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ON THE NORMALISED LAPLACIAN SPECTRUM, DEGREE-KIRCHHOFF INDEX AND SPANNING TREES OF GRAPHS

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Abstract

Given a connected regular graph G, let $l(G)$ be its line graph, $s(G)$ its subdivision graph, $r(G)$ the graph obtained from G by adding a new vertex corresponding to each edge of G and joining each new vertex to the end vertices of the corresponding edge and $q(G)$ the graph obtained from G by inserting a new vertex into every edge of G and new edges joining the pairs of new vertices which lie on adjacent edges of G. A formula for the normalised Laplacian characteristic polynomial of $l(G)$ (respectively $s(G)$, $r(G)$ and $q(G)$) in terms of the normalised Laplacian characteristic polynomial of G and the number of vertices and edges of G is developed and used to give a sharp lower bound for the degree-Kirchhoff index and a formula for the number of spanning trees of $l(G)$ (respectively $s(G)$, $r(G)$ and $q(G)$).

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1. Introduction

Distance is an important concept in graph theory (see [4]). In 1993, Klein and Randić [15] proposed a novel distance function, namely the *resistance distance*, on a graph. The term resistance distance was used because of the physical interpretation: place unit resistors on each edge of a graph G and take the resistance distance, r_{ij}, between vertices i and j of G to be the resistance between them. This new parameter is in fact intrinsic to the graph and has some nice interpretations and applications in chemistry (see [13, 14] for details). The resistance distance can be computed via the Moore–Penrose generalised inverse of the (*combinatorial*) *Laplacian matrix* $L = D - A$, where A is the adjacency matrix of G and D is the diagonal matrix of vertex degrees.

The traditional distance, d_{ij}, between vertices i and j is the length of a shortest path connecting them. An important parameter called the *Wiener index*, $W(G)$, is given by $W(G) = \sum_{i<j} d_{ij}$ (see [21]). As an analogue to the Wiener index, the sum

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$K(G) = \sum_{i<j} r_{ij}$ was proposed in [15] and later called the Kirchhoff index of G in [3]. In [10, 24], it is shown that

$$K(G) = \sum_{i<j} r_{ij} = n \sum_{i=2}^{n} \frac{1}{\mu_i},$$

where $0 = \mu_1 < \mu_2 \leqslant \cdots \leqslant \mu_n$ $(n \geqslant 2)$ are the eigenvalues of L.

In recent years, another matrix, the *normalised Laplacian*, which is consistent with the matrix in spectral geometry and random walks [7], has attracted attention. One of the original motivations for defining the normalised Laplacian was to deal more naturally with nonregular graphs. Chen and Zhang [6] showed that the resistance distance can be expressed naturally in terms of the eigenvalues and eigenvectors of the normalised Laplacian and proposed the degree-Kirchhoff index, which is closely related to the spectrum of the normalised Laplacian. There are many connections between the normalised Laplacian and its eigenvalues and the structural properties of graphs (see [1, 16] for recent examples).

In this paper, inspired by [6, 9, 20], we study the normalised Laplacian characteristic polynomial, the degree-Kirchhoff index and the enumeration of spanning trees of four types of graphs.

2. Normalised Laplacian and degree-Kirchhoff index

Throughout this paper, we only consider simple connected graphs $G = (V_G, E_G)$, where V_G is the vertex set and E_G is the edge set. We call $n = |V_G|$ the *order* of G and $m = |E_G|$ the *size* of G. For all graph theoretical terms the reader is referred to [7] and for matrix terms to [11].

For a graph G, we can define the *random walks* on G as the Markov chain $X_n, n \geqslant 0$, that from its current vertex i jumps to the adjacent vertex j with probability $p_{ij} = 1/d_i$, where d_i is the degree of the vertex i. Clearly, the transition probability matrix $P = (p_{ij}) = D^{-1}A$ is a stochastic matrix. The hitting time T_j of the vertex j is the number of jumps the walk takes to reach j. The expected value of T_j when the walk is started at the vertex i is denoted by E_iT_j. The expected commute time between vertices i and j is defined by $E_iT_j + E_jT_i$. In view of [5, 15], we know that there is an elegant relation between commute times and resistance distances:

$$E_iT_j + E_jT_i = 2mr_{ij}.$$

The *normalised Laplacian matrix* of G is defined to be

$$\mathcal{L} = I - D^{1/2}PD^{-1/2} = D^{-1/2}LD^{-1/2}. \tag{2.1}$$

It is easy to see that $\mathcal{L}$ is Hermitian and similar to $I - P$, so the eigenvalues of $\mathcal{L}$ are nonnegative and may be labelled $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Let $u_1, u_2, \ldots, u_n$ be the corresponding mutually orthogonal eigenvectors of unit length. For convenience, let

$$u_i = (u_{i1}, u_{i2}, \ldots, u_{in})^t,$$

where t denotes the transposition. Set $U = (u_1, u_2, \ldots, u_n)$. Then U is an orthogonal matrix, that is,

$$\sum_{k=1}^{n} u_{ik}u_{jk} = \sum_{k=1}^{n} u_{ki}u_{kj} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$U^t \mathcal{L} U = \operatorname{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$
$$\mathcal{L}_{ij} = \sum_{k=1}^{n} \lambda_k u_{ki}u_{kj} = \sum_{k=2}^{n} \lambda_k u_{ki}u_{kj}.$$

