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On the Stable Basin Theorem

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Abstract. The stable basin theorem was introduced by Basmajian and Miner as a key step in their necessary condition for the discreteness of a non-elementary group of complex hyperbolic isometries. In this paper we improve several of Basmajian and Miner's key estimates and so give a substantial improvement on the main inequality in the stable basin theorem.

1 Introduction

Jørgensen's inequality [4] gives a well known necessary condition for a non-elementary, two generator subgroup of PSL$(2,\mathbb{C})$ to be discrete. In [1] Basmajian and Miner generalised this condition to complex hyperbolic 2-space $\mathbf{H}^2_{\mathbb{C}}$ and its isometry group PU$(2,1)$. Their method involved first proving a result which they termed the *stable basin theorem*. (See Goldman's book [2] as well as the papers cited below for further information about complex hyperbolic geometry and the Heisenberg group.) Suppose that we are given a pair of points $p,\ q \in \partial\mathbf{H}^2_{\mathbb{C}}$ and neighbourhoods U_p and U_q in $\partial\mathbf{H}^2_{\mathbb{C}}$ of these points. Then the pair $(U_p,\ U_q)$ is said to be a *stable* with respect to the points $(p,\ q)$ and a set $\mathcal{S}$ of complex hyperbolic isometries if for all $A \in \mathcal{S}$ we have $A(p) \in U_p$ and $A(q) \in U_q$. We identify the boundary of complex hyperbolic space $\partial\mathbf{H}^2_{\mathbb{C}}$ with the one point compactification of the Heisenberg group $\mathcal{N} \cup \{\infty\}$. Following [1], we take p to be the origin $o = (0,0)$ in the Heisenberg group and U_p to be $\mathbf{B}_{r'}$, the ball in $\mathcal{N}$ centred at o with radius $r' > 0$ with respect to the Cygan metric (see below). Similarly, we take q to be ∞ and U_q to be $\overline{\mathbf{B}}^c_{1/r'}$, the exterior of the Cygan ball of radius $1/r'$. Given $0 < r < 1$ and $\epsilon > 0$, let $\mathcal{S}(r,\epsilon)$ be the collection of those loxodromic maps A with multiplier $\lambda = \lambda(A) \in \mathbb{C} - \{0\}$ satisfying $\left|\lambda - 1\right| < \epsilon$ and with fixed points in $\mathbf{B}_r$ and $\overline{\mathbf{B}}^c_{1/r}$. The stable basin theorem gives a condition on $\epsilon = \epsilon(r,r')$ that guarantees the pair $(\mathbf{B}_{r'},\ \overline{\mathbf{B}}^c_{1/r'})$ is stable with respect to the points $(o,\ \infty)$ and the set $\mathcal{S}(r,\epsilon)$. By refining the estimates used by Basmajian and Miner, Kamiya has given improved versions of the stable basin theorem [5, 6] which give a larger family of loxodromic transformations under which $(\mathbf{B}_{r'},\ \overline{\mathbf{B}}^c_{1/r'})$ is stable. In this note we improve these conditions yet further.

In order to prove a complex hyperbolic Jørgensen's inequality we need to find a pair of open sets that are stable only with respect to a sequence of distinct loxodromic maps rather than with respect to an entire family (see Theorem 9.1 of [1]). Thus we expect our conditions for the stable basin theorem to be more restrictive than those for Jørgensen's inequality. This is indeed the case, see Section 6 of [3].

In Figure 1 we compare the various results by plotting $\epsilon(r,r)$ from three versions of the stable basin theorem and a bound coming from the complex hyperbolic

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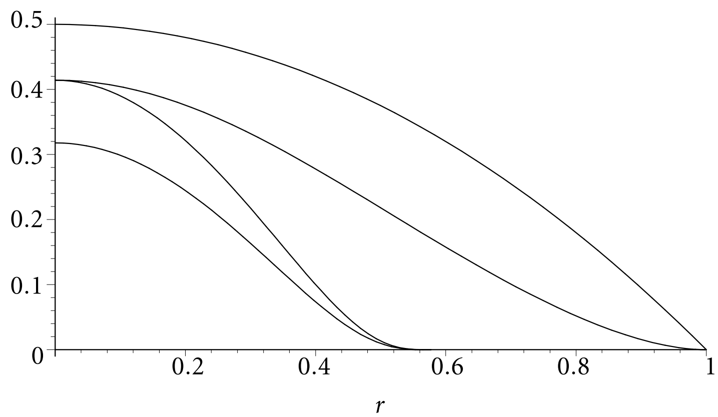

Figure 1: Comparing three versions of the stable basin theorem and Jørgensen's inequality.

