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The Kostrikin Radical and the Invariance of the Core of Reduced Extended Affine Lie Algebras

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Abstract. In this paper we prove that the Kostrikin radical of an extended affine Lie algebra of reduced type coincides with the center of its core, and use this characterization to get a type-free description of the core of such algebras. As a consequence we get that the core of an extended affine Lie algebra of reduced type is invariant under the automorphisms of the algebra.

1 Introduction

Extended affine Lie algebras are a class of infinite dimensional Lie algebras that were originally proposed by the physicists Hoegh-Krohn and Torresani [HT] under the name of irreducible quasi-simple Lie algebras. They are natural generalizations of finite-dimensional simple Lie algebras, affine Kac–Moody Lie algebras and toroidal Lie algebras. Our main reference for this class of algebras is the AMS-memoirs [AABGP]. Roughly speaking, extended affine Lie algebras are characterized by the existence of an invariant nondegenerate form and the fact that they have a decomposition into root spaces. The form gives rise to a partition of the root system into isotropic and non-isotropic roots, and the subalgebra generated by the root spaces corresponding to non-isotropic roots is called the *core*. The structure and representations of extended affine Lie algebras have been investigated in many papers, but it turns out that the whole structure of an extended affine Lie algebra can be recovered from its core as shown in [N2]. Moreover, for each extended affine Lie algebra, there exists a finite irreducible (non necessarily reduced) root system called its *type*, and it follows that the core of an extended affine Lie algebra is a so-called *Lie torus* of the same type, whose precise structure is known for the reduced types and for types BC_1 and BC_2.

An important question regarding the core of an extended affine Lie algebra is that of whether the core is invariant under the automorphisms of the algebra. A positive answer to this question was given by Krylyuk for the particular case of extended affine Lie algebras whose type is simply-laced of rank at least 2, by using notions of absolute zero divisors and the Kostrikin radical. The aim of this paper is to extend this positive answer to all extended affine Lie algebras of reduced type, and for that we also use notions of absolute zero divisors and the Kostrikin radical.

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Let us recall that the *Kostrikin radical* of a Lie algebra is the smallest ideal whose associated quotient algebra does not have *absolute zero divisors*, that is, elements satisfying $\mathrm{ad}_x^2 = 0$, and it is indeed a radical in the sense of Amitsur and Kurosh [Fi]. Absolute zero divisors of a Lie algebra have played an important role in several questions in mathematics, such as solving the restricted Burnside problem for a prime exponent [K2] or the characterization of the classical modular Lie algebras [S]. The main results of this paper are the following:

If E is an extended affine Lie algebra of reduced type, then

(i) $K(E) = \mathcal{Z}(E_c)$ *(Theorem* 3.3*),*

(ii) $E_c = [\mathcal{C}_E(K(E)), E]$ *(Theorem 5.1),*

(iii) E_c *is invariant under the automorphisms of E (Corollary 5.2),*

where E_c *is the core of E,* $\mathcal{Z}(E_c)$ *is its center,* $K(E)$ *is the Kostrikin radical of E, and* $\mathcal{C}_E(K(E))$ *is the centralizer of* $K(E)$ *in E.*

By using structure theory, Krylyuk proved the above (i) and (iii) and that $E_c = [\mathcal{C}_E(K(E)), \mathcal{C}_E(K(E))]$ for the particular case of E being an extended affine Lie algebra of simply laced type and rank at least 2 [Kr, Proposition 4.2(i), (ii) and Corollary 4.26].

This paper is organized as follows. We devote Section 2 to characterizing the Kostrikin radical of infinite dimensional Lie algebras that satisfy some fairly natural and not too restrictive conditions and we use this result in Section 3 to obtain (i) of the above. We point out that our proofs up to Section 3 are completely independent of structure theory, thus we also provide an alternative proof of [Kr, Proposition 4.2(i)]. On the other hand, in Section 4, we make use of the general construction of extended affine Lie algebras given in [N2] in order to prove that *the core of any extended affine Lie algebra E is* $E_c = [\mathcal{C}_E(\mathcal{Z}(E_c)), E]$, *where* $\mathcal{Z}(E_c) = K(E_c)$. In the last section, we specialize the previous characterization to extended affine Lie algebras of reduced type, which combined with (i) of the above, gives (ii). As an immediate consequence, we obtain the central result of this paper, namely, that the core of an extended affine Lie algebra of reduced type is invariant under the automorphisms of the algebra ((iii) of the above).

2 The Kostrikin Radical of a Lie Algebra

Throughout this section, L will be a Lie algebra over a ring of scalars Φ. We recall that an element $x \in L$ is called an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$, where as usual $\mathrm{ad}_x(y) := [x, y]$, for $y \in L$. We denote by $c(L)$ the set of all absolute zero divisors of L. Absolute zero divisors are also known as *crust of thin sandwiches* [K], since they satisfy the following property, whose proof is included here for completeness.

Lemma 2.1 *Suppose that* $\frac{1}{2} \in \Phi$ *and let* L' *be a subalgebra of L. If* $x \in L$ *is such that* $\mathrm{ad}_x^2(L') = 0$, *then* $\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x(L') = 0$ *for all* $y \in L'$.

Proof Let $x \in L$ such that $\mathrm{ad}_x^2(L') = 0$ and $y \in L'$. Then

$$0 = \mathrm{ad}_{[x,[x,y]]} = [\mathrm{ad}_x, [\mathrm{ad}_x, \mathrm{ad}_y]] = \mathrm{ad}_x^2 \mathrm{ad}_y - 2 \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x + \mathrm{ad}_y \mathrm{ad}_x^2 .$$

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But since $\operatorname{ad}_x^2(L') = 0$, we have

$$0 = (\operatorname{ad}_x^2 \operatorname{ad}_y - 2\operatorname{ad}_x \operatorname{ad}_y \operatorname{ad}_x + \operatorname{ad}_y \operatorname{ad}_x^2)(L') = -2\operatorname{ad}_x \operatorname{ad}_y \operatorname{ad}_x(L').$$

Hence $\operatorname{ad}_x \operatorname{ad}_y \operatorname{ad}_x(L') = 0$. ■

As a consequence, if the base ring contains $\frac{1}{2}$, then the set $c(L)$ is closed under the Lie product and therefore generates a subalgebra which is invariant under the automorphism group of L. A Lie algebra without nonzero absolute zero divisors is called *strongly nondegenerate.*

The *Kostrikin radical* of L, denoted $K(L)$, can be constructed as follows: put $K_0(L) = 0$, and let $K_1(L)$ be the ideal of L generated by $c(L)$. Using transfinite induction we define a nondecreasing chain of ideals $K_\alpha(L)$ by putting $K_\alpha(L) = \bigcup_{\beta<\alpha} K_\beta(L)$ for a limit ordinal α, and $K_\alpha(L)/K_{\alpha-1}(L) = K_1(L/K_{\alpha-1}(L))$ otherwise. Then $K(L) := \bigcup_\alpha K_\alpha(L)$ and $K(L)$ is the smallest ideal of L whose associated quotient algebra is strongly nondegenerate. The following trivial observation will be used later.

