COMMUTATORS OF CERTAIN FINITELY GENERATED SOLUBLE GROUPS

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1. Introduction. Groups in which the commutator subgroup coincides with the set of commutators have been studied to a certain extent by several authors. It was shown in (**2**; **4**; **6**; **7**) that various types of known simple groups have this property. In (**3**), Macdonald has considered certain soluble groups with this property, and Hall has shown that any group can be embedded as a subgroup of a simple group of this type. Here we shall be concerned with the class C of groups defined as follows.

For any positive integer n, denote by C_n the class of all groups in which every element of the commutator subgroup can be expressed as a product of at most n commutators. It is not difficult to show that C_n is a proper subclass of C_{n+1} for all n. Let $C = \bigcup_{n=1}^{\infty} C_n$ so that a group $G \in C$ if and only if $G \in C_n$ for some n. It was shown in (**5**) that no non-trivial free product apart from the infinite dihedral group can be a C-group. The main results in this note are as follows.

THEOREM 1. *If A is a normal abelian subgroup of a soluble group G such that G/A satisfies the maximal condition for normal subgroups, then $G \in C$.*

An immediate consequence of this result is the following.

COROLLARY 1. *Any soluble group satisfying the maximal condition for normal subgroups is a C-group.*

THEOREM 2. *Every finitely generated abelian $\times$ nilpotent $\times$ nilpotent group is a C-group.*

In particular, we have the following result.

COROLLARY 2. *Every finitely generated soluble group of length 3 is a C-group.*

THEOREM 3. *The wreath product of a C_1-group with a finite cyclic group is again a C_1-group.*

Repeated application of Theorem 3 shows that the group obtained by taking successive wreath product of finite cyclic groups is a C_1-group. In general, it is not possible to weaken the conditions in Theorem 3. Let G be a group generated by a, b, c with $a^2 = b^2 = c^2 = 1$ and with one further condition that G be nilpotent of class two, so that the order of G is 2^6. This is a C_1-group but the

Received March 27, 1968.

https://doi.org/10.4153/CJM-1969-126-4 Published online by Cambridge University Press

wreath product of G with an infinite cyclic group is no longer a C_1-group; nor is the wreath product of a cyclic group of order 2 with G. Both these results can be verified by easy computation and we shall omit the details. We have not been able to determine if there exists a finitely generated soluble group that is not a C-group.

Notation and definitions. For any group G, we denote by $l(G)$ the smallest integer n such that $G \in C_n$. We write $l(G) = \infty$ if G is not a C-group. For a subgroup K of G, $l(G|K)$ denotes the smallest integer m such that every element of $K \cap G'$ can be expressed as a product of m commutators of G. (G' as usual stands for the commutator subgroup of G.) We write $l(G|K) = \infty$ if no such integer exists.

The following standard notation will be used. $\mathscr{N}_r$ for nilpotent groups of class at most r; $\mathscr{A}^s$ for soluble groups of length at most s; $\mathscr{G}_n$ for groups generated by n elements; G^t for the tth derived subgroup of G; $\gamma_m(G)$ for the mth term of the lower central series of G; $Z(G)$ for the centre of G and $[A, B] = \langle [a, b]; a \in A, b \in B \rangle$. We denote by $gp_G \langle x_1, \ldots, x_\nu \rangle$ the normal closure in G of the subgroup $\langle x_1, \ldots, x_\nu \rangle$.

2. Proofs.

LEMMA 1. *If K is a normal subgroup of a group G, then*

$$l(G/K) \leqq l(G) \leqq l(G/K) + l(G|K).$$

Proof. The left-hand inequality is obvious since every homomorphic image of a C_n-group is again a C_n-group. To prove the second inequality, suppose that $l(G/K) = l_1$ and $l(G|K) = l_2$. (If l_1 or l_2 is ∞, then $l(G) = \infty$ and there is nothing to prove.) Any $g \in G'$ is a product of l_1 commutators with some element $h \in K \cap G'$, and h is a product of l_2 commutators so that g is the product of at most $l_1 + l_2$ commutators.

