* $L^2[0,1/2]$. *Particularly, for* $k=1$ *we have*

$$\lambda_1 = \inf \left\{ \mathrm{s}_1[\phi]: \ \phi \in \mathcal{D}(\mathrm{s}_1), \ \|\phi\|_2 = 1 \right\}. \tag{2.25}$$

The similar conclusions hold for problems (2.18), (2.21), *and* (2.22).

2.4. The generalized Lyapunov inequality

Consider the boundary problem of the Sturm–Liouville equation

$$-y'' + qy = \lambda w y, \ y = y(x), \ x \in [c,d] \tag{2.26}$$

subjected to the general separated boundary condition:

$$c_1 y(c) - c_2 y'(c) = 0 = d_1 y(d) - d_2 y'(d), \tag{2.27}$$

where $q, w \in L^1([c,d], \ \mathbb{R})$, $c, \ d, \ c_1, \ c_2, \ d_1, \ d_2 \in \mathbb{R}$, $c_1^2 + c_2^2 \neq 0$, $d_1^2 + d_2^2 \neq 0$. Note that the coefficients $c_j, \ d_j, \ j = 1,2$ are allowed to be infinity. For example, if $c_1 = \infty$, then the condition $c_1 y(c) - c_2 y'(c) = 0$ is understood in the form $y(c) = 0$.

Let $u(x)$ and $v(x)$ satisfy:

$$\begin{cases} -u'' + qu = 0, \ c_1 u(c) - c_2 u'(c) = 0, \\ -v'' + qv = 0, \ d_1 v(d) - d_2 v'(d) = 0. \end{cases}$$

If zero is not an eigenvalue of problems (2.26) and (2.27), then

$$G(x,t) = -\frac{1}{W(u,v)} \begin{cases} u(x)v(t), & c \leqslant x < t, \\ u(t)v(x), & t < x \leqslant d \end{cases}$$

is the Green function associated with (2.26) and (2.27) at $\lambda = 0$, where $W(u,v) = uv' - vu'$ is the Wronskian of u and v.

The following lemma is the generalized Lyapunov inequality for Sturm–Liouville problems, which yields the infimum of the L^1-norm of the weights by eigenvalues.

LEMMA 2.6 (cf. [5]). *Consider the eigenvalue problems* (2.26) *and* (2.27) *with* $w \geqslant 0$ *a.e. on* $[c,d]$. *Suppose that zero is not an eigenvalue and* $G(x,t)$ *is the associated*

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Green function at $\lambda = 0$*, then*

$$\int_c^d G(x,x)w(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{\lambda_n},$$

where λ_n *is the* n*-th eigenvalue of the problem. Furthermore,*

$$\int_c^d w(x)\,\mathrm{d}x \geqslant \frac{1}{G}\left|\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\right|, \; G = \max\{|G(x,x)| : \; x \in [c,d]\}.$$

Particularly, if $0 < \lambda_1 \leqslant 1$*, then*

$$\int_c^d w(x)\,\mathrm{d}x > \frac{1}{G}, \tag{2.28}$$

and $\frac{1}{G}$ *is the best constant.*

Notice that the generalized Lyapunov inequality (2.28) requires the positivity of both the first eigenvalue and the weight $w(x)$, it cannot be applied directly to the present situation of this paper. Recall that the positivity of w is not required in the classical Lyapunov inequality. In fact, one can prove that the positivity of w is not necessary while applying lemma 2.6.

LEMMA 2.7. *Consider the eigenvalue problems* (2.26) *and* (2.27)*. If* (2.26) *and* (2.27) *has a nontrivial solution for* $\lambda = 1$ *and all the eigenvalues of* (2.26) *and* (2.27) *with* $w(x) \equiv 1$ *are positive, then*

$$\int_c^d |w(x)|\,\mathrm{d}x > \frac{1}{G}, \; G = \max\{|G(x,x)| : \; x \in [c,d]\} \tag{2.29}$$

and the constant $\frac{1}{G}$ *is best, where* $G(x,t)$ *is the Green function of* (2.26) *and* (2.27) *at* $\lambda = 0$*.*

Proof. Since the quadratic form associated with (2.26) and (2.27) with $w \geqslant 0$ and $\int_c^d w\,\mathrm{d}x > 0$ is

$$\mathrm{t}[u] = \int_c^d \left[|u'|^2 + q|u|^2\right]\,\mathrm{d}x - \beta|u(d)|^2 + \alpha|u(c)|^2, \; \alpha = c_1/c_2, \; \beta = d_1/d_2 \tag{2.30}$$

with

$$\mathcal{D}(\mathrm{t}) := \left\{u \in L^2[c,d] : \; u \in AC[c,d], \; u' \in L^2[c,d]\right\}.$$