The following theorem gives some fundamental results about the spectrum of $\mathcal{L}$.

THEOREM 2.1 [7]. *For a graph $G = (V_G, E_G)$:*

(i) *if G is not a complete graph, then $1/(2md) < \lambda_2 \leqslant 1$, where d is the diameter of G;*
(ii) *$n/(n-1) \leqslant \lambda_n \leqslant 2$ with $\lambda_n = 2$ if and only if G is bipartite;*
(iii) *$\prod_{i=1}^{n} d_i \prod_{k=2}^{n} \lambda_k = 2m\tau(G)$, where $\tau(G)$ is the number of spanning trees of G.*

Using the notation as above, Lovász obtained the following result.

THEOREM 2.2 [18]. *For a graph $G = (V_G, E_G)$, for all $i, j \in V_G$,*

$$E_iT_j + E_jT_i = 2m \sum_{k=2}^{n} \frac{1}{\lambda_k}\Big(\frac{u_{kj}}{\sqrt{d_j}} - \frac{u_{ki}}{\sqrt{d_i}}\Big)^2.$$

From the earlier remarks, we have the following formula for the resistance distance of a graph $G = (V_G, E_G)$:

$$r_{ij} = \sum_{k=2}^{n} \frac{1}{\lambda_k}\Big(\frac{u_{kj}}{\sqrt{d_j}} - \frac{u_{ki}}{\sqrt{d_i}}\Big)^2 \quad \forall i, j \in V_G.$$

As pointed out above, the Kirchhoff index $K(G) = \sum_{i<j} r_{ij}$ is closely related to the spectrum of L. Chen and Zhang [6] introduced a new graph index related to resistance distance, defined by

$$K'(G) = \sum_{i<j} d_i d_j r_{ij},$$

called the *degree-Kirchhoff index* (see also [8, 19]). The following beautiful result obtained by Chen and Zhang [6] shows that it is closely related to the spectrum of the normalised Laplacian $\mathcal{L}$.

THEOREM 2.3 [6]. *For a graph $G = (V_G, E_G)$, for all $i, j \in V_G$,*

$$K'(G) = 2m \sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

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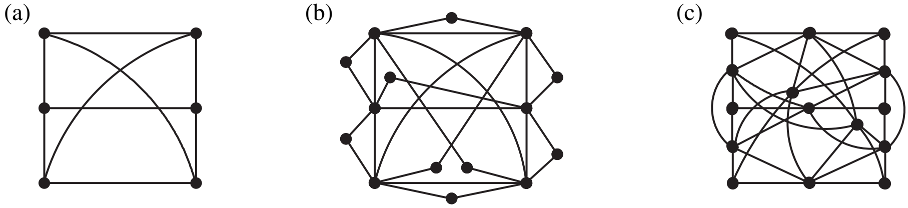

FIGURE 1. Graphs of (a) $K_{3,3}$, (b) $r(K_{3,3})$ and (c) $q(K_{3,3})$.

Given a k-regular graph G, if we denote its Laplacian spectrum by $\{\mu_1, \mu_2, \ldots, \mu_n\}$, then its normalised Laplacian spectrum is $\{\mu_1/k, \mu_2/k, \ldots, \mu_n/k\}$, but, for general graphs, there can be significant differences between the two spectra. Hence, it is interesting to study the normalised Laplacian of nonregular graphs. To conclude this section, we introduce the four classes of graphs which will be considered throughout this paper.

