ARTICLES

COMPUTATIONAL ASPECTS OF SUNDT'S GENERALIZED CLASS [1]

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ABSTRACT

Some practical applications of SUNDT'S (1992) generalized class of counting distributions are discussed. The numerical stabilities of some recursive formulas in Sundt's class are investigated.

KEYWORDS

Compound distributions; recursive method; stability.

1. INTRODUCTION

To model the claims from an insurance portfolio over an accounting period, assume that the claim frequency N is a non-negative integer-valued random variable with probability function (p.f.) $\{p_i\}_{i=0}^{\infty}$. Further assume that, conditional on N, the N claims $X_1, X_2, \ldots, X_N$ are positive integer-valued random variables, mutually independent and indentically distributed with common discrete density $\{f(x)\}_{x=1}^{\infty}$, called the claim severity p.f.. We are interested in the total claim amount

$$S = X_1 + \ldots + X_N,$$

which has a compound distribution with p.f.

$$g(x) = \sum_{n=0}^{\infty} p_n f^{*n}(x), \qquad x = 0, 1, 2, \ldots \tag{1}$$

Equation (1) may be difficult to use because of the high order of convolutions involved.

PANJER (1981) observed that the widely used Poisson, negative bionomial and binomial claim frequencies share the common recursive pattern:

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \qquad \mathrm{n} = 1, 2,3, \ldots \tag{2}$$

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and showed that the corresponding compound distribution can be evaluated recursively:

$$(3) \qquad g(x) = \sum_{j=1}^{x} \left(a + b\frac{j}{x}\right) f(j)\, g(x-j), \qquad x = 1, 2, 3, \ldots$$

$$g(0) = p_0.$$

The recursive formula (3) is very handy for computer programming and significantly reduces the computing time comparing with a direct convolution approach (1).

SUNDT and JEWELL (1981) showed that Poisson, negative binomial and binomial distributions are the only members of the class defined by equation (2).

DE PRIL (1985) derived recursions for the n-fold convolution of an arbitrary non-negative discrete distribution, which can be viewed as a variant of evaluating a compound binomial distribution.

SCHRÖTER (1990) generalized the recursion (3) to the class of counting distributions satisfying the recursion:

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1} + \frac{c}{n} p_{n-2}, \qquad \mathrm{n} = 1, 2, 3, \ldots, \qquad p_{-1} = 0;$$

which is further generalized by SUNDT (1992) to a class of counting distributions satisfying:

$$(4) \qquad p_n = \sum_{j=1}^{k} \left(a_j + \frac{b_j}{n}\right) p_{n-1}, \qquad \mathrm{n} = 1, 2, \ldots$$

with $p_n = 0$, for $n < 0$. It turns out (SUNDT, 1992, p. 65) that every discrete density $\{p_0, p_1, \ldots, p_k\}$ (k can be ∞) with $p_0 > 0$ can fit into (4) by choosing

$$(5) \qquad a_j = -\frac{p_j}{p_0}, \qquad b_j = 2j\frac{p_j}{p_0}, \qquad j = 1, 2, \ldots, k.$$

Sundt also discussed the properties of convolutions of members of this generalized class (4), giving a general result including the one in DE PRIL (1985) as a special case.

For a claim frequency distribution in the class (4), by a conditional probability argument, Sundt derived a recursion for the corresponding compound distribution:

$$(6) \qquad g(x) = \sum_{y=1}^{x} g(x-y) \sum_{j=1}^{k} \left(a_j + \frac{b_j y}{jx}\right) f^{*j}(y), \qquad x = 1, 2, \ldots$$

$$g(0) = p_0,$$

which unifies the results of PANJER (1981) (when $k = 1$) and SCHRÖTER (1990) (when $k = 2$ and $a_2 = 0$).

In this paper, SUNDT (1992) extended the class (4) even further by loosening the recursive range:

$$p_n = \sum_{j=1}^{k} \left(a_j + \frac{b_j}{n}\right) p_{n-1}, \qquad n = \omega+1, \omega+2, \ldots \tag{7}$$

and derived a recursion for the corresponding compound distributions:

$$\begin{aligned} g(x) &= \sum_{y=1}^{x} g(x-y) \sum_{j=1}^{k} \left(a_j + \frac{b_j y}{jx}\right) f^{*j}(y) + \\ &\quad + \sum_{n=1}^{\omega} \left[p_n - \sum_{j=1}^{k} \left(a_j + \frac{b_j}{n}\right) p_{n-j}\right] f^{*n}(x), \qquad x = 1, 2, \ldots \\ g(0) &= p_0 . \end{aligned} \tag{8}$$

However, the model fitting of the class (7) in practical applications and the computational aspects of the recursions (7) and (8) are not discussed in SUNDT (1992). Our present paper is devoted to addressing these concerns.