The proof is given in the Appendix. Then, according to lemma 2.5, all the eigenvalues of (2.26) and (2.27) with $w(x) \equiv 1$ are positive means that

$$\mathrm{t}[u] > 0, \; \forall u \in \mathcal{D}(\mathrm{t}).$$

From this fact, we must have $w_+(x) \not\equiv 0$. For otherwise, there would have $w(x) \leqslant 0$ on $[c,d]$ and $\int_c^d |w|\mathrm{d}x > 0$, which implies that -1 is the first eigenvalue of

$$-y'' + qy = \lambda w_- y, \; x \in [c,d], \; \mathbb{B}y = 0 \tag{2.31}$$

with an eigenfunction $\phi \in \mathcal{D}(\mathrm{t})$, and hence $\mathrm{t}[\phi] = -\int_0^1 w_-|\phi|^2 \mathrm{d}x \leqslant 0$, a contradiction. As a result, 1 is an eigenvalue of the eigenvalue problem:

$$-y'' + (q + w_-)y = \mu w_+ y, \ \mathbb{B}y = 0,$$

and hence the first eigenvalue $\mu(w_+)$ of the eigenvalue problem

$$-y'' + qy = \mu w_+ y, \ \mathbb{B}y = 0$$

satisfies that $0 < \mu_1(w_+) \leqslant 1$ by the monotonicity of eigenvalues on potentials. Now, applying lemma 2.6 we have

$$\int_c^d |w(x)|\,\mathrm{d}x \geqslant \int_c^d w_+(x)\,\mathrm{d}x > \frac{1}{G}. \qquad \square$$

3. The proof of theorem 1.2

This section presents the proof of theorem 1.2. In order to make the proof clearly, we provide the outline of the proof in this section.

We first find a $\mu_0 \in \Omega_0(\lambda_1, \lambda_2)$ such that $\|\mu_0\|_V = 2r$ in lemma 3.2, hence $2r \geqslant E_0(\lambda_1, \lambda_2)$. Since $E(\lambda_1, \lambda_2) = E_0(\lambda_1, \lambda_2)$ by theorem 1.1, it is sufficient to prove $E(\lambda_1, \lambda_2) \geqslant 2r$ to have $E(\lambda_1, \lambda_2) = 2r$. This is $\int_0^{1/2} |q(x)|\,\mathrm{d}x \geqslant r, \ \forall q \in \Omega(\lambda_1, \lambda_2)$ by the symmetry of the potentials. To this end, we will use lemmas 2.7 and 2.5. The accessibility in theorem 1.2 can be obtained by theorem 1.1 and lemma 3.2.

Firstly, the following lemma guarantees the existence and uniqueness of the point a defined in theorem 1.2.

Lemma 3.1. *The point a which is defined in theorem 1.2 exists uniquely.*

Proof. Set $H(x) = h_1(x) - h_2(x)$ for $x \in [0, 1/2]$, where

$$h_1(x) = \begin{cases} \rho_1 \left[\cot(\rho_1 x) - \tan \rho_1(1/2 - x)\right], & 0 \leqslant \lambda_1 < \pi^2, \ \rho_1 = \sqrt{\lambda_1}, \\ \tau_1 \left[\coth(\tau_1 x) + \tanh \tau_1(1/2 - x)\right], & \lambda_1 < 0, \ \tau_1 = \sqrt{-\lambda_1}. \end{cases} \tag{3.1}$$

$$h_2(x) = \begin{cases} \rho_2 \left[\cot(\rho_2 x) + \cot \rho_2(1/2 - x)\right], & 0 \leqslant \lambda_2 < 4\pi^2, \ \rho_2 = \sqrt{\lambda_2}, \\ \tau_2 \left[\coth(\tau_2 x) + \coth \tau_2(1/2 - x)\right], & \lambda_2 < 0, \ \tau_2 = \sqrt{-\lambda_2}. \end{cases} \tag{3.2}$$

Particularly, if $\lambda_1 = 0$, (3.1) is reduced to $h_1(x) = 1/x$ and if $\lambda_2 = 0$, (3.2) is reduced to $h_2(x) = 1/x + 1/(1/2 - x)$.

One can verify that $h_1(x)$ is decreasing on $(0, 1/2]$ for $\lambda_1 \in (-\infty, \pi^2)$ and

$$h_1(0+0) = +\infty, \ h_1(1/2) = \begin{cases} \rho_1 \cot(\rho_1/2), & 0 \leqslant \lambda_1 < \pi^2, \\ \tau_1 \coth(\tau_1/2), & \lambda_1 < 0, \end{cases}$$

$$h_1(1/4) = \begin{cases} \rho_1 \left[\cot(\rho_1/4) - \tan(\rho_1/4)\right], & 0 \leqslant \lambda_1 < \pi^2, \\ \tau_1 \left[\coth(\tau_1/4) + \tanh(\tau_1/4)\right], & \lambda_1 < 0. \end{cases}$$

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Similarly, $h_2(x)$ is decreasing and increasing on $(0, 1/4]$ and $[1/4, 1/2)$, respectively for $\lambda_2 \in (\lambda_1, 4\pi^2)$, and

$$h_2(0+0) = h_2(1/2-0) = +\infty,\ h_2(1/4) = \begin{cases} 2\rho_2 \cot(\rho_2/4), & 0 \leqslant \lambda_2 < 4\pi^2, \\ 2\tau_2 \coth(\tau_2/4), & \lambda_2 < 0. \end{cases}$$

As a result, condition (1.5) yields that $H(1/4) \geqslant 0$. This together with

$$H(1/2-0) = h_1(1/2) - h_2(1/2-0) = -\infty$$

implies that there exists $a \in [1/4, 1/2)$ such that $H(a) = 0$. The uniqueness comes from the monotonicity of h_1 and h_2 on $[1/4, 1/2)$. □

Since the potential q is symmetric on $[0, 1]$, the original problem is equivalent to that both of the following two problems

$$-y'' - \lambda_1 y = \mu w y,\ y(0) = 0 = y'(1/2), \qquad \textbf{(P1)}$$
$$-y'' - \lambda_2 y = \gamma w y,\ y(0) = 0 = y(1/2) \qquad \textbf{(P2)}$$

possess the first eigenvalue $\mu_1(w) = \gamma_1(w) = 1$, where $w = -q$. With these notations, one can prove the following lemma.