Remark 2.2 If $c(L)$ is an ideal of L, then $K_1(L) = c(L)$. If moreover $c(L/c(L)) = 0$, then $K(L) = c(L)$. Indeed, $c(L/K_1(L)) = 0$ implies that $K_1(L/K_1(L)) = 0$, that is, $K_2(L) = K_1(L)$ and then $K(L) = K_1(L)$.

Remark 2.3 If L is finite dimensional and the base ring is a field of characteristic 0, then L is semisimple if and only if it is strongly nondegenerate. Notice that if L is semisimple, then by Lemma 2.1, we have that if $x \in c(L)$, then $(\operatorname{ad}_x \operatorname{ad}_L)^2 = 0$, hence $c(L)$ is contained in the radical of the killing form of L. Conversely, if L is not semisimple, then it contains a nonzero abelian ideal I. Hence $0 \neq I \subseteq c(L)$ since for all $x \in I$, it follows that $[x,[x,L]] \subseteq [I,I] = 0$.

However, we are more interested in Lie algebras of infinite dimension.

Setting From now on and up to Theorem 2.9, $L = \bigoplus_{\delta\in G} L_\delta$ will be graded by an abelian group G. Put $R := \{\delta \in G : L_\delta \neq 0\} \cup \{0\}$. Then $L = \bigoplus_{\delta\in R} L_\delta$. We assume that there exists $R^a \subseteq R$ and denote $R^0 := R \setminus R^a$. For L as before, L^a will denote the subalgebra of L generated by $\{L_\alpha : \alpha \in R^a\}$. Also, for any subset $L' \subseteq L$, we will use the following standard notation: $c(L') := \{x \in L' : \operatorname{ad}_x^2(L') = 0\}$, $\mathcal{C}_L(L') := \{x \in L : \operatorname{ad}_x(L') = 0\}$ and $\mathcal{Z}(L') := \mathcal{C}_L(L') \cap L'$.

Remark 2.4 Let $\delta \in R$.

(a) If $\frac{1}{2} \in \Phi$ and $\{e_\delta, [e_\delta, e_{-\delta}], e_{-\delta}\}$ is an $\mathfrak{sl}_2$-triple, where $e_{\pm\delta} \in L_{\pm\delta}$, then $e_{\pm\delta} \in \operatorname{ad}^2_{e_{\pm\delta}}(L)$.

(b) If $e_\delta \in \operatorname{ad}^2_{e_\delta}(L)$ for all $e_\delta \in L_\delta$, then $c(L) \cap L_\delta = 0$.

(c) $\mathcal{Z}(I) \subseteq c(L)$ for all ideals I of L (note that if $x \in \mathcal{Z}(I)$, then $[x,[x,L]] \subseteq [x,I] = 0$). In particular, if $c(L) \cap L_\delta = 0$, then $\mathcal{Z}(I) \cap L_\delta = 0$.

Lemma 2.5 *If $(R^0 + R^a) \cap R \subseteq R^a$, then $R^a = -R^a$ is nonempty and the subalgebra L^a is an ideal of L. If moreover, $e_\alpha \in \operatorname{ad}^2_{e_\alpha}(L)$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$, then the ideal L^a is perfect,* i.e., $L^a = [L^a, L^a]$.

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Proof That $R^a \neq \emptyset$ is clear since otherwise $R^0 = R = \emptyset$. Also, if $R^a \neq -R^a$, then $0 \in R^a$ and then $\emptyset \neq R^0 \subseteq R^a$, which is a contradiction. Let $x = \sum_{\lambda\in F^x} x_\lambda \in L$, where $x_\lambda \in L_\lambda$, with F^x being a finite subset of R, and let $e_\alpha \in L_\alpha$ with $\alpha \in R^a$. Then $[x, e_\alpha] = \sum_{\lambda\in F^x\cap R^a}[x_\lambda, e_\alpha] + \sum_{\lambda\in F^x\cap R^0}[x_\lambda, e_\alpha]$. Clearly $\sum_{\lambda\in F^x\cap R^a}[x_\lambda, e_\alpha] \in L^a$ by definition of L^a, and $\sum_{\lambda\in F^x\cap R^0}[x_\lambda, e_\alpha] \in L^a$ because $(R^0 + R^a) \cap R \subseteq R^a$, therefore $[x, e_\alpha] \in L^a$. Now, by using the Jacobi identity we have that $[x, [e_\beta, e_\eta]] = [[x, e_\beta], e_\eta] + [e_\beta, [x, e_\eta]] \in L^a$ for all $e_\beta \in L_\beta, e_\eta \in L_\eta$, with $\beta, \eta \in R^a$, hence $[L, L^a] \subseteq L^a$. Note that $L^a = \bigoplus_{\delta\in R}(L^a \cap L_\delta)$, where if $\delta \in R^0$, then

$$L^a \cap L_\delta = \sum_{\substack{\alpha,\beta\in R^a\\ \alpha+\beta=\delta}} [L_\alpha, L_\beta] \subseteq [L^a, L^a].$$

If, on the contrary, $\delta \in R^a$, then $L^a \cap L_\delta = L_\delta$. Suppose that $e_\alpha \in \mathrm{ad}^2_{e_\alpha}(L)$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$. Let $e_\delta \in L_\delta$, $\delta \in R^a$. Then because of the grading, there exists $f_{-\delta} \in L_{-\delta}$ such that $e_\delta = [e_\delta, [e_\delta, f_{-\delta}]]$. Hence $L_\delta = [L_\delta, [L_\delta, L_{-\delta}]] \subseteq [L^a, L^a]$ and therefore $L^a = [L^a, L^a]$. ■

Proposition 2.6 *Assume that the abelian group G is ordered and satisfies*

(a) $c(L) \cap L_\alpha = 0$ *for all* $\alpha \in R^a$,

(b) *if* $\delta_1, \delta_2 \in R^0$ *and* $\alpha \in R$ *with* $\delta_1 < \alpha < \delta_2$, *then* $\alpha \in R^0$.