2. MODEL FITTING AND COMPUTING EFFORT

In fitting probability models, the parismony principle is observed. In practical applications, since it is desirable to try to fit a claim frequency model with relatively few (2 or 3) parameters, the recursive relation (7) is useful only when the claim frequency distribution can fit into (7) with small k and ω.

There are many well known counting distributions which can fit into (7) with $k \leq 2$ and $\omega \geq 1$.

The Delaporte distribution (RUOHONEN, 1988; WILLMOT and SUNDT, 1989), which is in the class of SCHRÖTER (1990), satisfies (7) with $k = 2$ and $\omega = 0$. The Pólya-Aeppli distribution (JOHNSON et al., 1992, p. 329-330), which is not in the classes of PANJER (1981) or SCHRÖTER (1990), satisfies (7) with $k = 2$ and $\omega = 0$. Other interesting examples for the general class (7) can be found among the mixed Poisson distributions in WILLMOT (1993). The Poisson-Pareto is obtained by mixing the Poisson mean λ over a Pareto density:

$$h(x) = \frac{\alpha\mu^{\alpha}}{(\mu+x)^{\alpha+1}} .$$

The Poisson-Pareto p.f. satisfies recursion (7) with $k = 2$ and $\omega = 1$:

$$p_n = \left(1 - \frac{1+\alpha+\mu}{n}\right) p_{n-1} + \frac{\mu}{n} p_{n-2}, \qquad n = 2, 3, \ldots, \tag{9}$$

with the boundary condition $p_1 = \alpha - (\alpha+\mu) p_0$.

The Poisson-Truncated-Normal p.f., where the Poisson mean λ has a Normal density left truncated at point zero satisfies recursion (7) with $k=2$ and $\omega=0$:

$$(10) \qquad p_n = \frac{\mu-\sigma^2}{n} p_{n-1} + \frac{\sigma^2}{n} p_{n-2}, \qquad n=1,2,\ldots$$

with $p_{-1}=0$.

On the other hand, many other counting distributions cannot fit into (7) with finite number of parameters. For exampple, the Poisson-Inverse-Gaussian (P-IG) is a two parameter distribution with a p.f. satisfying (WILLMOT, 1987):

$$(11) \qquad p_n = \frac{\beta(2n-3)}{(1+2\beta)n} p_{n-1} + \frac{\mu^2}{(1+2\beta)n(n-1)} p_{n-2}, \qquad n \geq 2,$$

with intial values:

$$p_0 = e^{\mu\beta^{-1}[1-(1+2\beta)^{1/2}]}, \qquad p_1 = \mu(1+2\beta)^{-1/2} p_0.$$

It is noted that recursion (11) is not of the same type as (7), and an infinite number of a_j and b_j's would be needed to fit it into (7).

Another example is the Generalized Poisson with a p.f. (GOOVAERTS and KAAS, 1991):

$$p_n = \frac{\theta(\theta+n\lambda)^{n-1} e^{-\theta-n\lambda}}{n!}, \qquad n=0,1,\ldots.$$

ISLAM and CONSUL (1992) suggested the Generalized Poisson distribution for automobile insurance claim data. On the other hand, ELVERS (1991) reported that the Generalized Poisson dit not fit well the data sets which he studied. GOOVAERTS and KAAS (1991) derived a recursive scheme for the compound Generalized Poisson distributions. Again, the two-parameter Generalized Poisson cannot fit it into (7) with finite number of a_j and b_j's.

Now consider the computing work needed in evaluating the compound distribution for an *arbitrary* frequency $\{p_0, p_1, \ldots, p_r\}$ (r can be ∞) with $p_0 > 0$. For this arbitrary frequency, most probably it would fit into class (4) with $k=r$ (as in (5)). Compare Sundt's recursive scheme (6) with the direct convolution approach (1). First, both (6) and (1) need to evaluate convolutions up to the k-th fold. After that, Sundt's recursive scheme (6) needs one more recursive evaluation, while (1) needs taking an weighted average of the obtained convolutions. Therefore, for an arbitrary claim frequency, the computing effort using Sundt's recursive scheme (6) is of the same order of magnitude as that needed by a direct convolutin approach.

