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ON THE SEMIGROUP OF PROBABILITY MEASURES OF A LOCALLY COMPACT SEMIGROUP

BY

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ABSTRACT. We show that a locally compact semigroup S is topological left amenable iff a certain space of left uniformly continuous functions on the convolution semigroup of probability measures $M_0(S)$ on S is left amenable or equivalently iff the convolution semigroup $M_0(S)$ has the fixed point property for uniformly continuous affine actions on compact convex sets.

1. **Introduction**. Let S be a locally compact semi-topological semigroup with convolution algebra $M(S)$ and probability measures $M_0(S)$. Let $X \subset M(S)^*$ be a topological left and right invariant linear subspace of the dual $M(S)^*$ containing the constant functional 1 (see Wong [17] for these definitions and terminologies). Under convolution and the topology $\sigma(M(S), X)$, $M_0(S)$ is a semi-topological semigroup. In what follows, we shall always consider $S_1 = M_0(S)$ in the topology $\sigma(M(S), X)$. Let $CB(S_1)$ denote the space of all real-valued bounded continuous functions on S_1 with supremum norm. Denote by $\mathcal{WLUC}(S_1)$ the space of all functions $f \in CB(S_1)$ such that the map $\mu \to \ell_\mu f$ of S_1 into $CB(S_1)$ is uniformly continuous with respect to the uniformity induced on S_1 by $\sigma(M(S), X)$ and the uniformity induced on $CB(S_1)$ by $\sigma(CB(S_1), CB(S_1)^*)$. That is, given $\varepsilon > 0$ and $m \in CB(S_1)^*$, there is some nbhd N of the origin in $\sigma(M(S), X)$ such that $\mu, \nu \in S_1$ and $\mu - \nu \in N$ implies $|m(\ell_\mu f) - m(\ell_\nu f)| < \varepsilon$. Here $\ell_\mu f(\theta) = f(\mu * \theta)$, $\theta \in S_1$. In general, $\mathcal{WLUC}(S_1) \subset WLUC(S_1)$, the weakly left uniformly continuous functions on S_1 as a topological semigroup (as defined in Mitchell [14]). Functions in $\mathcal{WLUC}(S_1)$ are sometimes called weakly left additively uniformly continuous. (See Ganeson [7] who first considered a similar but different type of additively uniformly continuous functions). Perhaps these functions should really be called weakly *uniformly* left *uniformly* continuous. (See Wong [18]). $\mathcal{WLUC}(S_1)$ is a norm closed left and right invariant linear subspace of $CB(S_1)$ containing the constants. It is also left introverted: $m_\ell(f) \in \mathcal{WLUC}(S_1)$ if $m \in CB(S_1)^*$ and $f \in \mathcal{WLUC}(S_1)$

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where $m_\ell(f)(\mu) = m(\ell_\mu f)$, $\mu \in S_1$. Because of this, $\mathcal{WLUC}(S_1)^*$ becomes a Banach algebra under Aren's product $m \odot n$, m, $n \in \mathcal{WLUC}(S_1)^*$. Here $m \odot n(f) = m(n_\ell(f))$, $f \in \mathcal{WLUC}(S_1)$. On the other hand, if $F \in M(S)^*$, $\mu \in M(S)$, the topological left translation (or left convolution) $\ell_\mu F$ is defined by $\ell_\mu F(\nu) = F(\mu * \nu)$, $\nu \in M(S)$. X is called topological left introverted if $M_L(F) \in X$ for any $F \in X$, $M \in M(S)^{**}$ where $M_L(F)(\mu) = M(\ell_\mu F)$, $\mu \in M(S)$. In this case, X^* becomes a Banach algebra under Aren's product $M \odot N$ where $(M \odot N)(F) = M(N_L(F))$. Note that the space $\mathcal{WLUC}(S_1)$ depends on X and is always left introverted even if X is not topological left introverted.

2. **Main results**

THEOREM 2.1 *Consider the following statements: (1) X has a topological left invariant mean (TLIM). (2)* $\mathcal{WLUC}(S_1)$ *has a left invariant mean (LIM).*

In general, (1) implies (2). Conversely, if X is also topological left introverted, then they are equivalent.

