ON CHARACTERIZING THE MULTIVARIATE LINEAR EXPONENTIAL DISTRIBUTION[1]

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1. Introduction and Summary. If x and y are independent p component column vectors, and the conditional distribution of x, given $x+y = z$, is known, what can be said about the distributions of x and y? This problem has been solved by Seshadri (1966) in the particular case when the conditional distribution of x, given $x+y = z$, is multivariate normal. In fact Seshadri's paper implicitly contains a characterization of the multivariate linear exponential distribution

$$f(x) = K\, A(x)\, \exp\{w'x\}, \tag{1}$$

where $A(x)$ is a function of x not involving the p component column vector w of constant terms. The normalizing constant K is determined by the condition

$$K \int A(x)\, \exp\{w'x\}dx = 1, \tag{2}$$

the integration (or the summation) being carried over the range of the values of x. The multivariate linear exponential distribution includes multivariate normal, positive and negative multinomial distributions.

1 This research work was done while the author held a summer (1967) research fellowship of the Canadian Mathematical Congress.

Hence $f(x)$ given by (1) may be a probability function. Our purpose in this paper is to give a characterization of the distribution (1). This characterization is, perhaps, foreseeable from a recent paper by Mathai (1967), who considers the univariate case of identically distributed variates. Our characterization, though for non-identically distributed variates, follows on parallel lines. However, we restrict our attention to the case of two vectors. The result for more than two vectors follows easily.

2. A Characterization. Let x and y be two p component column vectors whose probability functions do not vanish at the origin. Let the conditional probability of x, given $x+y = z$, be denoted by $C(x, z)$. If $C(x, z)$ is such that

$$\frac{C(x,z)\ C(y,z)\ C(0,z)}{C(0,z)\ C(0,z)\ C(z,z)} = \frac{h(x)\ h(y)}{h(x+y)}, \tag{3}$$

for some non-negative function $h(x)$, then x and y belong to the multivariate linear exponential distribution of the type (1). Note that in our case $C(y, z)$ does not represent the conditional probability function of y, given $x+y = z$, although in Mathai's paper it does.

The proof of the characterization follows on the same lines as in Mathai's paper (1967) in the univariate case. Let $f(x)$ and $g(y)$ be the probability functions of x and y respectively. Then we note that

$$f(x)\ g(y) = C(x,z)\ \Phi(z), \tag{4}$$

where $\Phi(z)$ is the marginal probability function of z. In (4) we set $x = 0$ and find that

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(5) $$f(0)\ g(y) = C(0,z)\ \Phi(z).$$

Note that the equation (5) expresses the left hand side probability, in terms of x and y, in terms of the right hand side probability in terms of $C(0,z)$ and $\Phi(z)$. On dividing (4) by (5) we find that

(6) $$\frac{f(x)}{f(0)} = \frac{C(x,z)}{C(0,z)}.$$

Now it follows from (3) and (6) that

(7) $$\frac{f(x)\ f(y)\ f(0)}{f(0)\ f(0)\ f(z)} = \frac{C(x,z)\ C(y,z)\ C(0,z)}{C(0,z)\ C(0,z)\ C(z,z)} = \frac{h(x)\ h(y)}{h(x+y)}.$$

By setting

(8) $$\psi(x) = f(x)/f(0)\ h(x)$$

and using (7) we may easily deduce that

(9) $$\psi(x+y) = \psi(x)\ \psi(y).$$

However, we know from Aczel (1965) that the solution of (9) is given by

(10) $$\psi(x) = \exp\{w'x\},$$

where w is an arbitrary p component vector. Thus it follows that

(11) $$f(x) = f(0)\ h(x)\ \exp\{w'x\}.$$

Now, to determine $g(x)$, we write (4) as

(12) $$f(x)\ g(y) = C(x,\ x+y)\ \Phi(x+y)$$

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and first set $x = 0$ in (12), then change y to $y+x$, and find that

(13) $$f(0)\ g(y+x) = C(0,\ y+x)\ \phi(y+x).$$

By using (12) and (13) we have that

(14) $$f(x)\ g(y)/f(0)\ g(x+y) = C(x,\ x+y)/C(0,\ x+y).$$

In (14) we set $y = 0$ and obtain that

(15) $$f(x)\ g(0)/f(0)\ g(x) = C(x,y)/C(0,x),$$

or that

(16) $$\frac{g(x)}{g(0)} = \frac{C(0,x)}{C(x,x)} \frac{f(x)}{f(0)} = \frac{C(0,x)}{C(x,x)} h(x) \exp\{w'\mathbf{x}\}.$$

Thus we have proved our characterization.

For more that two vectors we may state the characterization as follows. If $x, x_1, x_2, \ldots, x_N$ are independently distributed p component column vectors and the conditional distribution of x, given $x_1 + x_2 + \ldots + x_N = z$, is denoted by $C(x, z)$, then $x, x_1, x_2, \ldots, x_N$ each have multivariate exponential distribution of type (1), provided that

(17) $$\frac{C(x,z)\ C(x_1,z)\ \ldots\ C(x_N,z)\ C(0,z)}{C(0,z)\ C(0,z)\ \ldots\ C(0,z)\ C(z,z)} = \frac{h(x)\ h(x_1)\ \ldots\ h(x_N)}{h(x + x_1 + \ldots + x_N)}$$

for some non-negative function $h(x)$. Further, if $f(x)$, $f_1(x), f_2(x), \ldots, f_N(x)$ denote respectively the probability functions of $x, x_1, x_2, \ldots, x_N$, then

(18) $$f(x) = f(0)\ h(x)\ \exp\{w'x\},$$

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where w is an arbitrary p component column vector. The probability function of x_i, $i = 1,2,\ldots,N$, is

$$\frac{f_i(x)}{f_i(0)} = \frac{C(0,\ x + x_1 + \ldots + x_N - x_i)}{C(x,\ x + x_1 + \ldots + x_N - x_i)}\ \frac{f(x)}{f(0)}\ . \tag{19}$$

3. Illustrative Example. Take the example considered by Seshadri (1966). Here we have two vectors x and y, and

$$C(x,z) = (2\pi)^{-p/2}\ |V|^{-\frac{1}{2}}\ \exp\{-\tfrac{1}{2}(x-Cz)'\ V^{-1}(x-Cz)\}. \tag{20}$$

We may easily prove that

$$\frac{C(x,z)\ C(y,z)}{C(0,z)\ C(z,z)} = \exp\{-\tfrac{1}{2}(x'V^{-1}x + y'V^{-1}y - z'V^{-1}z)\}\ , \tag{21}$$

and find that

$$h(x) = \exp\{-\tfrac{1}{2}\ x'V^{-1}x\}\ . \tag{22}$$

If $f(x)$ and $g(x)$ denote the probability functions of x and y, then it follows from (11) that

$$f(x) = f(0)\ \exp\{-\tfrac{1}{2}\ x'V^{-1}x + w'x\}. \tag{23}$$

Further, by using (16) we find that

$$g(x)/g(0) = f(x)\ \exp\{\tfrac{1}{2}\ x'V^{-1}x - x'C'V^{-1}x\}/f(0). \tag{24}$$

The results (23) and (24) show that x and y are multivariate normal.

The conditions imposed by Seshadri on the matrices V and C are necessary for the existence of the multivariate normal distributions and not, per se, for their characterization.

Note that in (16) we may take $g(x) = [C(0,x)/C(x,x)]h(x) \exp\{\delta' x\}$, where δ is an arbitrary p component column vector. The results of this paper may not hold good in some discrete cases.

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