- The *line graph* of a graph G, denoted by $l(G)$, is the graph whose vertices correspond to the edges of G, with two vertices of $l(G)$ being adjacent if and only if the corresponding edges of G share a common vertex.
- The *subdivision graph* of G, denoted by $s(G)$, is the graph obtained by replacing every edge in G with a copy of P_3 (a path of length two).
- $r(G) = (V_{r(G)}, E_{r(G)})$ is the graph obtained from G by adding a new vertex e' corresponding to each edge $e = (a, b)$ of G and joining each new vertex e' to the end vertices a and b of the corresponding edge $e = (a, b)$. Thus, $V_{r(G)} = V_G \cup \{e' \mid e \in E_G\}$ and $E_{r(G)} = E_G \cup \{(v_i, e'), (v_j, e') \mid e = (v_i, v_j) \in E_G\}$; see Figure 1 for an example.
- $q(G) = (V_{q(G)}, E_{q(G)})$ is the graph obtained from G by inserting a new vertex e'_i into every edge e_i of G and joining by edges those pairs of these new vertices e'_i and e'_j which lie on adjacent edges e_i and e_j of G, $i, j = 1, 2, \ldots, m$. Denote by v_{i1} and v_{i2} the end vertices of edge e_i of G. Then $V_{q(G)} = V_G \cup \{e'_i \mid e_i \in E_G, i = 1, 2, \ldots, m\}$ and $E_{q(G)} = \{(v_{i1}, e'_i), (v_{i2}, e'_i) \mid i = 1, 2, \ldots, m\} \cup \{(e'_i, e'_j) \mid e_i$ and e_j are the adjacent edges of $G\}$; see Figure 1 for an example.

REMARK 2.4. The last two graphs considered are interesting because they might not be regular. A line graph of a regular graph is regular and a subdivision graph is biregular, that is, a bipartite graph where the degrees in the two respective parts have constant degrees. For biregular graphs it easily follows from the definition of the normalised Laplacian that the eigenvalues of the adjacency matrix of the subdivision graph when scaled and translated give the eigenvalues for the normalised Laplacian.

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3. The normalised Laplacian characteristic polynomials of $l(G), s(G), r(G)$ and $q(G)$

We denote by $\Phi(B) = \det(xI - B)$ the *characteristic polynomial* of the square matrix B. If $B = L(G)$, we write $\Gamma(G; x) = \Phi(L(G))$ and call it the *Laplacian characteristic polynomial* of G; if $B = \mathcal{L}(G)$, we write $\Psi(G; x) = \Phi(\mathcal{L}(G))$ and call it the *normalised Laplacian characteristic polynomial* of G.

In this section, for a regular graph G, we characterise the relationship between the normalised Laplacian polynomial of $l(G)$ (respectively $s(G), r(G)$ and $q(G)$) and the normalised Laplacian polynomial of G. We will need the following lemmas.

LEMMA 3.1 [11]. *Let M be a nonsingular square matrix. Then*

$$\det\begin{pmatrix} M & N \\ P & Q \end{pmatrix} = \det M \det(Q - PM^{-1}N).$$

LEMMA 3.2 [17]. *Let G be an undirected simple graph with n vertices and m edges. Then*

(i) $I(G)I(G)^t = D(G) + A(G)$;
(ii) $I(G)^t I(G) = 2I_m + A(l(G))$,

where $I(G)$ is the incidence matrix of G, I_p is the unit matrix of order p and t denotes the transposition.

LEMMA 3.3 [12]. *Let G be a k-regular graph with n vertices and m edges. Then*

$$\Gamma(l(G); x) = (x - 2k)^{m-n}\Gamma(G; x).$$

LEMMA 3.4 [7]. *Let G be the disjoint union of graphs $G_1, G_2, \ldots, G_k$; then*

$$\Psi(G; x) = \prod_{i=1}^{k} \Psi(G_i; x).$$

First, we determine the relationship between the normalised Laplacian characteristic polynomial of $l(G)$ and that of the regular graph G.

THEOREM 3.5. *Let G be a k-regular graph with n vertices and m edges. Then*

$$\Psi(l(G); x) = \frac{[(k-1)x - k]^{m-n} k^n}{2^n (k-1)^m} \Psi\Big(G; \frac{2(k-1)x}{k}\Big).$$

PROOF. By Lemma 3.3,

$$\det(xI_m - L(l(G))) = (x - 2k)^{m-n} \det(xI_n - L(G)). \tag{3.1}$$

Note that if G is a k-regular graph, then $l(G)$ is a $2(k-1)$-regular graph, which implies that

$$\mathcal{L}(G) = \frac{L(G)}{k}, \quad \mathcal{L}(l(G)) = \frac{L(l(G))}{2(k-1)}.$$

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Together with (3.1),

$$\det[xI_m - 2(k-1)\mathcal{L}(l(G))] = (x-2k)^{m-n}\det[xI_n - k\mathcal{L}(G)],$$

that is,

$$2^m(k-1)^m\Psi\Big(l(G); \frac{x}{2(k-1)}\Big) = (x-2k)^{m-n}k^n\Psi\Big(G; \frac{x}{k}\Big),$$

which gives

$$\Psi(l(G);x) = \frac{[(k-1)x-k]^{m-n}k^n}{2^n(k-1)^m}\Psi\Big(G; \frac{2(k-1)x}{k}\Big).$$

This completes the proof. □

Next, we determine the relationship between the normalised Laplacian characteristic polynomial of $s(G)$ and that of the regular graph G.