LEMMA 2. *If $G = \langle A, x_1, \ldots, x_n \rangle$, where A is an abelian normal subgroup of G, then the set*

$$S = \{[a_1, x_1][a_2, x_2] \ldots [a_n, x_n]; a_i \in A\}$$

is precisely the subgroup $[A, G]$.

Proof. For any fixed $g \in G$, the mapping $a \to [a, g]$, $a \in A$, is homomorphic since A is abelian and normal in G. Thus, the set $Sg = \{[a, g]; a \in A\}$ is a subgroup of A. Furthermore, g normalizes Sg since $g^{-1}[a, g]g = [g^{-1}ag, g] \in Sg$. Evidently, A/Sg is a central factor of $\langle A, g \rangle$, and thus $Sg \geqq [A, \langle g \rangle]$. However, trivially $Sg \leqq [A, \langle g \rangle]$, hence $Sg = [A, \langle g \rangle]$.

Now write S_i for Sx_i so that $S = S_1 S_2 \ldots S_n$. As shown above, x_i centralizes A/S_i. Since $S_i \leqq S \leqq A$, x_i centralizes A/S. This holds for all $i = 1, 2, \ldots, n$, so that $G = \langle A, x_1, \ldots, x_n \rangle$ centralizes A/S, whence $[A, G] \leqq S$. However, trivially $S \leqq [A, G]$, so that $S = [A, G]$.

LEMMA 3. *With A and G as in Lemma 2 above, $l(G) \leqq l(G/[A, G]) + n$.*

This follows directly from Lemmas 1 and 2.

LEMMA 4. *Let $G \in \mathscr{G}_n$ and $H = gp_G\langle y_1, \ldots, y_s\rangle$, s finite. Then G/H' is a C-group if and only if G/H^2 is a C-group.*

Proof. Implication one way is obvious since G/H' is a homomorphic image of G/H^2. In order to show that $G/H' \in C$ implies $G/H^2 \in C$, we can clearly assume that $H^2 = 1$, so that H' is abelian. By Lemma 3, $l(G) \leqq l(G/[H', G]) + n$ and we need only show that $l(G/H')$ finite implies $l(G/[H', G])$ finite. We may now assume, without loss of generality, that $[H', G] = 1$. Thus $H' \leqq Z(G)$ and $H \in \mathscr{N}_2$. Hence, the map

$$h \rightarrow [h, y_i], \qquad h \in H,$$

is a homomorphism. $H_i = \{[h, y_i]; h \in H\} \leqq H' \leqq Z(G)$ so that H_i is normal in G, and

$$K = \prod_{i=1}^{s} H_i$$

is a normal subgroup of G. Thus, the centralizer in H of H/K,

$$C_H(H/K) = \{h; h \in H, [h, H] \leqq K\},$$

is also normal in G. However, $y_i \in C_H(H/K)$ for all $i = 1, 2, \ldots, s$. Hence H/K is abelian and $H' \leqq K$. However, trivially $K \leqq H'$, so that $K = H'$. Since every element of K can be written as a product of s commutators, $l(G) \leqq l(G/H') + s$. This completes the proof.

Proof of Theorem 1. G/A is finitely generated since it satisfies the maximal condition for normal subgroups; see (**1**). Thus, $G = \langle A, x_1, \ldots, x_n\rangle$, where $x_1, \ldots, x_n$ are suitably chosen elements of G and n finite. Let $M = \langle x_1, \ldots, x_n\rangle$ so that $G = AM$. By Lemma 3, $l(G) \leqq l(G/[A, G]) + n$, and we may assume that $A \leqq Z(G)$. This makes $G' = [AM, AM] = [M, M] = M'$, and it is therefore sufficient to show that $l(M)$ is finite. Note that $M/M \cap A \cong G/A$ satisfies the maximal condition for normal subgroups, and $A \cap M \leqq Z(M)$.