To conclude, Sundt's recursive scheme (6) or (8) is practically useful only when the clam frequency can fit into (4) or (7) with relatively few parameters (i.e. when k and ω are small).

3. STABILITY AGAINST ROUND-OFF ERRORS

SUNDT (1992) did not discuss under which circumstances the recursion was of practical utility. When computational use is concerned, one needs to know the numerical stability of the recursions against round-off errors. Since computers can only represent finite number of digits, round-off errors are inevitable. The recursive nature makes computer evaluation very simple to implement, but it may also create problems due to the accumulation of round-off errors in the evaluation process.

PANJER and WANG (193) studied the numerical stability of recursive formulas against round-off errors. A recursive evaluation is said to be stable if the relative error grows linearly, and unstable otherwise. If a recursive evaluation is unstable, the accumulated error grows rapidly and makes the solution no longer useful. For example, one may get incorrect large negative numbers in evaluating a probability distribution when using unstable recursive evaluations.

In this Section, we try to give some general insight into stability theory.

Consider the linear homogeneous recursion in the forward direction

$$g(x) = \sum_{j=1}^{m} A_j(x)\, g(x-j), \qquad x > x_0, \qquad A_m(x) \neq 0, \tag{12}$$

where m is the *order* of the recursion. Under some regularity conditions (CASH, 1979, p. 2; WIMP, 1984, p. 19, p. 272), recursion (12) has a fundamental set of m basic solutions $\{g^{(h)}(x),\ h = 1, \ldots, m\}$ such that

- $g^{(1)}(x)$ outgrows all the other solutions

$$\lim_{x \to \infty} \frac{g^{(1)}(x)}{g^{(h)}(x)} = \infty \qquad \text{for} \qquad 2 \leq h \leq m;$$

- every solution $g(x)$ of (12) can be expressed as their linear combinations

$$g(x) = c_1 g^{(1)}(x) + \ldots + c_m g^{(m)}(x),$$

  where $g(x)$ is called a **dominant** solution if $c_1 \neq 0$, or a **subordinate** solution if $c_1 = 0$.

On the other hand, the round-off error propagation $\varepsilon(x)$, as a disturbance solution, can be written as a linear combination of the fundamental set:

$$\varepsilon(x) = \varepsilon_1 g^{(1)}(x) + \ldots + \varepsilon_m g^{(m)}(x), \tag{13}$$

where ε_1 is small, but with probaility 1 that $\varepsilon_1 \neq 0$.

Since $\varepsilon_1 \neq 0$, one has

$$\lim_{x \to \infty} \frac{\varepsilon(x)}{g(x)} = \begin{cases} \infty & \text{if} \quad g(x) \text{ is subordinate} \\[2ex] \dfrac{\varepsilon_1}{c_1} & \text{if} \quad g(x) \text{ is dominant} \end{cases} \tag{14}$$

where $\dfrac{\varepsilon_1}{c_1}$ can be made arbitrarily small by using sufficient number of digits.

It is the rate of relative growth of the desired solution with respect to other solutions that determines whether or not a recursive computation is successful. As a sufficient condition for stability, the recursive evaluation using (12) is stable if the desired solution $g(x)$ is a dominant solution and unstable if the desired solution is a subordinate solution.

4. Perron's theorem and Poincare's lemma

In this Section, we shall introduce Perron's theorem, which we believe is the most important asymptotic result for the solutions of recursive formulas of finite order.

Assume that

$$\lim_{x \to \infty} A_j(x) = \mu_j, \qquad j = 1, \ldots, m.$$

The polynomial equation:

$$\Phi(z) = z^m - \sum_{j=1}^{m} \mu_j z^{m-j} = 0, \tag{15}$$

is called the **characteristic equation** for recursion (12).

Theorem 1 (Perron): Let $t_1, t_2, \ldots, t_m$ be the roots of the characteristic equation (15) and assume that they all have distinct modulus. Then the recursion (12) has a fundamental set of solutions $\{g^{(h)}(x), h = 1, \ldots, m\}$ such that

$$\lim_{x \to \infty} \frac{g^{(h)}(x+1)}{g^{(h)}(x)} = t_h, \qquad h = 1, \ldots, m. \tag{16}$$

Proof: See Milne-Thompsn (1968, p. 548).

When the characteristic equation (15) has repeated roots, Poincaré's result may be useful:

Lemma 1 (Poincaré): Let c be a number whose modulus is greater than that of every root t_i of the characteristic equation (15), then for every solution $g(x)$ of the recursion (12),

$$\lim_{x \to \infty} \frac{g(x)}{c^x} = 0. \tag{17}$$

Proof: See Milne Thomson (1968, p. 551).