LEMMA 3.2. *Suppose that* λ_1 *and* λ_2 *satisfy* (1.5). *Let* a *and* r *be defined in theorem* 1.2 *and* $w := w_0 = r\delta(x-a)$. *Then, both of the problems* **(P1)** *and* **(P2)** *possess the first eigenvalue* $\mu_1(w_0) = \gamma_1(w_0) = 1$.

Proof. Using the definitions of r and a, that is $h_1(a) = h_2(a)$ and

$$r = h_1(a) = \begin{cases} \rho_1 \left[\cot(\rho_1 a) - \tan \rho_1 (1/2 - a)\right], & 0 \leqslant \lambda_1 < \pi^2, \\ \tau_1 \left[\coth(\tau_1 a) + \tanh \tau_1 (1/2 - a)\right], & \lambda_1 < 0, \end{cases} \tag{3.3}$$

it is easy to verify that

$$\psi_1(x) = \begin{cases} \sin(\rho_1 x), & x \in [0, a], \\ \alpha \cos \rho_1 (1/2 - x), & x \in [a, 1/2] \end{cases} \tag{3.4}$$

is the first eigenfunction of **(P1)** with $\mu_1(w_0) = 1$, where

$$\alpha = \sin(\rho_1 a) / \cos \rho_1 \left(1/2 - a\right). \tag{3.5}$$

Similarly, one can verify that

$$\psi_2(x) = \begin{cases} \sin(\rho_2 x), & x \in [0, a], \\ \beta \sin \rho_2 (1/2 - x), & x \in [a, 1/2] \end{cases} \tag{3.6}$$

is the first eigenfunction of **(P2)** with $\gamma_1(w_0) = 1$, where

$$\beta = \sin(\rho_2 a) / \sin \rho_2 (1/2 - a). \tag{3.7}$$

□

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The proof of theorem 1.2. By the explanation at the beginning of this section, we only need to prove:

$$\int_0^{1/2} |q(x)|\,\mathrm{d}x \geqslant r \text{ or } \int_0^1 |q(x)|\,\mathrm{d}x \geqslant 2r,\ \forall q \in \Omega(\lambda_1, \lambda_2).$$

Set $-w = q \in \Omega(\lambda_1, \lambda_2)$. Let ϕ_1 and ϕ_2 be the first positive eigenfunctions of **(P1)** and **(P2)**, respectively in the meaning that both of ϕ_1 and ϕ_2 are positive in $(0, 1/2)$. Let b_j be the biggest one of the maximum points of $\phi_j(x)$ on $[0, 1/2]$ for $j = 1, 2$. Note that although the first positive eigenfunction is not unique, the points b_1 and b_2 do not change, and hence the uniqueness of b_j is well-defined. Let a be defined as in theorem 1.2. The proof will be divided into three cases according to the relations between a and b_j. □

Case 1. $b_1 \leqslant a$. Consider the following two problems:

$$-y'' - \lambda_1 y = \mu w y,\ y(0) = 0 = y'(b_1), \qquad (\mathbf{P11})$$
$$-y'' - \lambda_1 y = \mu w y,\ y'(b_1) = 0 = y'(1/2). \qquad (\mathbf{P12})$$

Clearly, ϕ_1 is a non-trivial solution of the above problems with $\mu = 1$.

If $\lambda_1 < 0$, then all the first eigenvalue of **(P11)** and **(P12)** with w replaced by 1 are positive, and hence, lemma 2.7 can be applied to problems **(P11)** and **(P12)** such that the following inequalities

$$\int_0^{b_1} |w(x)|\,\mathrm{d}x > \frac{1}{G_{11}},\ \int_{b_1}^{1/2} |w(x)|\mathrm{d}x > \frac{1}{G_{12}} \tag{3.8}$$

hold, where G_{11} and G_{12} are the maximums of $|G_{11}(x, x)|$ and $|G_{12}(x, x)|$, respectively, and $G_{11}(x, t)$, $G_{12}(x, t)$ are the Green functions associated with **(P11)**, **(P12)** at $\mu = 0$, respectively. Since the equation $-y'' - \lambda_1 y = 0$ can be solved, one can verify that

$$\begin{cases} G_{11}(x, x) = \dfrac{1}{\tau_1 \cosh \tau_1 b_1} \sinh(\tau_1 x) \cosh \tau_1 (b_1 - x), & x \in [0, b_1], \\ G_{12}(x, x) = \dfrac{1}{\tau_1 \sinh \tau_1 (1/2 - b_1)} \cosh \tau_1 (x - b_1) \cosh \tau_1 (1/2 - x), & x \in [b_1, 1/2]. \end{cases} \tag{3.9}$$

Furthermore, a calculation yields that

$$\begin{cases} G_{11} = |G_{11}(b_1, b_1)| = \dfrac{1}{\tau_1} \tanh(\tau_1 b_1), \\ G_{12} = |G_{12}(b_1, b_1)| = \dfrac{1}{\tau_1} \coth \tau_1 (1/2 - b_1). \end{cases} \tag{3.10}$$