Jørgensen's inequality. The lowest curve is the stable basin theorem given in [5], Figure 2. The original curve of Basmajian and Miner would be a similar curve slightly below this one, intersecting the ϵ-axis in the same place, namely $\epsilon = \sqrt{3} - \sqrt{2}$, and meeting the r-axis at $r = 1/2$. The second curve is the stable basin theorem given in [6], Figure 1. The third curve is the stable basin theorem from Theorem 3.2 below. Finally, the top curve is the corresponding curve from Figure 3 of [3] arising from Jørgensen's inequality.

2 The Cygan Metric

Consider $\partial\mathbf{H}^2_{\mathbb{C}} = \mathcal{N} \cup \{\infty\}$. There is a natural metric, called the Cygan metric, on $\mathcal{N}$. This metric is given by

$$\rho_0\big((\zeta, v), (\xi, t)\big) = \left| -|\zeta|^2 - iv + 2\overline{\zeta}\xi - |\xi|^2 + it \right|^{1/2}.$$

We want to investigate how the Cygan metric scales when we apply certain isometries of $\mathbf{H}^2_{\mathbb{C}}$. First we consider a *complex dilation* map fixing the origin $o = (0, 0)$ and ∞ with *multiplier* $\lambda = \lambda(A) \in \mathbb{C} - \{0\}$. Such a map acts on $\mathcal{N}$ by $A(\zeta, v) = \big(\lambda\zeta, |\lambda|^2 v\big)$. Hence for all $z \in \mathcal{N}$:

$$\rho_0\big(o, A(\zeta, v)\big) = \left| -|\lambda\zeta|^2 + i|\lambda|^2 v \right|^{1/2} = |\lambda| \rho_0\big(o, (\zeta, v)\big).$$

A *loxodromic* map in $\mathrm{PU}(2, 1)$ is a map conjugate to a complex dilation with $|\lambda| \neq 1$. We now estimate the Cygan translation length of a complex dilation. In the proof of the stable basin theorem this estimate will replace the dilation bound lemma of Basmajian and Miner (Proposition 3.3 of [1]) and should be compared with Lemma 2.1 of [6].

Lemma 2.1 *Suppose that $A \in \mathrm{PU}(n, 1)$ fixes o and ∞ and has complex multiplier $\lambda = \lambda(A)$. Then $\rho_0(Az, z) \leq |\lambda - 1|^{1/2}\big(|\lambda| + 1\big)^{1/2} \rho_0(z, o)$ for all $z \in \partial\mathbf{H}^2_{\mathbb{C}} - \{\infty\}$.*

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Proof If $z=(\zeta,v)$ then $A(z)=(\lambda\zeta,|\lambda|^2v)$. So:

$$\begin{aligned}\rho_0(Az,z) &= \big|\,|\lambda|^2(-|\zeta|^2-iv)+2\overline{\lambda}|\zeta|^2-|\zeta|^2+iv\big|^{1/2}\\ &= \big|\overline{\lambda}\big(\lambda-1\big)(-|\zeta|^2-iv)-\big(\overline{\lambda}-1\big)(-|\zeta|^2+iv)\big|^{1/2}\\ &\le |\lambda-1|^{1/2}\big(\big|\lambda\big|+1\big)^{1/2}\big|-|\zeta|^2+iv\big|^{1/2}\\ &= |\lambda-1|^{1/2}\big(|\lambda|+1\big)^{1/2}\rho_0(z,o).\end{aligned}$$

This completes the proof. ∎

Next we consider how the Cygan metric behaves when we apply elements B of $\mathrm{PU}(2,1)$ that do not fix infinity. We use a result of Kamiya in place of the uniform Lipschitz bound of Basmajian and Miner (Theorem 5.22 of [1]). To use Kamiya's result we need the notion of an isometric sphere. In Proposition 1.6 of [7] it is shown that the Cygan spheres centred at $B^{-1}(\infty)$ are mapped to Cygan spheres centred at $B(\infty)$. Among these there is exactly one sphere I_B centred at $B^{-1}(\infty)$ so that I_B and $B(I_B)$ have the same radius. We call I_B the *isometric sphere* of B and denote its radius by r_B.