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Definition 1: For a function g defined on non-negative integers, we define the **tail index** as:

(18) $$\rho_g = \lim_{x \to \infty} \frac{|g(x+1)|}{|g(x)|} \qquad \text{(if exists)}.$$

If $g(x)$ has only finite support, we define $\rho_g = -1$. We say the $g(x)$ has a thicker tail than $h(x)$, if $\rho_g > \rho_h$.

5. CONVOLUTIONS OF MEMBERS IN PANJER'S CLASS

SUNDT (1992, p. 70-71) presents a nice argument on convolutions of the members of the class (4). The following is a special case of convolutions of members in Panjer's class. It may be useful in combining independent portfolios each having a claim frequency distribution in Panjer's class.

Let $R_1[\alpha, \beta]$ denote a member in Panjer's class (2) with parameters $a = \alpha$ and $b = \beta$. Since the convolution of $R_1[\alpha, \beta_1]$ and $R_1[\alpha, \beta_2]$ is $R_1[\alpha, \alpha + \beta_1 + \beta_2]$,we can drop this trivial case. In the following, we assume that α_j's are different from each other.

Lemma 2: The convolution of r distributions $R_1[\alpha_1, \beta_1], \ldots, R_1[\alpha_r, \beta_r]$ can be evaluated recursively as:

(19) $$p_n = \sum_{j=1}^{r} \left(a_j + \frac{b_j}{n} \right) p_{n-j},$$

with

(20) $$a_j = (-1)^{j+1} \sum_{1 \le l_1 < l_2 < \ldots < l_j \le r} \prod_{i=1}^{j} \alpha_{l_i}, \qquad (j = 1, \ldots, r),$$

(21) $$b_j = (-1)^{j+1} \sum_{s=1}^{r} \beta_s \sum_{\substack{1 \le l_1 < l_2 < \ldots < l_{j-1} \le r \\ l_i \ne s \, (i = 1, \ldots, j-1)}} \prod_{i=1}^{j-1} \alpha_{l_i}, \qquad (j = 2, \ldots, r),$$

(22) $$b_1 = \sum_{j=1}^{r} \beta_j.$$

Proof: See SUNDT (1992, p. 70-71).

Lemma 3 (Bender): Let $g * h$ denote the convolution of counting distributions $g(x)$ and $h(x)$. We have

$$\rho_{g*h} = \max\{\rho_g, \rho_h\}$$

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Proof: See BENDER (1974); WILLMOT (1989).

Theorem 2: In evaluating the convolution of r distributions

$$R_1[\alpha_1, \beta_1], \ldots, R_1[\alpha_r, \beta_r],$$

the recursion (19) is stable in the forward direction if the α_j which has the largest absolute value in $\{\alpha_1, \ldots, \alpha_r\}$ is positive; and is unstable in the forward direction if the α_j which has the largst absolute value in $\{\alpha_1, \ldots, \alpha_r\}$ is negative.

Proof: Here we assume that at least one α_j is non-negative and leave the discussion for the case where all the α_j's are negative to the next Section.

By Lemma 3, the tail for the convolution of $R_1[\alpha_1, \beta_1]$ and $R_1[\alpha_2, \beta_2]$ has the same tail index as the one with a thicker tail. Then the convolution of m distributions

$$R_1[\alpha_1, \beta_1], \ldots, R_1[\alpha_r, \beta_r],$$

has a tail index of $\max\{\alpha_1, \ldots, \alpha_r\}$.

The characteristic equation of recursion (19) is

$$\Phi(z) = z^k - \sum_{j=1}^{k} a_j z^{k-j} = 0. \tag{23}$$

From (20), we can factorize $\Phi(z)$ into:

$$\Phi(z) = \prod_{j=1}^{k} (z - \alpha_j) = 0. \tag{24}$$

Applying Perron's Theorem to the characteristic equation (24), recursion (19) has r basic solutions with tail indexes $|\alpha_1|, \ldots, |\alpha_r|$, respectively.

If the $\alpha_j$which has the largest absolute value in $\{\alpha_1, \ldots, \alpha_r\}$ is positive, then

$$\max\{\alpha_1, \ldots, \alpha_r\} = \max\{|\alpha_1|, \ldots, |\alpha_r|\},$$

the convolution of r distributions

$$R_1[\alpha_1, \beta_1], \ldots, R_1[\alpha_r, \beta_r],$$

is a dominant solution of (19) and the recursive evaluation by (19) is stable.