Then

(i) $c(L) \subseteq \oplus_{\delta\in R^0} L_\delta$.

(ii) *If* $(R^0 + R^a) \cap R \subseteq R^a$, *then*

$$\mathcal{Z}(L^a) \subseteq \{x \in c(L) : \mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(L^a) = 0, \forall e_\alpha \in L_\alpha, \alpha \in R^a\}.$$

If, moreover, $\frac{1}{2} \in \Phi$, *then*

$$\{x \in c(L) : \mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(L^a) = 0, \forall e_\alpha \in L_\alpha, \alpha \in R^a\} \subseteq \mathcal{C}_L(L^a).$$

Proof (i) It suffices to assume that $c(L) \neq 0$. Let $0 \neq x \in c(L)$ and write $x = \sum_{\lambda\in F^x} x_\lambda$, where $0 \neq x_\lambda \in L_\lambda$, with $F^x \subseteq R$. Let $\lambda_{\mathrm{m}} := \min(F^x)$ and $\lambda_{\mathrm{M}} := \max(F^x)$. For each $\delta \in R$, we have that $[x_{\lambda_{\mathrm{m}}}, [x_{\lambda_{\mathrm{m}}}, L_\delta]] \subseteq L_{\delta+2\lambda_{\mathrm{m}}}$, and by minimality of λ_{m}, this is the only term in $L_{\delta+2\lambda_{\mathrm{m}}}$ of the expansion of $[x, [x, L_\delta]]$. Therefore, since $[x, [x, L_\delta]] = 0$, we have that $[x_{\lambda_{\mathrm{m}}}, [x_{\lambda_{\mathrm{m}}}, L_\delta]] = 0$ for all $\delta \in R$, that is, $x_{\lambda_{\mathrm{m}}} \in c(L)$. So, if $\lambda_{\mathrm{m}} \in R^a$, then $x_{\lambda_{\mathrm{m}}} = 0$ by (a), which is a contradiction. Similarly, $\lambda_{\mathrm{M}} \in R^a$ leads to a contradiction. Thus, $\lambda_{\mathrm{m}}, \lambda_{\mathrm{M}} \in R^0$ and $F^x \subseteq R^0$ by (b).

(ii) Suppose that $(R^0 + R^a) \cap R \subseteq R^a$. Then L^a is an ideal of L by Lemma 2.5 and hence $\mathcal{Z}(L^a) \subseteq c(L)$ by Remark 2.4(c). Clearly $\mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(L^a) = 0$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$, $x \in \mathcal{Z}(L^a)$.

Suppose now that $\frac{1}{2} \in \Phi$ and let $x \in c(L)$ such that $\mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(L^a) = 0$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$. If $x = 0$, then $x \in \mathcal{C}_L(L^a)$. If $x \neq 0$, then $x = \sum_{\lambda\in F^x} x_\lambda \in \bigoplus_{\delta\in R^0} L_\delta$ by (i). Let $e_\alpha \in L_\alpha$, $\alpha \in R^a$. Then

$$\begin{aligned}\mathrm{ad}^2_{[x,e_\alpha]} &= (\mathrm{ad}_x \mathrm{ad}_{e_\alpha} - \mathrm{ad}_{e_\alpha} \mathrm{ad}_x)^2\\ &= \mathrm{ad}_x \mathrm{ad}_{e_\alpha} \mathrm{ad}_x \mathrm{ad}_{e_\alpha} - \mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x - \mathrm{ad}_{e_\alpha} \mathrm{ad}^2_x \mathrm{ad}_{e_\alpha} + \mathrm{ad}_{e_\alpha} \mathrm{ad}_x \mathrm{ad}_{e_\alpha} \mathrm{ad}_x,\end{aligned}$$

and since $\mathrm{ad}_x^2 = 0$, then $\mathrm{ad}_x \mathrm{ad}_{e_\alpha} \mathrm{ad}_x = 0$ by Lemma 2.1. So

$$\mathrm{ad}^2_{[x,e_\alpha]}(L^a) = -\mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(L^a) = 0.$$

Then we have that $\mathrm{ad}^2_{[x,e_\alpha]}(L^a) = 0$. Therefore, $[x, e_\alpha] \in c(L^a)$ with $[x, e_\alpha] = \sum_{\lambda\in F^x}[x_\lambda, e_\alpha] \in \bigoplus_{\delta\in R^a} L_\delta$ because $(R^0 + R^a) \cap R \subseteq R^a$. But now, applying (i) to $L^a = \bigoplus_{\delta\in R}(L^a \cap L_\delta)$ as L with respect to the same R^a, we get that $[x, e_\alpha] \in \bigoplus_{\delta\in R^0}(L^a \cap L_\delta)$. Hence $[x, e_\alpha] = 0$, that is, $[x, L_\alpha] = 0$ for all $\alpha \in R^a$. Then, since $\mathcal{C}_L(L^a) = \bigcap_{\alpha\in R^a} \mathcal{C}_L(L_\alpha)$, we have that $x \in \mathcal{C}_L(L^a)$. ■

Corollary 2.7 *Suppose that* $\frac{1}{2} \in \Phi$. *Let L be as in Proposition* 2.6 *and suppose that* $(R^0 + R^a) \cap R \subseteq R^a$. *If* $\mathrm{ad}^2_{e_\alpha}(L_\delta) = 0$ *for all* $e_\alpha \in L_\alpha$ *with* $\alpha \in R^a$ *and* $\delta \in R^0$, *then*

$$c(L) \subseteq \mathcal{C}_L(L^a).$$

Proof Let $x \in c(L)$. Then $x \in \bigoplus_{\delta\in R^0} L_\delta$ by Proposition 2.6(i). Note that $\mathrm{ad}^2_{e_\alpha}(L_\delta) = 0$ for all $\delta \in R^0$ implies that $\mathrm{ad}^2_{e_\alpha}(x) = 0$. Now by Proposition 2.6(ii), to see that $x \in \mathcal{C}_L(L^a)$, it is enough to show that $\mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(L^a) = 0$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$. Let $y \in L^a$. Then by using the Jacobi identity we get

$$\begin{aligned}\mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(y) &= \mathrm{ad}_x\big([e_\alpha, [[e_\alpha, x], y]] + [e_\alpha, [x, [e_\alpha, y]]]\big)\\ &= \mathrm{ad}_x\big([[e_\alpha, [e_\alpha, x]], y] + 2[[e_\alpha, x], [e_\alpha, y]] + [x, [e_\alpha, [e_\alpha, y]]]\big)\\ &= \mathrm{ad}_x\big([[e_\alpha, [e_\alpha, x]], y]\big) + 2\,\mathrm{ad}_x([[e_\alpha, x], [e_\alpha, y]]).\end{aligned}$$