Lemma 2.2 (Proposition 2.4 of [5]) *Let B be any element of* $\mathrm{PU}(2,1)$ *not fixing* ∞. *Then for all z, w in $\partial\mathbf{H}^2_{\mathbb{C}}-\{\infty,\,B^{-1}(\infty)\}$ we have:*

$$\begin{aligned}\rho_0\big(B(z),B(w)\big) &= \frac{{r_B}^2\rho_0(z,w)}{\rho_0\big(z,B^{-1}(\infty)\big)\,\rho_0\big(w,B^{-1}(\infty)\big)},\\ \rho_0\big(B(z),B(\infty)\big) &= \frac{{r_B}^2}{\rho_0\big(z,B^{-1}(\infty)\big)}.\end{aligned}$$

3 The Stable Basin Theorem

For a given $0<r<1$ consider the neighbourhoods $U_o=\mathbf{B}_r$ of $o=(0,0)$ and $U_\infty=\overline{\mathbf{B}}^c_{1/r}$ of ∞ given by

$$\begin{aligned}\mathbf{B}_r &= \{z\in\mathcal{N}\cup\{\infty\}:\rho_0(o,z)<r\}\,,\\ \overline{\mathbf{B}}^c_{1/r} &= \{z\in\mathcal{N}\cup\{\infty\}:\rho_0(o,z)>1/r\}\,.\end{aligned}$$

Consider the involution ι defined by

$$\iota(\zeta,v)=\left(\frac{-\zeta}{|\zeta|^2-iv},\frac{-v}{|\zeta|^4+v^2}\right),$$

which swaps o and ∞. It is easy to see that $\rho_0(o,\iota(p))=1/\rho_0(o,p)$ for any $p\in\mathcal{N}-\{o\}$. Thus ι interchanges $\mathbf{B}_r$ and $\overline{\mathbf{B}}^c_{1/r}$.

Lemma 3.1 (Lemma 3.2 of [1]) *Let $0 < r < 1$ be fixed and let $\mathcal{S}$ be a set of elements of* $\mathrm{PU}(2,1)$ *with the following properties. Each $A \in \mathcal{S}$ should be loxodromic and fix a point of $\mathbf{B}_r$ and a point of $\overline{\mathbf{B}}^c_{1/r}$. Suppose also that $\mathcal{S}$ is closed under conjugation by ι. Then the pair $\left(\mathbf{B}_{r'}, \overline{\mathbf{B}}^c_{1/r'}\right)$ is stable with respect to the points (o, ∞) and the family $\mathcal{S}$ if and only if $A(o) \in \mathbf{B}_{r'}$ for all $A \in \mathcal{S}$.*

We can now state the main theorem:

Theorem 3.2 (Stable basin theorem) *Let $0 < r,\ r' < 1$ be given. For any $\epsilon = \epsilon(r, r')$ let $\mathcal{S}(r, \epsilon)$ be the collection of all loxodromic maps in A in* $\mathrm{PU}(2,1)$ *so that (i) the multiplier $\lambda = \lambda(A)$ satisfies $\left|\lambda - 1\right| < \epsilon$ and (ii) A fixes a point of $\mathbf{B}_r$ and a point of $\overline{\mathbf{B}}^c_{1/r}$. Then the pair $\left(\mathbf{B}_{r'}, \overline{\mathbf{B}}^c_{1/r'}\right)$ is stable with respect to the points (o, ∞) and the family $\mathcal{S}(r, \epsilon)$ where*

$$\epsilon(r, r') = \frac{\sqrt{1 + (1 - r^4)s^2} - 1 - r^2(1 - r^2)s^2}{1 - r^4 s^2} \tag{1}$$

and s denotes r'/r.

Proof Suppose we are given A_{pq} fixing $p \in \mathbf{B}_r$ and $q \in \overline{\mathbf{B}}^c_{1/r}$. Choose a map B with $B(p) = o$ and $B(q) = \infty$. Thus $\rho_0(o, B^{-1}(o)) < r$ and $\rho_0(o, B^{-1}(\infty)) > 1/r$. From Lemma 2.2 we have

$$\rho_0(o, B^{-1}(o)) = \frac{{r_B}^2 \rho_0(o, B(o))}{\rho_0(o, B(\infty))\rho_0(B(o), B(\infty))},$$
$$\rho_0(o, B^{-1}(\infty)) = \frac{{r_B}^2}{\rho_0(B(o), B(\infty))}.$$

Hence
$$\frac{\rho_0(o, B(o))}{\rho_0(o, B(\infty))} = \frac{\rho_0(o, B^{-1}(o))}{\rho_0(o, B^{-1}(\infty))} < r^2.$$