Then, using the definition of h_1 in (3.1), the inequalities in (3.8) yield that

$$\int_0^{1/2} |w(x)|\,\mathrm{d}x = \int_0^{b_1} |w(x)|\,\mathrm{d}x + \int_{b_1}^{1/2} |w(x)|\,\mathrm{d}x > \frac{1}{G_{11}} + \frac{1}{G_{12}} = h_1(b_1). \tag{3.11}$$

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If $0 \leqslant \lambda_1 < \pi^2$, then the first eigenvalue of **(P11)** with w replaced by 1 is still positive. Then, applying lemma 2.7 again we arrive at

$$\int_0^{b_1} |w(x)|\mathrm{d}x > \frac{1}{G_{11}} = \rho_1 \cot(\rho_1 b_1),$$

where $G_{11} = |G_{11}(b_1, b_1)| = \tan(\rho_1 b_1)/\rho_1$ and $G_{11}(x,t)$ is the Green function associated with **(P11)** at $\mu = 0$. For this case, $G_{11}(x,x)$ is given by

$$G_{11}(x,x) = \frac{1}{\rho_1 \cos \rho_1 b_1} \sin(\rho_1 x) \cos \rho_1 (b_1 - x), \ \ x \in [0, b_1].$$

This together with $\tan \rho_1 (1/2 - b_1) \geqslant 0$ and the definition of h_1 in (3.1) yields that

$$\int_0^{1/2} |w(x)| \, \mathrm{d}x \geqslant \int_0^{b_1} |w(x)|\mathrm{d}x > \rho_1 \cot(\rho_1 b_1) \geqslant h_1(b_1). \tag{3.12}$$

From now on, we arrive at that for all cases of λ_1, it holds that:

$$\int_0^{1/2} |w(x)| \, \mathrm{d}x > h_1(b_1).$$

Since $0 < b_1 \leqslant a < 1/2$ and $h_1(x)$ is decreasing on $(0, 1/2]$ by lemma 3.1, one sees that $h_1(b_1) \geqslant h_1(a) = r$, and hence

$$\int_0^1 |q(x)|\mathrm{d}x = \int_0^1 |w(x)|\mathrm{d}x = 2 \int_0^{1/2} |w(x)|\mathrm{d}x > 2r.$$

Case 2. $b_2 \geqslant a$. Consider the following two problems:

$$\begin{aligned} &- y'' - \lambda_2 y = \mu w y, \ \ y(0) = 0 = y'(b_2), \quad \textbf{(P21)} \\ &- y'' - \lambda_2 y = \mu w y, \ \ y'(b_2) = 0 = y(1/2). \quad \textbf{(P22)} \end{aligned}$$

Clearly, ϕ_2 is a non-trivial solution of the above problems with $\mu = 1$. Let $G_{21}(x,t)$ and $G_{22}(x,t)$ be the Green functions associated with **(P21)** and **(P22)** at $\mu = 0$, respectively since zero is not an eigenvalue.

Since $4\pi^2 \geqslant (\pi/2b_2)^2$, one can verify that for $x \in [0, b_2]$:

$$G_{21}(x,x) = \begin{cases} \dfrac{1}{\rho_2 \cos \rho_2 b_2} \sin(\rho_2 x) \cos \rho_2 (b_2 - x), & 0 \leqslant \lambda_2 < (\pi/2b_2)^2, \\ \dfrac{1}{\tau_2 \cosh \tau_2 b_2} \sinh(\tau_2 x) \cosh \tau_2 (b_2 - x), & \lambda_2 < 0. \end{cases}$$

A calculation yields that

$$G_{21} = \begin{cases} |G_{21}(b_2, b_2)| = \dfrac{1}{\rho_2} \tan(\rho_2 b_2), & 0 \leqslant \lambda_2 < (\pi/2b_2)^2, \\ |G_{21}(b_2, b_2)| = \dfrac{1}{\tau_2} \tanh(\tau_2 b_2), & \lambda_2 < 0, \end{cases}$$

where G_{21} is the maximum of $|G_{21}(x,x)|$.

Similarly, by $(\pi/(2(1/2-b_2)))^2 \geqslant 4\pi^2$, one sees that for $x \in [b_2, 1/2]$:

$$G_{22}(x,x) = \begin{cases} \dfrac{1}{\rho_2 \cos\rho_2(1/2-b_2)} \cos\rho_2(x-b_2)\sin\rho_2(1/2-x), & 0 \leqslant \lambda_2 < 4\pi^2, \\ \dfrac{1}{\tau_2 \cosh\tau_2(1/2-b_2)} \cosh\tau_2(x-b_2)\sinh\tau_2(1/2-x), & \lambda_2 < 0, \end{cases}$$

$$G_{22} = \begin{cases} |G_{22}(b_2,b_2)| = \dfrac{1}{\rho_2}\tan\rho_2(1/2-b_2), & 0 \leqslant \lambda_2 < 4\pi^2, \\ |G_{22}(b_2,b_2)| = \dfrac{1}{\tau_2}\tanh\tau_2(1/2-b_2), & \lambda_2 < 0, \end{cases}$$

where G_{22} is the maximum of $|G_{22}(x,x)|$.