But we have that $\mathrm{ad}_x([[e_\alpha, x], [e_\alpha, y]]) = \mathrm{ad}_x \mathrm{ad}_{[e_\alpha,y]} \mathrm{ad}_x(e_\alpha) = 0$ by Lemma 2.1. Hence

$$\mathrm{ad}_x \mathrm{ad}^2_{e_\alpha} \mathrm{ad}_x(y) = \mathrm{ad}_x([[e_\alpha, [e_\alpha, x]], y]) = \mathrm{ad}_x \mathrm{ad}_{\mathrm{ad}^2_{e_\alpha}(x)}(y) = 0,$$

since $\mathrm{ad}^2_{e_\alpha}(x) = 0$. ■

As we will see in the next section, extended affine Lie algebras of reduced type are in the setting of the above corollary and we will be able to describe their Kostrikin radical. The next theorem goes in this direction, but first we need a lemma.

Lemma 2.8 *Suppose that* $(R^0 + R^a) \cap R \subseteq R^a$. *Then* $\mathcal{Z}(L^a)$ *is an ideal of L and, for* $L \to \overline{L} := L/\mathcal{Z}(L^a)$ *being the canonical epimorphism,* $\overline{L} = \bigoplus_{\delta\in\overline{R}} \overline{L}_\delta$, *where* $\overline{L}_\delta = L_\delta/(\mathcal{Z}(L^a) \cap L_\delta)$ *and* $\overline{R} = \{\delta \in R : \overline{L}_\delta \neq 0\} \cup \{0\}$. *If* $\mathcal{Z}(L^a) \cap L_\alpha = 0$ *for all* $\alpha \in R^a$, *then* $R^a = \overline{R}^a \subseteq \overline{R}$ *and* $\overline{L}^a = \overline{L^a}$. *If we replace the hypothesis* $\mathcal{Z}(L^a) \cap L_\alpha = 0$ *by the stronger one,* $e_\alpha \in \mathrm{ad}^2_{e_\alpha}(L)$ *for all* $e_\alpha \in L_\alpha$, *then* $\mathcal{C}_{\overline{L}}(\overline{L}^a) = \overline{\mathcal{C}_L(L^a)}$ *and* $\overline{e}_\alpha \in \mathrm{ad}^2_{\overline{e}_\alpha}(\overline{L})$ *for all* $\overline{e}_\alpha \in \overline{L}_\alpha$. *In particular,* $c(\overline{L}) \cap \overline{L}_\alpha = 0$, *for all* $\alpha \in R^a$.

Proof In general for Lie algebras, the center of an ideal is an ideal. By Lemma 2.5, L^a and hence $\mathcal{Z}(L^a)$ are ideals of L. Since $\mathcal{Z}(L^a)$ is a graded ideal, $\overline{L} = \bigoplus_{\delta\in\overline{R}} \overline{L}_\delta$, where $\overline{L}_\delta = L_\delta/(\mathcal{Z}(L^a) \cap L_\delta)$. Moreover, if $\mathcal{Z}(L^a) \cap L_\alpha = 0$ for all $\alpha \in R^a$, then

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$R^a = \overline{R}^a \subseteq \overline{R}$ and $\overline{L}^a = L^a/\mathcal{Z}(L^a) = \overline{L^a}$. Suppose $e_\alpha \in \mathrm{ad}^2_{e_\alpha}(L)$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$. If $\overline{x} \in \mathcal{C}_{\overline{L}}(\overline{L}^a)$, then $[x, L^a] \subseteq \mathcal{Z}(L^a)$ and since L^a is perfect by Lemma 2.5,

$$[x, L^a] = [x, [L^a, L^a]] \subseteq [[x, L^a], L^a] + [L^a, [x, L^a]] = 0,$$

so $x \in \mathcal{C}_L(L^a)$. Thus, $\mathcal{C}_{\overline{L}}(\overline{L}^a) \subseteq \overline{\mathcal{C}_L(L^a)}$, while the reverse inclusion is obvious. Now for $\alpha \in R^a$, since $e_\alpha \in \mathrm{ad}^2_{e_\alpha}(L)$ for all $e_\alpha \in L_\alpha$, then $\overline{e}_\alpha \in \mathrm{ad}^2_{\overline{e}_\alpha}(\overline{L})$ for all $\overline{e}_\alpha \in \overline{L}_\alpha$. Then by Remark 2.4(b), we have that $c(\overline{L}) \cap \overline{L}_\alpha = 0$. ■

Theorem 2.9 *Suppose that $\frac{1}{2} \in \Phi$. Let L be as in Proposition* 2.6 *and suppose that $(R^0 + R^a) \cap R \subseteq R^a$. If*

(i) $\mathrm{ad}^2_{e_\alpha}(L_\delta) = 0$ *for all* $e_\alpha \in L_\alpha$, $\alpha \in R^a$ *and* $\delta \in R^0$,
(ii) *L is tame, that is,* $\mathcal{C}_L(L^a) = \mathcal{Z}(L^a)$,

then $c(L) = K_1(L) = \mathcal{Z}(L^a)$. *If moreover,* $e_\alpha \in \mathrm{ad}^2_{e_\alpha}(L)$ *for all* $e_\alpha \in L_\alpha$, $\alpha \in R^a$, *then*

$$c(L) = K(L) = \mathcal{Z}(L^a) = K(L^a) = c(L^a).$$

Proof By Proposition 2.6(ii) and Corollary 2.7, we have that $\mathcal{Z}(L^a) \subseteq c(L) \subseteq \mathcal{C}_L(L^a)$. Then by (ii), $\mathcal{Z}(L^a) = c(L) = \mathcal{C}_L(L^a)$, which is an ideal of L by Lemma 2.8. Then $K_1(L) = c(L) = \mathcal{Z}(L^a)$ by Remark 2.2. Now let $\overline{L} := L/\mathcal{Z}(L^a)$.