If the largest α_j which has the largest absolute value in $\{\alpha_1, \ldots, \alpha_r\}$ is negative, then

$$\max\{\alpha_1, \ldots, \alpha_r\} < \max\{|\alpha_1|, \ldots, |\alpha_r|\},$$

the convolutions of r distributions

$$R_1[\alpha_1, \beta_1], \ldots, R_1[\alpha_r, \beta_r],$$

is a subordinate solution of (19) and the recursive evaluation by (19) is unstable. □

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Corollary 1: Let α_j and β_j be given as in Lemma 2.

1. The cursion (19) is stable in the forward direction in evaluating any finite number of convolutions of Poisson and/or negative binomial distributions.
2. The recursion (19) is unstable in the forward direction in evaluating convolutions of Poisson and binomial distributions.
3. For the convolution of a binomial $R_1(\alpha_1, \beta_1)$ and a negative binomial $R_1(\alpha_2, \beta_2)$, the recursion (19) is
   - stable in the forward direction if $|\alpha_1| < \alpha_2$;
   - unstable in the forward direction if $|\alpha_1| > \alpha_2$.

If the recursion (4) for the claim frequency is stable, then the recursion (6) for the compound distribution is likely to be stable, since it involves terms of the same form. As a special case, recursion (4) is stable in evaluating the Delaporte distribution which is a convolution of a Poisson with a negative binomial. It can also be viewed as a mixed Poisson with a shifted gamma mixing density (WILLMOT and SUNDT, 1989). The recursion (6) is also stable in evaluating compound Delaporte distributions.

6. CONVOLUTIONS OF BINOMIAL DISTRIBUTIONS

Generally, if the desired solution of a recursive evaluation has only finite support, in either direction, the desired solution grows up at the beginning points and damps out at the end points. Therefore, the recursive evaluation is only stable at the beginning and become more and more unstable when they move to the other end. Over any specified range, the more stable it is in forward direction, the more unstable it is in the backward direction, and vice versa.

As a direct application, for a probability function $\{p_0, p_1, \ldots, p_k\}$ with finite support ($k < \infty$) and $p_0 > 0$, the recursion (4) with a_j and b_j given in (5) is unstable.

A binomial distribution with parameters (N, θ) is defined as

$$p_n = \frac{N!}{(N-n)!\,n!}\,\theta^n (1-\theta)^{N-n} < 0, \qquad n = 0, 1, \ldots, N;$$

which has a finite support and satisfies (2) with

$$a = -\frac{\theta}{1-\theta}, \qquad b = \frac{(N+1)\,\theta}{1-\theta}.$$

Left undiscussed in the proof of Theorem 2 is the case where all the α_j's are negative, i.e., convolutions of r binomial distributions. Since the convolution of r binomial distributions with parameters (N_i, θ_i) ($i = 1, \ldots, r$) has only finite support, in either direction, the recursion (19) is only stable at the beginning and become more and more unstable when they move to the other end. In this case, one utilize two directions to get a stable evaluation.

To illustrate this, we consider the convolution of two binomial distributions with parameters (N_1, θ_1) and (N_2, θ_2) where $\theta_1 \neq \theta_2$. From Lemma 1, the convolution gives a p.f. satisfying:

$$p_n = \left(a_1 + \frac{b_1}{n}\right) p_{n-1} + \left(a_2 + \frac{b_2}{n}\right) p_{n-2}, \tag{25}$$

with

$$a_1 = -\frac{\theta_1}{1-\theta_1} - \frac{\theta_2}{1-\theta_2}, \qquad a_2 = -\frac{\theta_1 \theta_2}{(1-\theta_1)(1-\theta_2)},$$

$$b_1 = \frac{(N_1+1)\theta_1}{1-\theta_1} + \frac{(N_2+1)\theta_2}{1-\theta_2}, \qquad b_2 = -\frac{(N_1+N_2+2)\theta_1 \theta_2}{(1-\theta_1)(1-\theta_2)},$$

and initial values

$$g(-1) = 0, \qquad g(0) = (1-\theta_1)^{N_1} (1-\theta_2)^{N_2}.$$

One can easily re-write (25) into a backward recursion with starting values:

$$g(N_1+N_2) = \theta_1^{N_1} \theta_2^{N_2}, \qquad g(N_1+N_2+1) = 0.$$

Example 1: Assume that

$$\theta_1 = .3, \qquad \theta_2 = .7, \qquad N_1 = 100, \qquad N_2 = 200,$$

and 10 digits are used, then (25) is unstable in both directions. As in the compound binomial case (PANJER and WANG, 1993, p. 249-52), negative values are observed during the evaluation. However, both directions produce the same values in their first 8 digits over the middle range [165,199], which suggests that a combined range of two directions can given at least 8 significance digits over the whole range [0,300].