We will complete the proof by using induction on the solubility length of M. If $M \in \mathscr{A}$, then $l(M) = 1$. Assume the result in the case $M \in \mathscr{A}^r$ and consider the case when $M \in \mathscr{A}^{r+1}$. Let $B = M \cap A$. Since M/B satisfies the maximal condition for normal subgroups, there exists a subgroup

$$N = gp_M\langle y_1, \ldots, y_t\rangle, \quad t \text{ finite},$$

such that $BN = BM^{r-1}$. Now $M^{r+1} = 1$ so that $(BN)' = N' = M^r$ is abelian. Lemma 4 applies, with N replacing H and M replacing G. Hence, $l(M/N')$ finite implies $l(M/N^2)$ finite. However, $N^2 = M^{r+1} = 1$ and $M/N' \in \mathscr{A}^r$, and by hypothesis, M/N' is a C-group. This completes the proof.

LEMMA 5. *Let χ be a class of groups such that every finitely generated $\mathcal{N}_2 \times \chi$ group is a C-group. Then every finitely generated $\mathcal{N}_c \times \chi$ group is again a C-group for any integer $c \geqq 2$.*

Let us denote by $(\mathcal{G}_n \cap \mathcal{N}_c\chi)$ the class of all $\mathcal{G}_n$ groups that are at the same time $\mathcal{N}_c \times \chi$ groups. In this notation, we state a slightly different version of the above lemma as follows.

LEMMA 5*. *For any class χ of groups and any integer n, $(\mathcal{G}_n \cap \mathcal{N}_2\chi) \leqq C$ if and only if $(\mathcal{G}_n \cap \mathcal{N}_c\chi) \leqq C$, for every integer $c \geqq 2$.*

Proof of Lemma 5*. $(\mathcal{G}_n \cap \mathcal{N}_c\chi) \leqq C$ trivially implies that $(\mathcal{G}_n \cap \mathcal{N}_2\chi) \leqq C$. Implication the other way is by induction on c. If $c = 2$, then there is nothing to prove. Assume the result when $c = m - 1$ and suppose that a group $G \in (\mathcal{G}_n \cap \mathcal{N}_m\chi)$. Hence, there is a normal subgroup N of G such that $G/N \in \chi$ and $N \in \mathcal{N}_m$. Thus, $\gamma_{m+1}(N) = 1$. Since $m > 2$, $\gamma_{m-1}(N)$ is abelian so that by Lemma 3,

$$l(G) \leqq l(G/[\gamma_{m-1}(N), G]) + n \quad \text{and} \quad G/[\gamma_{m-1}(N), G] \in (\mathcal{G}_n \cap \mathcal{N}_{m-1}\chi).$$

The result follows by the induction hypothesis.

Proof of Theorem 2. If G is a finitely generated abelian $\times$ nilpotent $\times$ nilpotent group, then for some integers m, n, $\gamma_m(\gamma_n(G)) = A$ is abelian, and normal in G. Thus by Lemma 3, $l(G) \leqq l(G/[A, G]) + k$, where k is the number of elements of G required to generate G. Now

$$\gamma_{m+1}(\gamma_n(G)) = [A, \gamma_n(G)] \leqq [A, G]$$

so that $G/[A, G] \in (\mathcal{G}_k \cap \mathcal{N}_m\mathcal{N}_{n-1})$. It is sufficient to show that any finitely generated nilpotent $\times$ nilpotent group is a C-group. Lemma 5* reduces this to showing that any $(\mathcal{G}_k \cap \mathcal{N}_2\mathcal{N})$ group is a C-group.

We may now suppose that $G \in (\mathcal{G}_k \cap \mathcal{N}_2\mathcal{N})$ so that G contains a normal subgroup $H \in \mathcal{N}_2$ and $G/H \in \mathcal{N}$. Now G/H' is finitely generated abelian $\times$ nilpotent, and therefore satisfies the maximal condition for normal subgroups; see (**1**). Furthermore, H' is abelian and normal in G. Theorem 1 now applies and yields the required result.