Suppose that $e_\alpha \in \mathrm{ad}^2_{e_\alpha}(L)$ for all $e_\alpha \in L_\alpha$, $\alpha \in R^a$. We next check that $\overline{L}$ is in the setting of Corollary 2.7. By Lemma 2.8, $\overline{L}$ is as in Proposition 2.6 with respect to $\overline{R}$ and R^a, with $\mathcal{C}_{\overline{L}}(\overline{L}^a) = \overline{\mathcal{C}_L(L^a)}$. Also $(R^0 + R^a) \cap \overline{R} \subseteq R^a$ and $\frac{1}{2} \in \Phi$ by assumption. Note that $\mathrm{ad}^2_{\overline{e}_\alpha}(\overline{L}_\delta) = 0$ for all $\overline{e}_\alpha \in \overline{L}_\alpha$ with $\alpha \in R^a$ and $\delta \in R^0$ since $\mathrm{ad}^2_{e_\alpha}(L_\delta) = 0$ and $\overline{L}_\delta = L_\delta/(\mathcal{Z}(L^a) \cap L_\delta)$. Hence by Corollary 2.7, $c(\overline{L}) \subseteq \mathcal{C}_{\overline{L}}(\overline{L}^a)$. But $\mathcal{C}_{\overline{L}}(\overline{L}^a) = \overline{\mathcal{C}_L(L^a)} = \overline{\mathcal{Z}(L^a)} = 0$. Then $c(\overline{L}) = 0$ and therefore $K(L) = K_1(L) = \mathcal{Z}(L^a)$ again by Remark 2.2. Now again because L^a is an ideal of L, we have that $K(L^a) = K(L) \cap L^a = K(L)$ [Z, Corollary 1]. Hence $c(L^a) \subseteq K(L^a) = \mathcal{Z}(L^a)$. Finally, since $\mathcal{Z}(L^a) \subseteq c(L^a)$ is clear, the proof is completed. ■

Example 2.10 Suppose Φ is a field of characteristic 0 and $L = \bigoplus_{\mu \in \Delta, g \in G} L^g_\mu$, where the G and $Q(\Delta)$-gradings are compatible, G is a torsion free abelian group, Δ is a locally finite irreducible *reduced* (*i.e.*, $0 \neq \mu \in \Delta \Rightarrow 2\mu \notin \Delta$) root system as defined in [LN], and $Q(\Delta)$ is the root lattice generated by Δ, such that $L_\mu \neq 0$ for some $0 \neq \mu \in \Delta$. Assume also that

(i) $L_0 = \sum_{\mu \in \Delta^\times} [L_\mu, L_{-\mu}]$, where $\Delta^\times := \Delta \setminus \{0\}$,
(ii) $e^g_\mu \in \mathrm{ad}^2_{e^g_\mu}(L)$ for all $0 \neq \mu \in \Delta$, $g \in G$.

For instance, the above condition (ii) is fulfilled if L is *division*, *i.e.*, for any $0 \neq \mu \in \Delta$ and any $0 \neq e^g_\mu \in L^g_\mu$, there exists $f^{-g}_{-\mu} \in L^{-g}_{-\mu}$ such that $[e^g_\mu, f^{-g}_{-\mu}] \equiv h_\mu$ modulo the center $\mathcal{Z}(L)$, where $\{h_\mu \in \mathfrak{h} : \mu \in \Delta\}$ is the set of coroots (see Remark 2.4(a)).

Then $K(L) = c(L) = \mathcal{Z}(L)$. Hence, if L is centerless, we have that L is strongly nondegenerate. To see this, consider

$$R := \{(\mu, g), \mu \in \Delta, g \in G, \text{ such that } L^g_\mu \neq 0\},$$

$$R^a := \{(\mu, g) \in R,\ \mu \neq 0\}.$$

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Denote $L^g_\mu = L_{(\mu,g)}$. Then $L = \bigoplus_{(\mu,g)\in R} L_{(\mu,g)}$, where $(0,0) \in R$ by (ii) and $R \subseteq Q(\Delta) \times G$. In $Q(\Delta) \times G$ we consider the lexicographic order (both $Q(\Delta)$ and G can be ordered since $Q(\Delta)$ is a free abelian group, see [LN, 7.5], and G is torsion free by assumption). Clearly, for $R^0 := R \setminus R^a$, $(R^0 + R^a) \cap R \subseteq R^a$. With respect to the lexicographic order in $Q(\Delta) \times G$, if $(0, g_1), (0, g_2) \in R^0$ and $(\mu, g') \in R$, with $(0, g_1) < (\mu, g') < (0, g_2)$, then $\mu = 0$ and $(\mu, g') \in R^0$. Moreover, $L = L^a$ by (i), which implies that L is tame, and because Δ is reduced, we also have that $\mathrm{ad}^2_{e^g_\mu}(L^{g'}_0) = 0$ for all $e^g_\mu \in L^g_\mu$ with $(\mu, g) \in R^a$ and $g' \in G$. Then L is in the setting of Theorem 2.9 satisfying that $e^g_\mu \in \mathrm{ad}^2_{e^g_\mu}(L)$ for all $(\mu, g) \in R^a$ by (ii), and we thus have that $K(L) = c(L) = \mathcal{Z}(L)$.

Remark 2.11 If L is *division* ΔG*-graded* as introduced in [Y2] (recall that Δ is then a finite irreducible reduced root system), where G is torsion free, then L is in the setting of the above example and $K(L) = c(L) = \mathcal{Z}(L)$.

3 The Kostrikin Radical of Extended Affine Lie Algebras of Reduced Type

Throughout, $\mathbb{F}$ will be a field of characteristic 0 and Lie algebras are always considered to be over $\mathbb{F}$. Let E be a Lie algebra satisfying the following two properties:

(EA1) E has a nondegenerate symmetric bilinear form $(\,\cdot\,)\colon E \times E \to \mathbb{F}$ which is invariant in the sense that $([x, y] \mid z) = (x \mid [y, z])$ for all $x, y, z \in E$,

(EA2) E contains a nontrivial finite-dimensional, self-centralizing subalgebra H which is ad-diagonalizable.