As a result, if $4\pi^2 \geqslant (\pi/2b_2)^2 > \lambda_2$, the similar argument as in case 1 yields that

$$\int_0^{1/2} |w(x)|\,\mathrm{d}x = \int_0^{b_2} |w(x)|\,\mathrm{d}x + \int_{b_2}^{1/2} |w(x)|\,\mathrm{d}x > \frac{1}{G_{21}} + \frac{1}{G_{22}} = h_2(b_2),$$

where $h_2(x)$ is defined in (3.2).

If $(\pi/2b_2)^2 \leqslant \lambda_2 < 4\pi^2$, then one can verify that the first eigenvalue of **(P22)** with w replaced by 1 is still positive by the fact that $b_2 \geqslant a \in [1/4, 1/2)$. It follows from lemma 2.7, the definition of h_2 and $\cos(\rho_2 b_2) \leqslant 0$ that

$$\int_0^{1/2} |w(x)|\,\mathrm{d}x \geqslant \int_{b_2}^{1/2} |w(x)|\,\mathrm{d}x > \frac{1}{G_{22}} \geqslant h_2(b_2).$$

Then, for all cases of λ_2, it holds that

$$\int_0^{1/2} |w(x)|\,\mathrm{d}x > h_2(b_2).$$

Since $1/2 > b_2 \geqslant a \geqslant 1/4$ and $h_2(x)$ is increasing on $[1/4, 1/2)$ by lemma 3.1, one sees that $h_2(b_2) \geqslant h_2(a) = r$, and hence $\int_0^1 |q(x)|\,\mathrm{d}x > 2r$.

Case 3. $a \in (b_2, b_1)$. Since ϕ_1 and ϕ_2 are the first eigenfunctions of **(P1)** and **(P2)**, respectively, they satisfy $\phi_1 \in \mathcal{D}(\mathrm{s}_1)$, $\phi_2 \in \mathcal{D}(\mathrm{s}_2)$ and

$$\lambda_1 \int_0^{1/2} |\phi_1|^2\,\mathrm{d}x = \int_0^{1/2} \left(|\phi_1'|^2 + q|\phi_1|^2\right)\mathrm{d}x = \mathrm{s}_1[\phi_1], \tag{3.13}$$

$$\lambda_2 \int_0^{1/2} |\phi_2|^2\,\mathrm{d}x = \int_0^{1/2} \left(|\phi_2'|^2 + q|\phi_2|^2\right)\mathrm{d}x = \mathrm{s}_2[\phi_2], \tag{3.14}$$

where s_1, s_2 are defined in (2.19) and (2.23), respectively.

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From lemma 3.2, λ_1 is the first eigenfunctions of the problem:

$$\begin{cases} -y'' - r\delta(x-a)y = \lambda y, \ y = y(x), \ x \in [0, 1/2], \\ y(0) = 0 = y'(1/2), \end{cases}$$

i.e. problem (2.18). And the min-max principle in lemma 2.5 yields that

$$\lambda_1 = \inf \{ \mathrm{t}_1[u] : \ u \in \mathcal{D}(\mathrm{t}_1), \ \|u\|_2 = 1 \} \leqslant \frac{\mathrm{t}_1[\phi_1]}{\|\phi_1\|_2^2},$$

where t_1 is defined in (2.20), and hence we have

$$\mathrm{s}_1[\phi_1] \leqslant \mathrm{t}_1[\phi_1],$$

that is

$$\int_0^{1/2} \left(|\phi_1'|^2 + q|\phi_1|^2 \right) \mathrm{d}x \leqslant \int_0^{1/2} |\phi_1'|^2 \,\mathrm{d}x - r|\phi_1(a)|^2.$$

Then (recall that $w = -q$)

$$\int_0^{1/2} w|\phi_1|^2 \,\mathrm{d}x \geqslant r|\phi_1(a)|^2. \tag{3.15}$$

Similarly,

$$\int_0^{1/2} w|\phi_2|^2 \,\mathrm{d}x \geqslant r|\phi_2(a)|^2. \tag{3.16}$$

Set

$$A(x, t) = \phi_1^2(x) + t^2\phi_2^2(x), \ x \in [0, 1/2], \ t \geqslant 0.$$

Let $M(t)$ be the biggest one of the maximum points of $A(x, t)$ on $[0, 1/2]$. Clearly, $M(t)$ is continuous on t, $M(0) = b_1$ and $M(\infty) = b_2$. Then, there exists $t_0 \in (0, \infty)$ such that $M(t_0) = a$. From (3.15) and (3.16), there has

$$\begin{aligned} r\left(|\phi_1(a)|^2 + t_0^2 |\phi_2(a)|^2 \right) &\leqslant \int_0^{1/2} w \left(|\phi_1|^2 + t_0^2 |\phi_2|^2 \right) \mathrm{d}x \\ &\leqslant \int_0^{1/2} |w| \left(|\phi_1|^2 + t_0^2 |\phi_2|^2 \right) \mathrm{d}x. \end{aligned}$$

Since $M(t_0) = a$, the above inequality yields that

$$rA(a, t_0) \leqslant A(a, t_0) \int_0^{1/2} |w| \mathrm{d}x,$$

which yields that $\int_0^{1/2} |w| \mathrm{d}x \geqslant r$, or $\int_0^1 |q| \mathrm{d}x \geqslant 2r$. This completes the proof of theorem 1.2.