PROOF: Suppose X has a *TLIM*. As is well-known (see Greenleaf [10, Theorem 2.4.2]), there is a net $\mu_\alpha \in S_1$ such that $\mu * \mu_\alpha - \mu_\alpha \to 0$ in the topology $\sigma(M(S), X)$ for each $\mu \in S_1$. Hence $\mu * \mu_\alpha - \nu * \mu_\alpha \to 0$ in $\sigma(M(S), X)$ for any μ, $\nu \in S_1$. Consider $Q(\mu_\alpha)$ where $Q(\mu_\alpha)f = f(\mu_\alpha)$, $f \in \mathcal{WLUC}(S_1)$. By weak* compactness of the set of means on $\mathcal{WLUC}(S_1)$, some subnet $Q(\mu_\beta)$ converges weak* to some mean m on $\mathcal{WLUC}(S_1)$. Put $n = m \odot m$. We claim that $n(\ell_\mu f) = n(\ell_\nu f)$ for any $f \in \mathcal{WLUC}(S_1)$ and μ, $\nu \in S_1$. Now $n(\ell_\mu f) = (m \odot m)(\ell_\mu f) = m(m_\ell(\ell_\mu f)) = \lim_\beta m_\ell(\ell_\mu f)(\mu_\beta) = \lim_\beta m(\ell_{\mu * \mu_\beta} f)$. Since $f \in \mathcal{WLUC}(S_1)$, given $\varepsilon > 0$, there is some nbhd N in $\sigma(M(S), X)$ such that γ, $\psi \in S$, and $\gamma - \psi \in N$ implies $|m(\ell_\gamma f) - m(\ell_\psi f)| < \varepsilon$. Also, there is some β_0 such that $\beta \geq \beta_0$ implies $\mu * \mu_\beta - \nu * \mu_\beta \in N$. Hence $|\mathrm{m}(\ell_{\mu * \mu_\beta} f) - m(\ell_{\nu * \mu_\beta} f)| < \varepsilon \ \forall\ \beta \geq \beta_0$ and $n(\ell_\mu f) = n(\ell_\nu f)$. This implies that $n_\ell(f)$ is a constant function on S_1. Consequently $n \odot n$ is a *LIM* on $\mathcal{WLUC}(S_1)$.

Conversely, assume that X is also topological left introverted. Define the map τ: $X \to CB(S_1)$ by $\tau(F)(\mu) = F(\mu)$, $F \in X$, $\mu \in S_1$. It is easy to see that τ is a linear isometry of X into $CB(S_1)$ which commutes with left translations ($\tau(\ell_\mu F) = \ell_\mu \tau(F)$, $F \in X$, $\mu \in S_1$). Moreover, $\tau \geq 0$ and $\tau(1) = 1$. We claim that $\tau(X) \subset \mathcal{WLUC}(S_1)$. Take $F \in X$. Given $\varepsilon > 0$ and $m \in CB(S_1)^*$. Extend $\tau^* m \in X^*$ to a mean M on $M(S)^*$ by the Hahn Banach Theorem. Since X is topological left introverted, $M_L(F) \in X$ and so as a $\sigma(M(S), X)$-continuous linear functional on $M(S)$, is $\sigma(M(S), X)$-continuous at the origin. Therefore, there is a nbhd N in $\sigma(M(S), X)$ such that $|M_L(F)(\mu)| < \varepsilon$ for all $\mu \in N$. Now if μ, $\nu \in S_1$ and $\mu - \nu \in N$, then $|m(\ell_\mu \tau F) - m(\ell_\nu \tau F)| = |m(\tau \ell_\mu F) - m(\tau \ell_\nu F)| = |\tau^* m(\ell_\mu F) - \tau^* m(\ell_\nu F)| = |M_L(F)(\mu) - M_L(F)(\nu)| < \varepsilon$. That is, $\tau(X) \subset \mathcal{WLUC}(S_1)$ and so $\tau^* : \mathcal{WLUC}(S_1)^* \to X^*$. If m is any *LIM* on $\mathcal{WLUC}(S_1)$, then $\tau^* m$ is a *TLIM* on X. This completes the proof.

REMARKS. (a) Let $S = G$ be a locally compact group and $P(G) = \{\gamma \in L_1(G) : \gamma \geq 0, \|\gamma\|_1 = 1\}$. Then $S_0 = P(G)$ is a semigroup under convolution of functions in

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L_1. In [7], S. Ganeson considered the space of additively uniformly continuous functions $AUC(S_0)$ on S_0 consisting of all $f \in CB(S_0)$ (S_0 with L_1-norm topology) such that f is uniformly continuous with respect to the uniformity on S_0 induced by the L_1-norm and the usual uniformity for the real numbers. That is, given $\varepsilon > 0$, there is some $\delta > 0$ such that $\gamma, \psi \in S_0$ and $\|\gamma - \psi\|_1 < \delta$ implies $|f(\gamma) - f(\psi)| < \varepsilon$. He also showed that $CB(G)$ has a LIM iff $AUC(S_0)$ has a LIM, using heavily properties of a locally compact group. Theorem 2.1 is an analogue of Ganeson's result. However, our proof is different and is valid for locally compact semigroups and amenability of other subspaces X in $M(S)^*$, for examples, $X = LUC(S)$ or $WLUC(S)$ or $M(S)^*$ itself. (b) For discrete semigroups, see Mitchell [13] for a related result. (c) Theorem 2.1 is also valid if we use the Mackey topology $\tau(M(S), X)$ or any topology in between σ and τ. All we need is a topology on $M(S)$ whose dual is X (see Greenleaf [10, Theorem 2.4.2]). In particular, if $X = M(S)^*$, then we can use the norm topology of $M(S)$.