Let H^* be the dual space of H. Then E has a root space decomposition

$$E = \bigoplus_{\delta\in H^*} E_\delta, \qquad E_0 = H,$$

where, as usual, $E_\delta = \{e \in E : [h, e] = \delta(h)e \text{ for all } h \in H\}$. The invariance of $(\,\cdot\,)$ implies that $(E_\delta \mid E_\eta) = 0$ for $\delta + \eta \neq 0$. It follows that $(\,\cdot\,)$ restricted to $H \times H$ is nondegenerate. We can therefore transfer this restricted form to a nondegenerate symmetric bilinear form on H^* by setting $(\delta \mid \eta) = (t_\delta \mid t_\eta)$, where $t_\delta \in H$ is defined by $(t_\delta \mid h) = \delta(h)$ for all $h \in H$. We define the *root system* of E as $R = \{\delta \in H^* : E_\delta \neq 0\}$, and

$$R^0 = \{\delta \in R : (\delta \mid \delta) = 0\} \quad \text{(isotropic roots)},$$
$$R^{\mathrm{an}} = \{\delta \in R : (\delta \mid \delta) \neq 0\} \quad \text{(anisotropic roots)}.$$

The subalgebra E_c of E generated by $\{E_\delta : \delta \in R^{\mathrm{an}}\}$ is called the *core* of E.

Definition 3.1 An *extended affine Lie algebra of nullity n*, or extended affine Lie algebra (EALA) for short, is a Lie algebra E satisfying (EA1), (EA2) of above and, in addition, the following axioms:

(EA3) For $\delta \in R^{\mathrm{an}}$ and $x_\delta \in E_\delta$, $\mathrm{ad}_{x_\delta} \in \mathrm{End}_{\mathbb{F}} E$ is locally nilpotent.

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(EA4) R^{an} is *irreducible, i.e.,* $R^{\text{an}} = R_1 \cup R_2$ and $(R_1 \mid R_2) = 0$ imply $R_1 = \varnothing$ or $R_2 = \varnothing$.

(EA5) E is *tame* in the sense that $\mathcal{C}_E(E_c) = \mathcal{Z}(E_c)$.

(EA6) If V is the real space spanned by R and Λ is the subgroup of V generated by R^0, denoted $\Lambda = \langle R^0 \rangle$, then Λ is a free abelian group of rank n.

If E is an EALA with root system R, and $\Lambda = \langle R^0 \rangle$, then there exists a finite (possibly non-reduced) irreducible root system Δ (containing 0), an imbedding $\Delta_{\text{ind}} \hookrightarrow R$, where $\Delta_{\text{ind}} = \{0\} \cup \{\alpha \in \Delta \setminus 0 : \alpha/2 \notin \Delta\}$, and a family $(\Lambda_\alpha : \alpha \in \Delta) \subseteq \Lambda$ such that

$$V = \operatorname{span}_{\mathbb{Q}}(\Delta) \oplus \operatorname{span}_{\mathbb{Q}}(R^0) \quad \text{and} \quad R = \bigcup_{\alpha \in \Delta} (\alpha \oplus \Lambda_\alpha)$$

and then E has a Δ-grading $E = \bigoplus_{\alpha \in \Delta} E_\alpha$, where $E_\alpha := \bigoplus_{\lambda \in \Lambda_\alpha} E_{\alpha \oplus \lambda}$. If we denote $E_{\alpha \oplus \lambda} = E_\alpha^\lambda$, then E also has a Λ-grading $E = \bigoplus_{\lambda \in \Lambda} E^\lambda$, where $E^\lambda := \bigoplus_{\alpha \in \Delta} E_\alpha^\lambda$. The *type* of an extended affine Lie algebra E is said to be the type of its associated finite irreducible root system Δ. The core E_c of an extended affine Lie algebra E is indeed a perfect ideal of E and the subspaces $(E_c)_\alpha^\lambda := E_c \cap E_{\alpha \oplus \lambda}$ give E_c the structure of a *Lie torus* of type (Δ, Λ) as defined in [N] (see Proposition 3.2 below).

Note that if $\mathbb{F} = \mathbb{C}$ and E is an EALA such that its associated root system R is a discrete subset of H^*, then E is a tame extended affine Lie algebra in the usual sense [AABGP]. See [N2, 7] for a more detailed discussion on the relation between the two definitions.

Let us recall the following properties of extended affine Lie algebras.

Proposition 3.2 **([N2, 3])** *Let $E = \bigoplus_{\alpha \in \Delta} E_\alpha$, where $E_\alpha := \bigoplus_{\lambda \in \Lambda_\alpha} E_\alpha^\lambda$, be an EALA as described above. Let $\alpha \in \Delta^\times = \Delta \setminus \{0\}$ and $\lambda \in \Lambda_\alpha$. Then $\dim E_\alpha^\lambda = 1$ and $E_\alpha^\lambda \oplus [E_\alpha^\lambda, E_{-\alpha}^{-\lambda}] \oplus E_{-\alpha}^{-\lambda}$ is a Lie subalgebra of E isomorphic to $\mathfrak{sl}_2(\mathbb{F})$.*

By using the above properties and as a corollary of Theorem 2.9, we can characterize the Kostrikin radical of an EALA of reduced type as follows.

Theorem 3.3 *If E is an extended affine Lie algebra of reduced type Δ, that is, $\Delta \neq BC_l$, then $K(E) = c(E) = \mathcal{Z}(E_c) = K(E_c) = c(E_c)$, where E_c is the core of E.*

Proof Let E be an EALA. In order to apply Theorem 2.9, consider

$$R := \{(\alpha, \lambda), \alpha \in \Delta, \lambda \in \Lambda, \dim E_\alpha^\lambda \neq 0\},$$
$$R^a := \{(\alpha, \lambda) \in R, \alpha \neq 0\}.$$

Then $E = \bigoplus_{(\alpha,\lambda) \in R} E_{(\alpha,\lambda)}$ with the notation $E_{(\alpha,\lambda)} = E_\alpha^\lambda$. By Proposition 3.2, for all $(\alpha, \lambda) \in R^a$, and every nonzero $e_\alpha^\lambda \in E_\alpha^\lambda$, we can take $f_{-\alpha}^{-\lambda} \in L_{-\alpha}^{-\lambda}$ such that $\{e_\alpha^\lambda, [e_\alpha^\lambda, f_{-\alpha}^{-\lambda}], f_{-\alpha}^{-\lambda}\}$ is an $\mathfrak{sl}_2$-triple, hence $e_\alpha^\lambda \in \operatorname{ad}^2_{e_\alpha^\lambda}(E)$ and $(0,0) \in R$. Next, we argue as in Example 2.10. Clearly $((R \setminus R^a) + R^a) \cap R \subseteq R^a$ and $R \subseteq Q(\Delta) \times \Lambda$, for $Q(\Delta)$ being the root lattice generated by Δ. We consider the lexicographic order in $Q(\Delta) \times \Lambda$ and, with respect to this order, if $(0, \delta_1), (0, \delta_2) \in R \setminus R^a$ and $(\alpha, \lambda) \in R$, with $(0, \delta_1) < (\alpha, \lambda) < (0, \delta_2)$, then $\alpha = 0$ and $(\alpha, \lambda) \in R \setminus R^a$. Note that E^a

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as defined in the previous section is indeed E_c. Then E is tame by definition (EA5). Finally, if E is of reduced type, then it also satisfies that $\operatorname{ad}^2_{e^\lambda_\alpha}(E^\delta_0) = 0$ for all $e^\lambda_\alpha \in E^\lambda_\alpha$ with $(\alpha, \lambda) \in R^a$ and $\delta \in \Lambda$.