THEOREM 3.6. *Let G be a k-regular graph with n vertices and m edges. Then*

$$\Psi(s(G);x) = (-\tfrac{1}{2})^n(x-1)^{m-n}\Psi(G; 2x(2-x)).$$

PROOF. Denote the incidence matrix of G by $I(G)$. Then

$$A(s(G)) = \begin{pmatrix} \mathbf{0}_m & I(G)^t \\ I(G) & \mathbf{0}_n \end{pmatrix}, \quad D(s(G)) = \begin{pmatrix} 2I_m & \mathbf{0} \\ \mathbf{0} & kI_n \end{pmatrix}$$

and

$$L(s(G)) = \begin{pmatrix} 2I_m & -I(G)^t \\ -I(G) & kI_n \end{pmatrix}.$$

In view of (2.1),

$$\begin{aligned}\mathcal{L}(s(G)) &= D(s(G))^{-1/2}L(s(G))D(s(G))^{-1/2} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}I_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{k}}I_n \end{pmatrix}\begin{pmatrix} 2I_m & -I(G)^t \\ -I(G) & rI_n \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}I_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{k}}I_n \end{pmatrix} \\ &= \begin{pmatrix} I_m & -\frac{1}{\sqrt{2k}}I(G)^t \\ -\frac{1}{\sqrt{2k}}I(G) & I_n \end{pmatrix}.\end{aligned}$$

It follows that

$$\Psi(s(G);x) = \det\begin{pmatrix} (x-1)I_m & \frac{1}{\sqrt{2k}}I(G)^t \\ \frac{1}{\sqrt{2k}}I(G) & (x-1)I_n \end{pmatrix}.$$

Case 1. $k=1$. In this case, G is a disjoint union of copies of K_2 and so $s(G)$ is a disjoint union of copies of P_3, that is, $G = (n/2)K_2$ and $s(G) = (n/2)P_3$, where n is even. Note that $\Psi(K_2;x) = x(x-2)$ and $\Psi(P_3;x) = x(x-1)(x-2)$. Hence, by Lemma 3.4, our result follows immediately in this case.

Case 2. $k \geq 2$. Based on Lemmas 3.1 and 3.2,

$$\begin{aligned}
\Psi(s(G);x) &= \det\begin{pmatrix} (x-1)I_m & \frac{1}{\sqrt{2k}}I(G)^t \\ \frac{1}{\sqrt{2k}}I(G) & (x-1)I_n \end{pmatrix} \\
&= (x-1)^m \det\Big[(x-1)I_n - \frac{1}{\sqrt{2k}}I(G)\frac{I_m}{\sqrt{2k}(x-1)}I(G)^t\Big] \quad \text{(by Lemma 3.1)} \\
&= (x-1)^{m-n}\det\Big[(x-1)^2 I_n - \frac{1}{2k}(2kI_n - L(G))\Big] \quad \text{(by Lemma 3.2)} \\
&= (x-1)^{m-n}\det[x(x-2)I_n + \tfrac{1}{2}\mathcal{L}(G)] \\
&= (-\tfrac{1}{2})^n (x-1)^{m-n}\Psi(G; 2x(2-x)).
\end{aligned}$$

Cases 1 and 2 together establish the result. □

Now, we determine the relationship between the normalised Laplacian characteristic polynomial of $r(G)$ and that of the regular graph G.

THEOREM 3.7. *Let G be a k-regular graph with n vertices and m edges. Then*

$$\Psi(r(G);x) = \frac{(x-1)^{m-n}(2x-3)^n}{4^n}\Psi(G;2x).$$

PROOF. It is routine to check that

$$A(r(G)) = \begin{pmatrix} \mathbf{0}_m & I(G)^t \\ I(G) & A \end{pmatrix}, \quad D(r(G)) = \begin{pmatrix} 2I_m & \mathbf{0} \\ \mathbf{0} & 2kI_n \end{pmatrix}$$

and

$$L(r(G)) = \begin{pmatrix} 2I_m & -I(G)^t \\ -I(G) & kI_n + L(G) \end{pmatrix}.$$

In view of (2.1),

$$\begin{aligned}
\mathcal{L}(r(G)) &= D(r(G))^{-1/2}L(r(G))D(r(G))^{-1/2} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}}I_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{2k}}I_n \end{pmatrix}\begin{pmatrix} 2I_m & -I(G)^t \\ -I(G) & kI_n + L(G) \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}}I_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{2k}}I_n \end{pmatrix} \\
&= \begin{pmatrix} I_m & -\frac{1}{2\sqrt{k}}I(G)^t \\ -\frac{1}{2\sqrt{k}}I(G) & \frac{1}{2}I_n + \frac{1}{2k}L(G) \end{pmatrix} \\
&= \begin{pmatrix} I_m & -\frac{1}{2\sqrt{k}}I(G)^t \\ -\frac{1}{2\sqrt{k}}I(G) & \frac{1}{2}I_n + \frac{1}{2}\mathcal{L}(G) \end{pmatrix}.
\end{aligned}$$