The map B has been chosen so that $A = BA_{pq}B^{-1}$ is a complex dilation fixing o and ∞ with the same complex multiplier as A_{pq}, namely $\lambda = \lambda(A_{pq}) = \lambda(A)$. A brief computation shows that

$$\epsilon = \frac{\sqrt{1 + (1 - r^4)s^2} - 1 - r^2(1 - r^2)s^2}{1 - r^4 s^2} < \frac{1 - r^2}{r^2}.$$

Thus when $|\lambda - 1| < \epsilon$ we have $|\lambda| \le |\lambda - 1| + 1 < 1/r^2$ and so:

$$\rho_0(o, B(\infty)) - \rho_0(o, AB(o)) > \left(1/r^2 - |\lambda|\right) \rho_0(o, B(o)) > 0.$$

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We now estimate $\rho_0\big(o, A_{pq}(o)\big)$ as follows:

$$\begin{aligned}
\rho_0(o, A_{pq}(o)) &= \rho_0(o, B^{-1}AB(o)) \\
&= \frac{r_B{}^2 \rho_0(B(o), AB(o))}{\rho_0(B(o), B(\infty))\rho_0(AB(o), B(\infty))} \\
&\le \frac{r_B{}^2 \big(|\lambda| + 1\big)^{1/2} |\lambda - 1|^{1/2} \rho_0(o, B(o))}{\rho_0(B(o), B(\infty))\big(\rho_0(o, B(\infty)) - \rho_0(o, AB(o))\big)} \\
&= \frac{\big(|\lambda| + 1\big)^{1/2} |\lambda - 1|^{1/2} \rho_0(o, B^{-1}(o))\rho_0(o, B(\infty))}{\rho_0(o, B(\infty)) - |\lambda|\rho_0(o, B(o))} \\
&= \frac{\big(|\lambda| + 1\big)^{1/2} |\lambda - 1|^{1/2} \rho_0(o, B^{-1}(o))}{1 - |\lambda|\rho_0(o, B(o))/\rho_0(o, B(\infty))} \\
&< \frac{\big(|\lambda| + 1\big)^{1/2} |\lambda - 1|^{1/2} r}{1 - |\lambda| r^2} \\
&\le \frac{\big(|\lambda - 1| + 2\big)^{1/2} |\lambda - 1|^{1/2} r}{1 - r^2 - |\lambda - 1| r^2}.
\end{aligned}$$

In order for $A_{pq}(o)$ to be in $\mathbf{B}_{r'}$ it suffices to impose the condition

$$\frac{\big(|\lambda - 1| + 2\big)^{1/2} |\lambda - 1|^{1/2} r}{1 - r^2 - |\lambda - 1| r^2} < r'.$$

Writing $s = r'/r$ and rearranging this is equivalent to

$$|\lambda - 1|^2 (1 - r^4 s^2) + 2|\lambda - 1|(1 + r^2(1 - r^2)s^2) - (1 - r^2)^2 s^2 < 0.$$

Solving for $|\lambda - 1|$ gives

$$|\lambda - 1| < \frac{\sqrt{1 + (1 - r^4)s^2} - 1 - r^2(1 - r^2)s^2}{1 - r^4 s^2} = \epsilon.$$

Hence $A_{pq}(o)$ is in $\mathbf{B}_{r'}$ whenever $|\lambda - 1| < \epsilon$. It is clear that $\mathcal{S}(r, \epsilon)$ is mapped to itself under conjugation by ι. Thus, using Lemma 3.1, we see that this proves the theorem. ■

References

[1] A. Basmajian & R. Miner, *Discrete subgroups of complex hyperbolic motions*. Invent. Math. **131**(1998), 85–136.
[2] W. M. Goldman, *Complex Hyperbolic Geometry*. Oxford University Press, 1999.

https://doi.org/10.4153/CMB-2004-043-9 Published online by Cambridge University Press

[3] Y. Jiang, S. Kamiya & J. R. Parker, *Jørgensen's inequality for complex hyperbolic space*. Geom. Dedicata **97**(2003), 55–80.
[4] T. Jørgensen, *On discrete groups of Möbius transformations*. Amer. J. Math. **98**(1976), 739–749.
[5] S. Kamiya, *On discrete subgroups of* $PU(1,2;\mathbb{C})$ *with Heisenberg translations*. J. London Math. Soc. **62**(2000), 827–842.
[6] S. Kamiya & J. R. Parker, *On discrete subgroups of* $PU(1,2;\mathbb{C})$ *with Heisenberg translations II*. Rev. Roumaine Math. Pures Appl. **47**(2002), 687–693.
[7] J. R. Parker, *Uniform discreteness and Heisenberg translations*. Math. Z. **225**(1997), 485–505.

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