Hence for an EALA of reduced type, we have that $c(E) = K(E) = \mathcal{Z}(E_c) = K(E_c) = c(E_c)$ by applying Theorem 2.9. ■

Note that it follows from the above corollary that the Kostrikin radical of an EALA of reduced type is Λ-graded since it is the center of a graded algebra. We also want to point out that the above result holds in the more general setting described in [Yo].

4 The Core of Extended Affine Lie Algebras

As mentioned in the previous section, the core of an EALA of type Δ and nullity n is a Lie torus of type (Δ, Λ), where Λ is a free abelian group of rank n. Attending to their type, the precise structure of Lie tori is known for the case of a reduced Δ and for BC_1 and BC_2 as shown in the following examples.

Examples 4.1 (i) By [BGK, Theorem 1.37], every Lie torus of type $\Delta = D_l, l \geq 4$, or $E_l, l = 6, 7, 8$ and nullity n is a central extension of $\mathfrak{g} \otimes \mathbb{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra of type Δ and $\mathbb{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is the ring of Laurent polynomials in n variables. Actually $\mathfrak{g} \otimes \mathbb{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra of type Δ is always a Lie torus of type Δ and nullity n.

(ii) Let $q = (q_{ij}) \in M_n(\mathbb{F})$ such that $q_{ii} = 1 = q_{ij}q_{ji}$ for $1 \leq i, j \leq n$ and let $\mathbb{F}_q$ be the associated *quantum torus*, which, by definition, is the unital associative algebra with $2n$ generators $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$ and defining relations $t_i t_i^{-1} = 1 = t_i^{-1} t_i$ and $t_i t_j = q_{ij} t_j t_i$ for $1 \leq i, j \leq n$. Denote by $[\mathbb{F}_q, \mathbb{F}_q]$ the span of all commutators $[a, b] = ab - ba$ with $a, b \in \mathbb{F}_q$. Then $\mathfrak{sl}_{l+1}(\mathbb{F}_q) = \{x \in M_{l+1}(\mathbb{F}_q) : \operatorname{tr}(x) \in [\mathbb{F}_q, \mathbb{F}_q]\}$ is a Lie torus of type $A_l, l \geq 1$ and nullity n. Conversely, by [BGK, Theorem 2.65], every Lie torus of type $A_l, l \geq 3$ and nullity n is a central extension of $\mathfrak{sl}_{l+1}(\mathbb{F}_q)$ for some quantum torus $\mathbb{F}_q$.

(iii) Lie tori of type A_2 are classified in [BGK, BGKN]. The centerless Lie tori of type A_1 are the Tits–Kantor–Koecher algebras of the so-called Jordan tori, classified in [Y]. Lie tori of type B_l, C_l, F_4 or G_2 are described in [AG], of type BC_1 in [AFY, AY], and of type BC_2 in [F].

For a centerless Lie torus L of type (Δ, Λ) and nullity n, recall that the *skew centroidal derivations* of L form the subalgebra of $\operatorname{Der}_{\mathbb{F}} L$ defined by

$$\operatorname{SCDer}_{\mathbb{F}} L = \bigoplus_{\mu \in \Gamma} t^\mu \{\partial_\theta \in \mathcal{D} : \theta(\mu) = 0\},$$

where Γ is the so-called *centroidal grading group* of L and is a subgroup of Λ of rank m, so $0 \leq m \leq n$, $t^\mu = t_1^{\mu_1} \cdots t_m^{\mu_m}$ for $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{Z}^m$, and $\mathcal{D} = \{\partial_\theta : \theta \in \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{F})\} \cong \mathbb{F}^n$, with the product

$$[t^\mu \partial_\theta, t^\eta \partial_\psi] = t^{\mu+\eta}(\theta(\eta)\partial_\psi - \psi(\mu)\partial_\theta).$$

Proposition 4.2 **([N2])** *Let E be an EALA of nullity n with nondegenerate symmetric bilinear form $(\cdot)$. Then there exists a unique subalgebra D of the algebra of skew centroidal derivations of the centerless Lie torus $L := E_c/\mathcal{Z}(E_c)$ of nullity n inducing the Λ-grading of L and a 2-cocycle $\tau\colon D\times D \to D^{\mathrm{gr}\,*}$ (the graded dual space of D) such that*

$$E \cong L \oplus \mathcal{Z}(E_c) \oplus D,$$

where $L := E_c/\mathcal{Z}(E_c)$ and $\mathcal{Z}(E_c) \cong D^{\mathrm{gr}\,*}$,

$$[x_1 \oplus f_1 \oplus d_1, x_2 \oplus f_2 \oplus d_2] = ([x_1, x_2] + d_1(x_2) - d_2(x_1)) \\ \oplus (\sigma_D(x_1, x_2) + d_1 \cdot f_2 - d_2 \cdot f_1 + \tau(d_1, d_2)) \oplus [d_1, d_2]_D,$$

where $\sigma_D(x_1, x_2)(d) := (dx \mid y)$, $d \in D$, $d \cdot f$ is the contragredient action and $[d_1, d_2]_D$ denotes the commutator of d_1 and d_2 in D. Moreover

$$(x_1 \oplus f_1 \oplus d_1, x_2 \oplus f_2 \oplus d_2) = (x_1 \mid x_2) + f_1(d_2) + f_2(d_1).$$

By using Neher's results, we can compute the Kostrikin radical of the core of an EALA without any assumptions on the type.