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It follows that

$$\Psi(r(G);x) = \det\begin{pmatrix} (x-1)I_m & \frac{1}{2\sqrt{k}}I(G)^t \\ \frac{1}{2\sqrt{k}}I(G) & \left(x-\frac{1}{2}\right)I_n - \frac{1}{2}\mathcal{L}(G) \end{pmatrix}. \tag{3.2}$$

Case 1. $k = 1$. In this case, G is a disjoint union of copies of K_2 and hence $r(G)$ is a disjoint union of copies of K_3, that is, $G = n/2K_2$ and $s(G) = n/2K_3$, where n is even. Note that $\Psi(K_2;x) = x(x-2)$ and $\Psi(K_3;x) = \frac{1}{4}x(2x-3)^2$. Hence, by Lemma 3.4, our result holds in this case.

Case 2. $k \geq 2$. By (3.2) and Lemmas 3.1 and 3.2,

$$\begin{aligned}
\Psi(r(G);x) &= (x-1)^m \det\Big[\Big(x-\frac{1}{2}\Big)I_n - \frac{1}{2}\mathcal{L}(G) - \frac{1}{2\sqrt{k}}I(G)\frac{I_m}{2\sqrt{k}(x-1)}I(G)^t\Big] \\
&= (x-1)^m \det\Big[\Big(x-\frac{1}{2}\Big)I_n - \frac{1}{2}\mathcal{L}(G) - \frac{1}{4k(x-1)}(2kI_n - L(G))\Big] \\
&= (x-1)^m \det\Big[\Big(x-\frac{1}{2}\Big)I_n - \frac{1}{2}\mathcal{L}(G) - \frac{1}{2(x-1)}I_n + \frac{1}{4(x-1)}\mathcal{L}(G)\Big] \\
&= (x-1)^{m-n} \det\Big\{\Big[(x-1)\Big(x-\frac{1}{2}\Big) - \frac{1}{2}\Big]I_n - \Big[\frac{1}{2}(x-1) - \frac{1}{4}\Big]\mathcal{L}(G)\Big\} \\
&= \frac{(x-1)^{m-n}}{4^n}\det[2x(2x-3)I_n - (2x-3)\mathcal{L}(G)] \\
&= \frac{(x-1)^{m-n}(2x-3)^n}{4^n}\Psi(G;2x).
\end{aligned}$$

By Cases 1 and 2, our result holds. □

Finally, we determine the relationship between the normalised Laplacian characteristic polynomial of $q(G)$ and that of the regular graph G.

THEOREM 3.8. *Let G be a k-regular graph of order n and size m. Then*

$$\Psi(q(G);x) = \frac{(kx-k-1)^m}{2^n k^m}\Psi(G;2x).$$

PROOF. Observe that

$$A(q(G)) = \begin{pmatrix} A(l(G)) & I(G)^t \\ I(G) & \mathbf{0}_n \end{pmatrix}, \quad D(q(G)) = \begin{pmatrix} 2kI_m & \mathbf{0} \\ \mathbf{0} & kI_n \end{pmatrix},$$

$$L(q(G)) = \begin{pmatrix} 2kI_m - A(l(G)) & -I(G)^t \\ -I(G) & kI_n \end{pmatrix}$$

and, in view of (2.1),

$$\begin{aligned}
\mathcal{L}(q(G)) &= D(q(G))^{-1/2}L(q(G))D(q(G))^{-1/2}\\
&= \begin{pmatrix} \frac{1}{\sqrt{2k}}I_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{k}}I_n \end{pmatrix} \begin{pmatrix} 2kI_m - A(l(G)) & -I(G)^t \\ -I(G) & kI_n \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2k}}I_m & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{k}}I_n \end{pmatrix}\\
&= \begin{pmatrix} I_m - \frac{1}{2k}A(l(G)) & -\frac{1}{\sqrt{2k}}I(G)^t \\ -\frac{1}{\sqrt{2k}}I(G) & I_n \end{pmatrix}.
\end{aligned}$$

Since a permutation similarity transformation preserves the determinant, we can interchange rows to give

$$\Psi(q(G);x) = \det\begin{pmatrix} (x-1)I_n & \frac{1}{\sqrt{2k}}I(G) \\ \frac{1}{\sqrt{2k}}I(G)^t & (x-1)I_m + \frac{1}{2k}A(l(G)) \end{pmatrix}.$$