In Example 1, one can also first calculate the discrete density of each of the two binomial distributions and then convolute them. It is numerically stable to do convolutions.

7. PREFERABLE RECURSIVE SCHEMES

A probability function can satisfy many different recursions. In applications, among various recursive schemes, it would be good to know which one is preferable based on the following criteria:

(i) stability,
(ii) simplicity,
and
(iii) computing effort.

The Generalized Pólya-Aeppli frequency models: Let the claim frequency be in $R_1[\alpha, \beta]$ and the claim severity have a geometric distribution:

$$f(x) = pq^{x-1}, \qquad (q = 1 - p, \qquad x = 1, 2, \ldots). \tag{26}$$

then the total claim has a compound distribution, which is called the Generalized Pólya-Aeppli distribution (JOHNSON et al., 1992, p. 329-330).

MILIDIU (1985, p. 10) generalized a result of EVANS (1953) and gave a recursion for the Generalized Pólya-Aeppli distribution:

$$g(x) = \left(a_1 + \frac{b_1}{x}\right) g(x-1) + \left(a_2 + \frac{b_2}{x}\right) g(x-2), \qquad x \geq 1, \tag{27}$$

with

$$\begin{cases} a_1 = 2q + \alpha(1-q), & b_1 = -2q + \beta(1-q), \\ a_2 = -q(q + \alpha(1-q)), & b_2 = 2q(q + \alpha(1-q)), \end{cases} \tag{28}$$

and initial values

$$g(0) = (1-\alpha)^{\left(\frac{\beta}{\alpha} - 1\right)}, \qquad g(-1) = 0. \tag{29}$$

To evaluate the Generalized Póly-Aeppli distribution, recursion (27) is stable in the forward direction if $\alpha \geq 0$; in this case, (27) is preferable to Panjer's recursion with a geometric severity.

The recursion (27) can be verified being unstable for $\alpha < 0$ (i.e. binomial frequency). It is interesting that, by a re-parametrization, a compound binomial geometric can be turned into a compound negative binomial geometric (PANJER and WILLMOT, 1992, p. 270), which can be evaluated stably by recursion (27).

The compound Generalized Pólya-Aeppli distribution can be evaluated using Sundt's recursion (6); on the other hand, one can evaluate the compound Generalized Pólya-Aeppli distribution by a two-stage Panjer's recursion (3). Again, we would say that Sundt's recursive scheme (6) is preferable.

The Poisson Inverse Gaussian (P-IG) frequency models: WILLMOT (1987) fits 6 sets of claim frequency data and finds that the P-IG provides superior fit. He also discussed the parameter estimation in the model fitting.

Even though one can view the P-IG as a compound Poisson ETNB (Extended truncated negative binomial, WILLMOT, 1988), the preferable method for generating the probability function of the P-IG is the recursion (11). For the compound P-IG, as a member in the Sichel family, WILLMOT and PANJER (1987) derived recursive formulas in terms of auxiliary functions. Based on our experience, the preferable method is a two-stage Panjer's recursion (3) by viewing the P-IG as a compound Poisson ETNB, since the computation is both simple and stable.

Combining insurance portfolios: Assume that two independent insurance portfolios have the same severity distribution and have claim frequency distributions in Panjer's class. When combining these two portfolios, the claim frequency distribution is a convolution of two members in Panjer's class, and the total claim distribution can be evaluated by recursion (6). Comparing with the approach of first evaluating each portfolio by using Panjer's recursion and then taking convolutions, Sundt's recursive scheme (6) generally requires less computing efforts.

The numerical stability of Sundt's class (4) has many interesting structures. In WANG and PANJER (1993), some critical points are identified for the recursion (9); only starting from these critical points one can get a stable evaluation of the Poisson Pareto probabilities; Also, the stability of the recursion (10) for the Poisson-Truncated-Normal probabilities depends directly on whether $\mu > \sigma^2$ or $\mu < \sigma^2$. A complete account for the numerical aspects of the recursions (7) and (8) remains open for further studies.

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