Proposition 4.3 *If E_c is the core of an extended affine Lie algebra E, then*

$$K(E_c) = c(E_c) = \mathcal{Z}(E_c).$$

Proof Note that if E is of reduced type Δ, then E_c is a $\Delta\Lambda$-division graded Lie algebra as in Remark 2.11. Hence the result follows from that remark, so it only remains to prove the nonreduced case. In general, if we prove that $c(E_c) = \mathcal{Z}(E_c)$ and $c(E_c/\mathcal{Z}(E_c)) = 0$, since $\mathcal{Z}(E_c)$ is an ideal of E_c, we get that $K(E_c) = c(E_c) = \mathcal{Z}(E_c)$ by Remark 2.2. On the other hand, it is straightforward that $\mathcal{Z}(E_c) \subseteq c(E_c)$. Thus, the proof reduces to showing that $c(E_c) \subseteq \mathcal{Z}(E_c)$ and $c(E_c/\mathcal{Z}(E_c)) = 0$, where E is not of reduced type.

Suppose that E is not of reduced type. Let $L := E_c/\mathcal{Z}(E_c)$. Then, since L is not of type A_l, it follows from [N, Remarks] that the centroid $\mathrm{Cent}(L)$ of L is an integral domain acting without torsion on L and, if $\mathbb{K}$ is the quotient field of $\mathrm{Cent}(L)$, then the central closure $\tilde{L} = L \bigotimes_{\mathrm{Cent}(L)} \mathbb{K}$ is a simple finite-dimensional Lie algebra over $\mathbb{K}$. Since L is torsion-free over $\mathrm{Cent}(L)$, we have that L embeds in $\tilde{L}$ via $x \mapsto x \otimes 1$. Now let $x \in c(E_c)$ and $\pi\colon E_c \to L$ be the canonical epimorphism. Then $\pi(x)$ is an absolute zero divisor of L and $\pi(x) \otimes 1$ is thus an absolute zero divisor of $\tilde{L}$. But $\tilde{L}$ has no nonzero absolute zero divisors by Remark 2.3. Then $\pi(x) = 0$, that is, $x \in \mathcal{Z}(E_c)$. Finally, by using the same argument we also get that $c(L) = 0$. ■

From Proposition 4.3 and as an immediate consequence of the fact that Lie tori are central extensions of centerless cores of extended affine Lie algebras [Y3, Theorem 7.3], we have the following.

Corollary 4.4 *A Lie torus is centerless if and only if it is strongly nondegenerate.*

Since a Lie torus L has a (unique up to a nonzero scalar) nonzero invariant Λ-graded symmetric bilinear form that is nondegenerate if and only if L is centerless [Y3], then the above corollary can be seen as an infinite-dimensional version of Remark 2.3.

On the other hand, Proposition 4.2 tells us that the portion of an EALA which lies outside of the core is the part that is nondegenerately paired with the center of the core under the invariant bilinear form on the algebra and this is the clue to prove the following result.

Proposition 4.5 *Let E_c be the core of an extended affine Lie algebra $E \cong L \oplus \mathcal{Z}(E_c) \oplus D$, where $E_c = L \oplus \mathcal{Z}(E_c)$ and $\mathcal{Z}(E_c) \cong D^{gr*}$ as in Proposition 4.2. Then $\mathcal{C}_E(\mathcal{Z}(E_c)) \cong L \oplus \mathcal{Z}(E_c) \oplus \mathcal{Z}(D)$ and $E_c = [\mathcal{C}_E(\mathcal{Z}(E_c)), E]$, where $\mathcal{Z}(E_c) = K(E_c) = c(E_c)$.*

Proof By the definition of $\mathcal{Z}(E_c)$, it is clear that $E_c \subseteq \mathcal{C}_E(\mathcal{Z}(E_c))$. Using Proposition 4.2, we make the identifications $E = L \oplus D^{\mathrm{gr}\,*} \oplus D$, $E_c = L \oplus D^{\mathrm{gr}\,*}$ and $\mathcal{Z}(E_c) = D^{\mathrm{gr}\,*}$. We claim that $\mathcal{C}_E(D^{\mathrm{gr}\,*}) = L \oplus D^{\mathrm{gr}\,*} \oplus \mathcal{Z}(D)$. Since $E_c = L \oplus D^{\mathrm{gr}\,*} \subseteq \mathcal{C}_E(D^{\mathrm{gr}\,*})$, it suffices to show $D \cap \mathcal{C}_E(D^{\mathrm{gr}\,*}) = \mathcal{Z}(D)$. If $d \in D$, we have

$$([D^{\mathrm{gr}\,*}, d] \mid D) = (D^{\mathrm{gr}\,*} \mid [d, D]).$$

Since pairing between $D^{\mathrm{gr}\,*}$ and D is nondegenerate, we see that $[D^{\mathrm{gr}\,*}, d] = 0$ if and only if $[d, D] = 0$. Thus, $D \cap \mathcal{C}_E(D^{\mathrm{gr}\,*}) = \mathcal{Z}(D)$. Since E_c is perfect, we have

$$\begin{aligned} E_c = [E_c, E_c] &\subseteq [\mathcal{C}_E(\mathcal{Z}(E_c)), \mathcal{C}_E(\mathcal{Z}(E_c))] \\ &\subseteq [\mathcal{C}_E(\mathcal{Z}(E_c)), E] = [L \oplus D^{\mathrm{gr}\,*} \oplus \mathcal{Z}(D), L \oplus D^{\mathrm{gr}\,*} \oplus D] \\ &\subseteq L \oplus D^{\mathrm{gr}\,*} = E_c. \end{aligned}$$

Finally, by Proposition 4.3, $\mathcal{Z}(E_c) = K(E_c) = c(E_c)$. ■

5 Invariance of the Core of Extended Affine Lie Algebras of Reduced Type

In order to prove the invariance of the core of extended affine Lie algebras of reduced type under the automorphism group of the algebra, we need to find a characterization of the core which is independent of the data $(H, (\cdot))$. This characterization is given in the next theorem.

Theorem 5.1 *Let E be an extended affine Lie algebra of reduced type. Then*

$$E_c = [\mathcal{C}_E(K(E)), E]$$

where $K(E) = c(E)$.

Proof The proof follows directly from Theorem 3.3 and Proposition 4.5. ■

Hence, as a direct consequence of the above theorem and the fact that $\mathcal{C}_E(K(E))$ is invariant under the automorphisms of E, we get the following.

Corollary 5.2 *If E and E' are extended affine Lie algebras of reduced type and $f: E \to E'$ is an isomorphism of Lie algebras, then f, by restriction, induces an isomorphism $f_c: E_c \to E'_c$ of the corresponding cores E_c and E'_c of E and E'. In particular, the core of an extended affine Lie algebra E of reduced type is invariant under the automorphisms of E.*

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