By Lemmas 3.1 and 3.2,

$$\begin{aligned}
\Psi(q(G);x) &= (x-1)^n \det\Big[(x-1)I_m + \frac{1}{2k}A(l(G)) - \frac{1}{\sqrt{2k}}I(G)^t \frac{I_n}{\sqrt{2k}(x-1)}I(G)\Big]\\
&= (x-1)^n \det\Big[(x-1)I_m + \frac{1}{2k}A(l(G)) - \frac{1}{2k^2}\frac{1}{x-1}(2I_m + A(l(G)))\Big]\\
&= (x-1)^{n-m} \det\Big\{\Big[(x-1)^2 - \frac{1}{k^2}\Big]I_m + \Big(\frac{x-1}{2k} - \frac{1}{2k^2}\Big)A(l(G))\Big\}.
\end{aligned}$$

Since $A(l(G)) = D(l(G)) - L(l(G)) = 2(k-1)I_m - 2(k-1)\mathcal{L}(l(G))$,

$$\begin{aligned}
\Psi(q(G);x) = (x-1)^{n-m} \det\Big\{&\Big[(x-1)^2 - \frac{1}{k^2}\Big]I_m\\
&+ \Big(\frac{x-1}{2k} - \frac{1}{2k^2}\Big)[2(k-1)I_m - 2(k-1)\mathcal{L}(l(G))]\Big\}\\
= &\ \frac{(x-1)^{n-m}(kx-k-1)^m}{k^{2m}} \det[kxI_m - (k-1)\mathcal{L}(l(G))]\\
= &\ \frac{(k-1)^m(x-1)^{n-m}(kx-k-1)^m}{k^{2m}}\Psi\Big(l(G); \frac{k}{k-1}x\Big).
\end{aligned}$$

Then, using Theorem 3.5 and simplifying,

$$\Psi(q(G);x) = \frac{(kx-k-1)^m}{2^n k^m}\Psi(G;2x).$$

This completes the proof. □

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4. Degree-Kirchhoff index of $l(G)$, $s(G)$, $r(G)$ and $q(G)$

In this section, we study the relationship between the degree-Kirchhoff index of $l(G)$ (respectively $s(G)$, $r(G)$ and $q(G)$) and the degree-Kirchhoff index of the regular graph G. Consequently, some sharp lower bounds on $K'(l(G))$, $K'(s(G))$, $K'(r(G))$ and $K'(q(G))$ are determined.

First, we give a lower bound on the degree-Kirchhoff index of a general connected graph G.

Theorem 4.1. *Let G be a connected graph on $n \geqslant 3$ vertices and m edges. Then*

$$K'(G) \geqslant \frac{2m(n-1)^2}{n}.$$

The equality holds if and only if G is complete.

Proof. Recall that $\sum_{k=2}^{n} \lambda_k = n$. By the Cauchy–Schwarz inequality,

$$\sum_{k=2}^{n-1} \frac{1}{\lambda_k} \geqslant \frac{(n-2)^2}{\sum_{k=2}^{n-1} \lambda_k} = \frac{(n-2)^2}{n-\lambda_n}$$

and then

$$K'(G) = \frac{2m}{\lambda_n} + 2m \sum_{k=2}^{n-1} \frac{1}{\lambda_k} \geqslant \frac{2m}{\lambda_n} + 2m\frac{(n-2)^2}{n-\lambda_n}$$

with equality if and only if $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1}$.

Note that $f(x) = ((2m)/x) + (2m(n-2)^2)/(n-x)$ is increasing for $x \geqslant n/(n-1)$ and, by Theorem 2.1(ii), $\lambda_n \geqslant n/(n-1)$. Hence,

$$K'(G) \geqslant f(\lambda_n) \geqslant f\Big(\frac{n-1}{n}\Big) = \frac{2m(n-1)^2}{n}.$$

Equality holds if and only if $\lambda_2 = \lambda_3 = \cdots = \lambda_n = n/(n-1)$, that is, G is complete. This completes the proof. □

Lemma 4.2. *Let G be a connected graph with $n \geqslant 2$ vertices and $\Psi(G; x) = x^n + a_1 x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x$. Then*

$$\frac{K'(G)}{2m} = -\frac{a_{n-2}}{a_{n-1}} \quad (a_{n-2} = 1 \text{ whenever } n = 2).$$

Proof. Let $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the normalised Laplacian eigenvalues of G. Then $\lambda_i\,(i = 2, 3, \ldots, n)$ satisfy the equation $x^{n-1} + a_1 x^{n-2} + \cdots + a_{n-2}x + a_{n-1} = 0$ and so $1/\lambda_i\,(i = 2, 3, \ldots, n)$ satisfy the reciprocal equation and, by Theorem 2.4 and Vieta's theorem,

$$\frac{K'(G)}{2m} = \sum_{i=2}^{n} \frac{1}{\lambda_i} = -\frac{a_{n-2}}{a_{n-1}},$$

as desired. □

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THEOREM 4.3. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices. Then*