Proof of Theorem 3. Let $W = G \wr T$ be the wreath product of a group $G \in C_1$ with a cyclic group T of order n. W is then the semi-direct product BT, where $B = \prod_{i=1}^n G_i$ is the direct product of n copies of G, $T = \langle t \rangle$, and for any $g_i \in G_i$, $t^{-1}g_it = g_i{}^t = g_{i+1} \in G_{i+1}$.

Modulo B', the mapping $g \to [g, t]$, $g \in B$, is homomorphic, and every element of W'/B' has the form $B'[g, t]$ with $g \in B$. (See the proof of Lemma 2, where the situation is analogous.) Since $G \in C_1$ and B is the direct product of n copies of G, every element of B' is a commutator. Thus every element of W' has the form

(1) $$[\alpha, \beta][\gamma, t] \quad \text{with } \alpha, \beta, \gamma \in B.$$

Consider the commutator $[xt, y]^{t^{-1}}$ with $x, y \in B$:

(2) $$[xt, y]^{t^{-1}} = [x, y][t, y]^{t^{-1}} = (y^{-1})^x y^{t^{-1}}.$$

To prove the result, it is enough to show that any expression (1) may be written in the form (2). Write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_i)$, with $\alpha_i \in G$. In the same way write $\beta = (\beta_i)$, $\gamma = (\gamma_i)$, $x = (x_i)$, $y = (y_i)$. Then

$$[\alpha, \beta][\gamma, t] = (y^{-1})^x y^{t^{-1}}$$

holds if and only if

(3) $$\begin{cases} x_i^{-1} y_i^{-1} x_i y_{i+1} = [\alpha_i, \beta_i]\gamma_i^{-1}\gamma_{i-1}, & i = 1, 2, \ldots, n, \\ \gamma_0 = \gamma_n, \quad y_1 = y_{n+1}. \end{cases}$$

Write $[\alpha_i, \beta_i]\gamma_i^{-1}\gamma_{i-1} = \eta_i$ for brevity, and $\eta = \eta_1\eta_2 \ldots \eta_n$. Now $\eta \in G'$ since $(\gamma_1^{-1}\gamma_n)(\gamma_2^{-1}\gamma_1) \ldots (\gamma_n^{-1}\gamma_{n-1}) \in G'$. By hypothesis, η is a commutator. Choose x_1 and y_1 in G such that $[x_1, y_1] = \eta$. Put $x_2 = x_3 = \ldots = x_n = 1$. Now (3) takes the form

(4) $$x_1^{-1}y_1^{-1}x_1y_2 = \eta_1, \quad y_i^{-1}y_{i+1} = \eta_i, \qquad i = 2, 3, \ldots, n.$$

Let

(5) $$y_2 = y_1^{x_1}\eta_1 \quad \text{and} \quad y_{i+1} = y_i\eta_i, \qquad i = 2, 3, \ldots, n-1.$$

Equations (5) serve as definitions of $y_2, \ldots, y_n$, y_1 having been chosen already. We now have:

$$x_1^{-1}y_1^{-1}x_1y_2 = (y_1^{-1})^{x_1}y_1^{x_1}\eta_1 = \eta_1, \quad y_i^{-1}y_{i+1} = \eta_i, \qquad i = 2, 3, \ldots, n-1.$$

Thus, all but possibly the last of equations (4) are satisfied. However,

$$\begin{aligned} y_n^{-1}y_{n+1} &= y_n^{-1}y_1 \\ &= (y_1^{x_1}\eta_1\eta_2 \ldots \eta_{n-1})^{-1}y_1, \quad \text{by (5)} \\ &= (y_1^{x_1}\eta\eta_n^{-1})^{-1}y_1 \\ &= \eta_n\eta^{-1}(y_1^{-1})^{x_1}y_1 \\ &= \eta_n\eta^{-1}[x_1, y_1] \\ &= \eta_n. \end{aligned}$$

This completes the proof.

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