Remark 3.3. The set of λ_1 and λ_2 that meets the restrictions in (1.5) is not empty.

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Proof. Set

$$R_1(z) = z\,[\cot(z/4) - \tan(z/4)] = 2z\cot(z/2),\ \ z \in [0,\pi],$$

and

$$R_2(z) = 2z\cot(z/4),\ \ z \in [0,2\pi].$$

Define

$$R_1(0) = \lim_{z\to 0} R_1(z) = 4,\ \ R_2(0) = \lim_{z\to 0} R_2(z) = 8.$$

Clearly, $R_1(z)$ is decreasing on $[0,\pi]$ and $R_1(\pi) = 0$. $R_2(z)$ is decreasing on $[0,2\pi]$ and $R_2(2\pi) = 0$.

It follows that if $0 \leqslant \lambda_1 < \pi^2$ is fixed, then $R_1(\rho_1) > 0$ for $\rho_1 = \sqrt{\lambda_1}$, and hence, there exists $\delta(\rho_1) > 0$ such that for $\sqrt{\lambda_2} = \rho_2 \in (2\pi - \delta(\rho_1), 2\pi)$, the inequality

$$\rho_1\left(\cot\frac{\rho_1}{4} - \tan\frac{\rho_1}{4}\right) \geqslant 2\rho_2\cot\frac{\rho_2}{4}$$

holds. This means that the admissible set of the restrictions on λ_1 and λ_2 in (1.5) is not empty.

If λ_2 is fixed. Set $\tau_1 = \sqrt{-\lambda_1}$ for $\lambda_1 < 0$ and let $z = i\tau$ with $\tau > 0$, then it follows from

$$\sin z = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{2i},\ \ \cos z = \frac{\mathrm{e}^{iz} + \mathrm{e}^{-iz}}{2}$$

that

$$R_1(z) = R_1(i\tau) = 2\tau\coth(\tau/2),\ \ \tau > 0.$$

$R_1(i\tau)$ is clearly increasing on $\tau > 0$, and

$$R_1(0) = \lim_{\tau\to 0} R_1(i\tau) = 4,\ \ R_1(i\tau) \to \infty,\ \ \tau \to \infty.$$

Therefore, for fixed $\lambda_2 \in \mathbb{R}$, there exists $M(\lambda_2) > 0$ such that for $\tau_1 > M(\lambda_2)$, the inequality

$$2\tau_1\coth(\tau_1/2) \geqslant 2\tau_2\coth(\tau_2/4)$$

holds for $\lambda_2 < 0$, $\tau_2 = \sqrt{-\lambda_2}$ or

$$\tau_1\left(\coth\frac{\tau_1}{4} + \tanh\frac{\tau_1}{4}\right) \geqslant 2\rho_2\cot\frac{\rho_2}{4}$$

for $\lambda_2 \geqslant 0$, which means condition (1.5) holds. □

REMARK 3.4. In this remark, we compare our results with the results of the case $m = 1$.

For $m = 1$, the infimum of the L^1-norm of q is given by

$$E_n(\lambda) = \inf\{\|q\|_1 :\ q \in \Omega_n(\lambda)\},$$

where

$$\Omega_n(\lambda) = \{q :\ q \in L^1[0,1],\ \lambda_n(q) = \lambda\},\ n \geqslant 1.$$

Recall that in the present paper:

$$\Omega(\lambda_1, \lambda_2) = \{q :\ q \in L^1[0,1],\ \lambda_1(q) = \lambda_1,\ \lambda_2(q) = \lambda_2\},$$

$$E(\lambda_1, \lambda_2) = \inf\{\|q\|_1 :\ q \in \Omega(\lambda_1, \lambda_2)\}.$$

It follows that

$$\Omega(\lambda_1, \lambda_2) = \Omega_1(\lambda_1) \cap \Omega_2(\lambda_2),$$

and hence, it must hold:

$$E(\lambda_1, \lambda_2) \geqslant \min\{E_1(\lambda_1),\ E_2(\lambda_2)\}. \tag{3.17}$$

Specific numerical explanation is given below. In fact $E(\lambda_1, \lambda_2) = 2r$, where

$$\begin{aligned} r &= \rho_1[\cot(\rho_1 a) - \tan\rho_1(1/2 - a)] = h_1(a) \\ &= \rho_2[\cot(\rho_2 a) + \cot\rho_2(1/2 - a)] = h_2(a). \end{aligned}$$

From the result of [14] (see theorem 1.1):

$$E_n(\lambda) = 2n\sqrt{\lambda}\cot\frac{\sqrt{\lambda}}{2n},\ \lambda \leqslant n^2\pi^2,\ n \geqslant 1,$$

it follows that

$$E_1(\lambda_1) = 2\rho_1\cot(\rho_1/2) = 2h_1(1/2),\ E_2(\lambda_2) = 4\rho_2\cot(\rho_2/4) = 2h_2(1/4).$$

Note that $h_1(x)$ is strictly decreasing, and $h_2(x)$ is strictly increasing for $x \in [1/4, 1/2]$. Since $1/2 > a \geqslant 1/4$, then

$$E(\lambda_1, \lambda_2) = 2r = 2h_1(a) > 2h_1(1/2) = E_1(\lambda_1),$$

$$E(\lambda_1, \lambda_2) = 2r = 2h_2(a) \geqslant 2h_2(1/4) = E_2(\lambda_2).$$

This means inequality (3.17).