(i) $K'(l(G)) = ((2(k-1)^2)/k)K'(G) + (n^2(k-1)^2(k-2))/2$;

(ii) $K'(s(G)) = 8K'(G) + nk(nk - 2n + 1)$;

(iii) $K'(r(G)) = 6K'(G) + (n^2k(3k-2))/2$;

(iv) $K'(q(G)) = 2(k+1)K'(G) + (n^2k^3)/2$.

PROOF. (i) Assume that the size of G is m. Let $\Psi(G; x) = x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x$ and $S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the normalised Laplacian spectrum of G, where $\lambda_1 = 0$. By Lemma 4.2,

$$\frac{K'(G)}{2m} = -\frac{a_{n-2}}{a_{n-1}} \tag{4.1}$$

and it follows from Theorem 3.5 that the coefficient of x^2 in $\Psi(l(G); x)$ is

$$\frac{k^n}{2^n(k-1)^m}\Big[(m-n)(k-1)(-k)^{m-n-1}a_{n-1}\frac{2(k-1)}{k} + (-k)^{m-n}a_{n-2}\frac{4(k-1)^2}{k^2}\Big], \tag{4.2}$$

whereas the coefficient of x in $\Psi(l(G); x)$ is

$$\frac{k^n}{2^n(k-1)^m}\Big[(-k)^{m-n}a_{n-1}\frac{2(k-1)}{k}\Big]. \tag{4.3}$$

Note that $l(G)$ has $m(k-1)$ edges. From (4.2), (4.3) and Lemma 4.2,

$$\frac{K'(l(G))}{2m(k-1)} = -\frac{2(k-1)}{k}\frac{a_{n-2}}{a_{n-1}} + \frac{(m-n)(k-1)}{k}.$$

Substituting (4.1) and $m = (kn)/2$ into the above equation yields

$$K'(l(G)) = \frac{2(k-1)^2}{k}K'(G) + \frac{n^2(k-1)^2(k-2)}{2},$$

as desired. By a similar discussion, we can also show that (ii)–(iv) hold. □

The following results are a direct consequence of Theorems 4.1 and 4.3.

COROLLARY 4.4. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices. Then*

$$K'(l(G)) \geqslant 2(k-1)^2(n-1)^2 + \frac{n^2(k-1)^2(k-2)}{2},$$
$$K'(s(G)) \geqslant 8k(n-1)^2 + nk(nk-2n+1),$$
$$K'(r(G)) \geqslant 6k(n-1)^2 + \frac{n^2k(3k-2)}{2},$$
$$K'(q(G)) \geqslant 2k(k+1)(n-1)^2 + \frac{n^2k^3}{2}.$$

Equality holds in each case if and only if G is a complete graph.

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5. The number of spanning trees of $l(G)$, $s(G)$, $r(G)$ and $q(G)$

Let G be a regular graph. In this section, we give some formulae for the number of spanning trees of $l(G)$ (respectively $s(G)$, $r(G)$ and $q(G)$). Our results are motivated directly from [7, 22, 23]. We use $\tau(G)$ to denote the total number of spanning trees of a graph G.

The following result is a known formula for the number of spanning trees in the line graph $l(G)$ (see, for example, [2, page 36]). We use a new method to prove this result.

Theorem 5.1 [2]. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices and m edges. Then*

$$\tau(l(G)) = 2^{m-n+1}k^{m-n-1}\tau(G).$$

Proof. Let $\Psi(G; x) = x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x$ and $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the normalised Laplacian eigenvalues of $l(G)$. Then

$$\prod_{k=2}^{n} \lambda_k = (-1)^{n-1}a_{n-1}$$

and, by Theorem 2.1(iii),

$$k^n(-1)^{n-1}a_{n-1} = 2m\tau(G). \tag{5.1}$$

In view of (4.3), the coefficient of x in $\Psi(l(G); x)$ is

$$\frac{k^n}{2^n(k-1)^m}\bigg[(-k)^{m-n}a_{n-1}\frac{2(k-1)}{k}\bigg].$$

Note that $\prod_{i=1}^{m} d_i(l(G)) = 2^m(k-1)^m$ and $l(G)$ has $m(k-1)$ edges, so, by Theorem 2.1(iii),

$$2^m(k-1)^m\bigg[(-1)^{m-1}\frac{k^n}{2^n(k-1)^m}(-k)^{m-n}a_{n-1}\frac{2(k-1)}{k}\bigg] = 2m(k-1)\tau(l(G)). \tag{5.2}$$