Next, we shall unify Theorem 2.1 with some interesting fixed point theorems. An affine action of the semigroup $S_1 = M_0(S)$ on a compact convex subset K of a separated locally convex space E is a map $T : S_1 \times K \to K$ (where $(\mu, x) \to T_\mu(x)$, $\mu \in S$, $x \in K$) such that $T_\mu : K \to K$ is continuous affine for each $\mu \in S_1$ and $T_\mu \circ T_\nu = T_{\mu * \nu}$, $\mu, \nu \in S_1$. It is called $\sigma(M(S), X)$-uniformly continuous on S_1 at $x_0 \in K$ if the map $\mu \to T_\mu x_0$ of S_1 into K is uniformly continuous with respect to the uniformity induced on S_1 by $\sigma(M(S), X)$ and the unique uniformity induced on K by the separated convex space E. T is called $\sigma(M(S), X)$-uniformly continuous on S_1 if it is so at each $x \in K$. We say that S_1 has the fixed point property for $\sigma(M(S), X)$-uniformly continuous affine action on compact convex sets if every such action of S_1 has a common fixed point. (See Wong [18]).

THEOREM 2.2. *The following statements are equivalent: (2) $\mathcal{WLUC}(S_1)$ has a LIM. (3) S_1 has the fixed point property for $\sigma(M(S), X)$-uniformly continuous actions on compact convex sets.*

PROOF: (2) implies (3). Uniform continuity of the affine action $T : S_1 \times K \to K$ implies that T is an A-representation of the pair S_1, $\mathcal{WLUC}(S_1)$ in the sense of Argabright [1]. Hence by [1, Theorem 1], T has a common fixed point.

(3) implies (2). The affine action $T : S_1 \times K \to K$ defined by $T_\mu(m) = \ell_\mu^* m$, $\mu \in S_1$, $m \in K$, the set of all means on $\mathcal{WLUC}(S_1)$ with the weak* topology, is uniformly continuous. Any fixed point is a left invariant mean on $\mathcal{WLUC}(S_1)$.

REMARKS. (a) Theorem 2.2 is an analogue of Ganeson's fixed point theorem [7, Theorem 2.7] for groups. (b) If $X = M(S)^*$, Theorem 2.2 remains valid if S_1 has the norm topology of $M(S)$. In fact, all we need is a topology in which S_1 is a uniform semigroup in the sense of Wong [18].

We next consider another interesting fixed point property introduced in Wong [16] which is closely related to (3).

An action of the measure algebra $M(S)$ is a *bilinear* map $T : M(S) \times E \to E$ where $(\mu, x) \to T_\mu x$, $\mu \in M(S)$, $x \in E$, a separated convex space such that $T_\mu : E \to E$ is continuous for each $\mu \in M(S)$ and $T_\mu \circ T_\nu = T_{\mu * \nu}$, $\mu, \nu \in M(S)$. It is called $\sigma(M(S), X)$-separately continuous if $\mu \to T_\mu x$ is continuous for each $x \in E$ when $M(S)$ has the topology $\sigma(M(S), X)$. If K is any compact convex subset of E which is $M_0(S)$-invariant under T (i.e. $T_\mu(K) \subset K$ for all $\mu \in M_0(S)$), then the action T of $M(S)$ induces an affine action $T : M_0(S) \times K \to K$ of the semigroup $S_1 = M_0(S)$ which is $\sigma(M(S), X)$-uniformly continuous on S_1. (A linear map is continuous iff it is continuous at the origin). However, not every such affine action of S_1 comes from an action of the algebra $M(S)$. We say that S has the fixed point property for a $\sigma(M(S), X)$-separately continuous action of the algebra $M(S)$ if the induced action on $S_1 = M_0(S)$ has a fixed point in K. This fixed point property is also equivalent to X having a *TLIM* (see Wong [16] for a locally compact group analogue):

THEOREM 2.3. *Let X be topological left introverted. Then the following statements are equivalent:*

(1) X has a *TLIM*. (2) $\mathcal{WLUC}(S_1)$ has a *LIM*. (3) S_1 has the fixed point property for $\sigma(M(S), X)$-uniformly continuous affine actions on compact convex sets. (4) S has the fixed point property for $\sigma(M(S), X)$-separately continuous (bilinear) actions of the measure algebra $M(S)$.

PROOF: We have already seen that (1), (2), and (3) are all equivalent. The proof for the equivalence of (1) and (4) is similar to that in Wong [16, Theorem 3.1, p. 573–574] for locally compact groups with suitable modifications. We omit the details.

REMARKS. If G is a locally compact group, then as is well-known, amenability of G is equivalent to $LUC(G) = WLUC(G)$ having a left invariant mean (see Greenleaf [10]). Theorem 2.1 can be regarded as a "generalization" in this direction since it implies that a locally compact semigroup S is topological left amenable (i.e. $M(S)^*$ has a topological left invariant mean) iff a space $\mathcal{WLUC}(S_1) \subset LUC(S_1)$ of left uniformly continuous functions on S_1 (where S is embedded algebraically) has a left invariant mean.

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