Acknowledgements
This research was partially supported by the NSF of China [Grant numbers 12271299 and 12071254]. The authors are grateful to Professor Xiaoping Yuan for his helpful discussions and guidance.

https://doi.org/10.1017/prm.2024.28 Published online by Cambridge University Press

Appendix A.

The appendix explains the supplementary statement of §2.3. Some basic facts for the form t are briefly reviewed, see [9]. t is densely defined if $\mathcal{D}(\mathrm{t})$ is dense in Hilbert space H. t is said to be symmetric if

$$\mathrm{t}[u,v]=\overline{\mathrm{t}[v,u]},\ u,v\in\mathcal{D}(\mathrm{t}).$$

A symmetric form t is said to be bounded from below if

$$\mathrm{t}[u]\geqslant\gamma\|u\|^2,\ u\in\mathcal{D}(\mathrm{t}),$$

where $\gamma\in\mathbb{R}$ is a constant.

A form t is closed if $u_n\in\mathcal{D}(\mathrm{t}),\ u_n\to u$ in H and $\mathrm{t}[u_n-u_m]\to 0,\ n,m\to\infty$, then there has $u\in\mathcal{D}(\mathrm{t})$ and $\mathrm{t}[u_n-u]\to 0,\ n\to\infty$.

For the problem

$$\begin{cases}\tau_1 y:=-y''+qy=\lambda y,\ y=y(x),\ x\in[0,1/2],\\ y(0)=0=y'(1/2),\end{cases}\tag{A.1}$$

and

$$\begin{cases}\tau_2 y:=-y''=\lambda y,\ y=y(x),\ a\neq x\in[0,1/2],\\ y(0)=0=y'(1/2),\ y'(a-0)-y'(a+0)=ry(a),\end{cases}\tag{A.2}$$

where $q\in L^1[0,1/2]$ and $a,\ r$ are defined in theorem 1.1, let S_1 and T_1 be the corresponding operators to (A.1) and (A.2), respectively, i.e. $S_1y=f,\ y\in\mathcal{D}(S_1)$ if $\tau_1y=f$ for some $f\in L^2[0,1/2]$ and

$$\mathcal{D}(S_1):=\left\{y\in L^2[0,1/2]:\ y,\ y'\in AC[0,1/2],\ y(0)=0=y'(1/2)\right\}.$$

Similarly, $T_1y=f,\ y\in\mathcal{D}(T_1)$ if $\tau_2y=f,\ x\neq a$ for some $f\in L^2[0,1/2]$ and

$$\mathcal{D}(T_1):=\left\{\begin{array}{c}y\in L^2[0,1/2]:\ y\in AC[0,1/2],\ y'\in AC[0,a)\cup(a,1/2],\\ y(0)=0=y'(1/2),\ y'(a-0)-y'(a+0)=ry(a)\end{array}\right\}.$$

Consider the form:

$$\mathrm{s}_1[u,v]=\int_0^{1/2}(u'\overline{v}'+qu\overline{v})\,\mathrm{d}x,\tag{A.3}$$

$$\mathrm{t}_1[u,v]=\int_0^{1/2}u'\overline{v}'\,\mathrm{d}x-ru(a)\overline{v}(a),\tag{A.4}$$

where

$$u,v\in\mathcal{D}(\mathrm{s}_1)=\mathcal{D}(\mathrm{t}_1):=\left\{u\in L^2[0,1/2]:\ u\in AC[0,1/2],\ u'\in L^2[0,1/2],\ u(0)=0\right\}.$$

LEMMA A.1. *$S_1,\ T_1$ is respectively the self-adjoint operator associated with the quadratic form $\mathrm{s}_1,\ \mathrm{t}_1$.*

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Proof. From [**9**, IV-(1.19)], for $\forall\epsilon > 0$, there exists $\Gamma(\epsilon) > 0$ such that

$$\|y\|_\infty^2 \leqslant \epsilon\|y'\|_2^2 + \Gamma(\epsilon)\|y\|_2^2,\ y \in \left\{y \in AC[0,1/2],\ y' \in L^2[0,1/2]\right\}.$$

Then

$$\mathrm{s}_1[u] = \int_0^{1/2} \left(|u'|^2 + q|u|^2\right) \mathrm{d}x \geqslant \int_0^{1/2} |u'|^2 \,\mathrm{d}x - \epsilon\|q\|_1\|u'\|_2^2 - \Gamma(\epsilon)\|q\|_1\|u\|_2^2. \tag{A.5}$$

We first show that s_1 is closed. Let $u_n \in \mathcal{D}(\mathrm{s}_1),\ u_n \to u$ in $L^2[0,1/2]$, $\mathrm{s}_1[u_n - u_m] \to 0,\ n,m \to \infty$, and $\epsilon\|q\|_1 < 1/2$. From (A.5), we have:

$$\mathrm{s}_1[u_n - u_m] \geqslant \frac{1}{2}\int_0^{1/2} |u_n' - u_m'|^2 \,\mathrm{d}x - \Gamma(\epsilon)\|q\|_1\|u_n - u_m\|_2^2.$$