Substituting (5.1) into (5.2) yields the desired conclusion. □

Theorem 5.2. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices and m edges. Then*

$$\tau(s(G)) = 2^{m-n+1}\tau(G).$$

Proof. Let $\Psi(G; x) = x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x$. From Theorem 3.6,

$$\Psi(s(G); x) = (-\tfrac{1}{2})^n(x-1)^{m-n}[2^nx^n(2-x)^n + \cdots + a_{n-2}2^2x^2(2-x)^2 + a_{n-1}2x(2-x)].$$

Consequently, the coefficient of x in $\Psi(s(G); x)$ is

$$(-\tfrac{1}{2})^n(-1)^{m-n}4a_{n-1}.$$

Note that $\prod_{i=1}^{m+n} d_i(s(G)) = 2^mk^n$ and $s(G)$ has $2m$ edges, so, by Theorem 2.1(iii),

$$2^mk^n[(-1)^{m+n-1}(-\tfrac{1}{2})^n(-1)^{m-n}4a_{n-1}] = 4m\tau(s(G)). \tag{5.3}$$

Substituting (5.1) into (5.3) yields the desired conclusion. □

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REMARK 5.3. For Theorem 5.2, we may give an easy combinatorial proof. It is easy to see that $\tau(s(G)) = 2^{m-n+1}\tau(G)$, by noting that for every spanning tree of G, there are 2^{m-n+1} spanning trees for $s(G)$ by looking at each vertex which came from an edge of G. If that edge of G is part of the spanning tree, use both adjacent edges in $s(G)$. If it is not part of a spanning tree, then use only one of the two adjacent edges in $s(G)$. Note that there are $m - (n - 1)$ of the latter (that is, m edges and $n - 1$ of them have already been accounted for by the spanning tree edges coming from G).

Zhang *et al.* [23] proved that if G is a k-regular graph with n vertices and m edges, then

$$\tau(l(s(G))) = k^{m-n-1}(k + 2)^{m-n+1}\tau(G),$$

which inspired us to consider the formulae for $\tau(s(l(G)))$ with the same constraints on G. Note that the line graph of a regular graph is still regular, so that our results can be used to determine $s(l(G))$. But the edge subdivision graph is generally not regular and so $l(s(G))$ is not determined by the result in [23]. Combining Theorems 5.1 and 5.2 yields the following result.

COROLLARY 5.4. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices and m edges. Then*

$$\tau(s(l(G))) = 2^{m(k-1)-n+2}k^{m-n+1}\tau(G).$$

THEOREM 5.5. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices and m edges. Then*

$$\tau(r(G)) = 2^{m-n+1}3^{n-1}\tau(G).$$

PROOF. Let $\Psi(G; x) = x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x$. From Theorem 3.7, the coefficient of x in $\Psi(r(G); x)$ is

$$\frac{1}{4^n}[(-1)^{m-n}(-3)^n 2a_{n-1}].$$

Note that $\prod_{i=1}^{m+n} d_i(r(G)) = 2^m(2k)^n = 2^{m+n}k^n$ and $r(G)$ has $3m$ edges, so, by Theorem 2.1(iii),

$$2^{m+n}k^n\Big[(-1)^{m+n-1}\frac{1}{4^n}((-1)^{m-n}(-3)^n 2a_{n-1})\Big] = 6m\tau(r(G)). \tag{5.4}$$

Substituting (5.1) into (5.4) yields the desired result. □

THEOREM 5.6. *Let G be a connected k-regular graph with $n \geqslant 2$ vertices and m edges. Then*

$$\tau(q(G)) = 2^{m-n+1}(k + 1)^{m-1}\tau(G).$$

PROOF. Let $\Psi(G; x) = x^n + a_1x^{n-1} + \cdots + a_{n-2}x^2 + a_{n-1}x$. From Theorem 3.8, the coefficient of x in $\Psi(q(G); x)$ is

$$\frac{1}{2^n k^m}[(-k - 1)^m 2a_{n-1}].$$

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As $\prod_{i=1}^{m+n} d_i(q(G)) = (2k)^m k^n = 2^m k^{m+n}$ and $q(G)$ has $m(k+1)$ edges, Theorem 2.1(iii) gives

$$2^{m+n}k^n\left[(-1)^{m+n-1}\frac{1}{2^n k^m}(-k-1)^m 2a_{n-1}\right] = 2m(k+1)\tau(q(G)). \tag{5.5}$$

Substituting (5.1) into (5.5) yields the result. □

REMARK 5.7. While Theorem 5.2 has an easy combinatorial proof, the results of Section 5 should all have combinatorial explanations and it might be interesting to find them.

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