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BOUNDARY INTEGRAL METHODS FOR THE LAPLACE EQUATION

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In recent years there has been a revival of interest in the method of integral equations for the solution of boundary value problems. On the theoretical side, classical results have been extended by relaxing assumptions on the smoothness of the boundary, and in applications the boundary element method has become an important tool for the computational solution of partial differential equations. Both these developments are obviously of great interest to numerical analysts, especially since the integral equation formulations lead to numerical schemes which are completely independent of finite difference and finite element methods.

This thesis treats the Dirichlet and Neumann problems for the Laplace equation, and their reduction to boundary integral equations by means of single and double layer potentials. The overall aim is to prove asymptotic error estimates for numerical schemes based on these integral equations. Although numerical solutions are discussed only for two dimensional problems, much of the background theory is presented for higher dimensions as well.

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When designing a numerical scheme for solving an operator equation, the first step is to identify function spaces for which the problem is well posed, since such information is crucial when discussing stability. Consequently, after introducing some notation and terminology in Chapter 0, the thesis proper begins in Chapter 1 with a survey of the existence and uniqueness theory for the integral equations of the second kind which arise when the Neumann and Dirichlet problems are solved using single and double layer potentials respectively. These results are relatively elementary when one is dealing with a Lyapunov boundary, since the integral operators involved are then only weakly singular and hence are compact on L^p and C . However, in applications one frequently deals with boundaries which fail to satisfy the Lyapunov condition because they have corners. It is then far from trivial to show that the boundary integral equations are well posed, especially in higher dimensions. The key references used in Chapter 1 are Kral [7] for solvability in the space C (and its dual) and Verchota [8] for solvability in L^2 .

The numerical analysis starts in Chapter 2, where the simplest case is treated, namely that of a smooth boundary (in two dimensions). This chapter is quite short since it deals basically with the numerical solution of one dimensional Fredholm integral equations of the second kind having smooth kernels, a topic which has been covered in detail elsewhere, for example, Atkinson [1]. There are however some special features of boundary integral equations which make it possible to achieve surprisingly fast rates of convergence with very simple numerical schemes.

In Chapter 3 a detailed study is made of the boundary integral equations for an infinite sector, the aim being to prove regularity theorems involving function spaces with weights, thereby establishing the precise nature of the singular behaviour of solutions in a neighbourhood of a corner point. The Mellin transform techniques involved in this analysis have been used by various authors including Kondrat'ev [6], Fabes, Jodeit and Lewis [5] and Costabel and Stephan [4].

Chapter 4 is devoted to a collocation method for the numerical solution of the double layer potential equation in the case of a bounded, piecewise smooth domain with corners. Using an argument of Atkinson and de Hoog [2], this problem is reduced to an integral equation on the unit interval. The singular behaviour of the solution is taken care of by employing a graded mesh, an approach used by Chandler [3], who solved the same problem by using the Galerkin method. The results of Chapter 3 are used to determine the mesh gradings for which optimal rates of convergence hold in $L^{\infty}(0,1)$, and it is shown that with a stronger grading super-convergence can be achieved at the collocation points. Unfortunately, stability could not be proved for all cases, although the predicted rates of convergence are confirmed by computational experiments.

The final chapter deals with the integral equation of the first kind which arises when a single layer potential is used to represent the solution to the Dirichlet problem. The results of Verchota presented in Chapter 1 are used to prove a basic existence and uniqueness theorem for the integral equation, then a more detailed study is made of the two dimensional case. Next, a spectral Galerkin method is presented, and it is shown that very fast rates of convergence are attained provided sufficiently strong regularity conditions hold. The chapter ends with a brief discussion of the direct boundary integral method (or boundary element method) in which an integral identity for harmonic functions provides a representation of the solution to either the Dirichlet or the Neumann problem as the sum of a single and a double layer potential.

There is a short appendix containing proofs of some technical results which are used in the main part of this thesis.

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