Since $\mathrm{s}_1[u_n - u_m] \to 0,\ \|u_n - u_m\|_2^2 \to 0,\ n,m \to \infty$, then $\|u_n' - u_m'\|_2^2 \to 0,\ n,m \to \infty$, which means $\{u_n'\}$ is a Cauchy sequence in $L^2[0,1/2]$. Then, there exists $\widehat{u} \in L^2[0,1/2]$ such that $u_n' \to \widehat{u},\ n \to \infty$ in $L^2[0,1/2]$ and hence

$$u_n(x) = u_n(0) + \int_0^x u_n' \,\mathrm{d}t = \int_0^x u_n' \,\mathrm{d}t \to \int_0^x \widehat{u} \,\mathrm{d}t,\ n \to \infty.$$

Since $u_n \to u$ in $L^2[0,1/2]$, then $u(x) = \int_0^x \widehat{u}\,\mathrm{d}t,\ \ x \in [0,1/2]$. Hence $u' = \widehat{u} \in L^2[0,1/2],\ u(0) = 0$, i.e. $u \in \mathcal{D}(\mathrm{s}_1)$. From $\mathrm{s}_1[u_n - u] \to 0,\ n \to \infty$, it follows that s_1 is closed.

Since q is real-valued, s_1 is symmetric. It follows from (A.5) that s_1 is bounded from below. And s_1 is also densely defined. Then, according to Theorem 2.1 of chapter VI in [**9**], there is exactly one self-adjoint operator associated with s_1. We show that is S_1.

Since $\mathcal{D}(S_1) \subset \mathcal{D}(\mathrm{s}_1)$ and $\mathrm{s}_1[u,v] = \langle S_1 u, v\rangle,\ u \in \mathcal{D}(S_1),\ v \in \mathcal{D}(\mathrm{s}_1)$, then it only needs to prove that for $\forall u \in \mathcal{D}(\mathrm{s}_1)$, if there exists $h \in L^2[0,1/2]$ such that

$$\mathrm{s}_1[u,v] = \langle h, v\rangle,\ v \in \mathcal{D}(\mathrm{s}_1), \tag{A.6}$$

then $u \in \mathcal{D}(S_1)$ and $S_1 u = h$. From (A.6), there has

$$\int_0^{1/2} (u'\overline{v}' + qu\overline{v}) \,\mathrm{d}x = \int_0^{1/2} h\overline{v} \,\mathrm{d}x. \tag{A.7}$$

Let z be an indefinite integral of $h - qu$, then $z' = h - qu$. (A.7) together with $v \in \mathcal{D}(\mathrm{s}_1)$ yields that

$$\int_0^{1/2} (u' + z)\,\overline{v}' \,\mathrm{d}x - z(1/2)\overline{v}(1/2) = 0. \tag{A.8}$$

Let $G = span\{1\} \subset L^2[0,1/2]$, then for $\forall g_0 \in G^\perp$, there has $g(x) = \int_0^x g_0 \mathrm{d}s \in \mathcal{D}(s_1)$ and $g(1/2) = 0$. Substituting $g(x)$ into (A.8):

$$\int_0^{1/2} (u' + z)\,\overline{g_0} \,\mathrm{d}x = 0. \tag{A.9}$$

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From $g_0 \in G^{\perp}$, it follows that $u' + z \in G$, then $u'(x) + z(x) = c$, where c is a constant. Substituting this into (A.8), we arrive at

$$(c - z(1/2))\,\overline{v}(1/2) = 0. \tag{A.10}$$

Since $\overline{v}(1/2)$ varies over all complex numbers when v varies over $\mathcal{D}(s_1)$, then $c = z(1/2) = u'(1/2) + z(1/2)$, so $u'(1/2) = 0$. From $u'(x) + z(x) = c$, it follows that $u' \in AC[0, 1/2]$ and $u'' = -z' = qu - h$ or $-u'' + qu = h$. From the definition of the operator S_1, it follows that $u \in \mathcal{D}(S_1)$ and $S_1 u = h$.

Similar to above, the form t_1 is also symmetric, densely defined, closed and bounded from below. And it also only needs to prove for $\forall u \in \mathcal{D}(\mathrm{t}_1)$, if there exists $h \in L^2[0, 1/2]$ such that

$$t_1[u, v] = \langle h, v\rangle,\ \ v \in \mathcal{D}(\mathrm{t}_1), \tag{A.11}$$

then $u \in \mathcal{D}(T_1)$ and $T_1 u = h$. From (A.11):

$$\int_0^{1/2} u'\overline{v}'\,\mathrm{d}x - ru(a)\overline{v}(a) = \int_0^{1/2} h\overline{v}\,\mathrm{d}x. \tag{A.12}$$

Let

$$w(x) = \begin{cases} -ru(a), & x \in [0, a], \\ 0, & x \in (a, 1/2], \end{cases} \tag{A.13}$$

and $z' = h$, then (A.12) turns to

$$\int_0^{1/2} (u' + w)\overline{v}'\,\mathrm{d}x = \int_0^{1/2} h\overline{v}\,\mathrm{d}x. \tag{A.14}$$

The following proof is similar to above. □

In §2.4, the associated quadratic form of problems (2.26) and (2.27) with $w \geqslant 0$ and $\int_c^d w\,\mathrm{d}x > 0$ can be similarly given in (2.30).

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