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STATISTICAL STUDIES OF ORBITS IN STELLAR SYSTEMS

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ABSTRACT

This work is devoted to research of an evolution of dynamical systems with the factors, changed by accident, are investigated by statistical methods. The method of stochastic equations are applied for the study of the probe ensemble of stars, working in a field, where the regular component has spherical symmetry.

INTRODUCTION

With the development of natural sciences, the dynamical systems with the factors, changed by accident, are investigated on a large scale. The statistical approach is the fundamental one in the explanation of such systems.

This work is devoted to research of an evolution of such systems by statistical methods. Gravitational field is presented as a sum of a regular part and fluctuating irregular component. Let's consider the motion of a test particle (star) in such a system. Under the action of a regular component of gravitational force, defined by smooth distribution of mass density of all stars of the system, the test particle (star) will move along the regular unperturbed orbit. As a result of an action of an irregular component of force, caused by descretness of the system on relatively small scale and having stochastic nature, the star will move along trajectory, with fluctuating characteristics relative to the value, determined by the regular part of the force. So it is clear, that the motion of a star in real systems can be described by determined equations in the main only, approximately, neglecting the action of irregular forces. More correct description can be achieved with an application of stochastic differential equations of motion. We can receive such equations, including in addition random forces with certain statistical properties into determined equations of motion. At present the approach based on the description of various dynamical systems with stochastic equations, has received the considerable development [1-3]. In this work, the method of stochastic equations are applied for the study of the probe ensemble of stars, moving in a field, where the regular component has spherical symmetry.

The equation of motion of probe particle in spherical system of coordinate are the following:

$$\begin{aligned} &\ddot{r} - (r\dot{\psi}^2 \sin^2\theta + r\dot{\theta}^2) = -\frac{du}{dr} + A_r(t), \\ &\frac{1}{r}\frac{d}{dt}(r^2\dot{\psi}\ \sin^2\theta) = A_\psi(t), \\ &\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\psi}^2 \sin\theta\cos\theta = A_\theta(t), \end{aligned} \tag{1}$$

where r-polar radius, ψ-azimuthal angle, θ-angle between the polar radius and the z-axis, $(\dot{\ }) = d/dt$, u-potential energy of regular field on the unity of mass, A_i-components of random force ($i = r,\psi,\theta$). The latter is defined by setting their statistical properties, concerning to which we'll make the next assumption. The mean value with respect to ensemble of realization of the random force is

$$\langle A_i(t) \rangle = 0 \tag{2}$$

This condition means that in average the motion of the test particle is satisfied to determine equations of motion:

$$\begin{aligned} &\ddot{r} - (r\dot{\psi}^2 \sin^2\theta + r\dot{\theta}^2) = -\frac{du}{dr}, \\ &\frac{1}{r}\frac{d}{dt}(r^2\dot{\psi}\ \sin^2\theta) = 0, \\ &\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\dot{\psi}^2 \sin\theta\cos\theta = 0. \end{aligned} \tag{3}$$

The following concrete definition of statistical properties of a random force will be made later. The system of equations (3) without a random force in the right hand side permits the conservation of the angular moment, owing to which the motion is flat. We shall select this orbit plane of orbit so, that $\theta = \pi/2$. The trajectory of a star motion we can described in the form:

$$r(t) = r_o(t) + r_1(t),$$
$$\psi(t) = \psi_o(t) + \psi_1(t), \qquad (4)$$
$$\theta(t) = \pi/2 + \theta(t),$$

where the indexes "0" are related to an unperturbed orbit, defined by the regular part of total gravitational force.

Let the mean value of the module of a random force is a finite quantity and is considerably less than the value of a regular force and $r_1/r_o \ll 1$, $\psi_1/\psi_o \ll 1$, $2\theta_1/\pi \ll 1$. In this case, expanding the system of equations (1) in MacLauren series relatively r_o, ψ_o, $\pi/2$ and taking into account that r_o, ψ_o are satisfied to unperturbed equations of motion:

$$\ddot{r}_o - \psi_o r_o + u_o' = 0, \quad r_o\ddot{\psi}_o + 2\dot{r}_o\psi_o = 0,$$

we can receive the next nonlinear system of equations for r_1, ψ_1, θ_1:

$$\ddot{r}_1 - (\dot{\psi}_o^2 - u_o'')r_1 - 2r_o\dot{\psi}_o\dot{\psi}_1 +$$
$$r_o\dot{\psi}_o^2\theta_1^2 - 2\dot{\psi}_o r_1\dot{\psi}_1 - r_o\dot{\theta}_1^2 - r_o\dot{\psi}_1^2 + \frac{u_o'''}{2} r_1^2 + \ldots = A_r(t),$$

$$r_o\ddot{\psi}_1 + 2\dot{r}_o\dot{\psi}_1 + 2\dot{\psi}_o\dot{r}_1 + \ddot{\psi}_o r_1 + \qquad (5)$$
$$r_1\ddot{\psi}_1 + 2\dot{r}_1\dot{\psi}_1 - r_o\ddot{\psi}_o\theta_1^2 - 2\dot{r}_o\dot{\psi}_o\theta_1^2 - 2\dot{\psi}_o r_o\dot{\theta}_1\theta_1 + \ldots = A\ (t),$$

$$r_o\ddot{\theta}_1 + 2\dot{r}_o\theta_1 + r_o\dot{\psi}_o^2\theta_1 +$$
$$r_1\ddot{\theta}_1 + 2\dot{r}_1\dot{\theta}_1 + 2r_o\dot{\psi}_o\dot{\psi}_1\theta_1 + \dot{\psi}_o^2 r_1\theta_1 + \ldots = A_\theta(t),$$

where $u_o'' = (d^2u/dr^2)_{r=r_o(t)}$, $u''' = (d^3u/dr^3)_{r_o}$ etc.

In this work we consider only the system of equation linearized relative to an unperturbed orbit (equations in variation)

$$\ddot{r}_1 - (\dot{\psi}_o^2 - u'')r_1 - 2r_o\dot{\psi}_o\dot{\psi}_1 = A_r(t),$$
$$r_o\ddot{\psi}_1 + 2\dot{r}_o\dot{\psi}_1 + 2\dot{\psi}_o r_1 + \ddot{\psi}_o r_1 = A_\psi(t), \qquad (6)$$
$$r_o\ddot{\theta}_1 + 2\dot{r}_o\theta_1 + r_o\dot{\psi}_o^2\theta_1 = A_\theta(t).$$

It is to notice that the linear system (6) may describe only the general direction and the character of evolution of a probe ensemble of stars. Since $A_i(t)$ are random functions, the analytic solution of nonhomogeneous system of equations (6) resulting from these functions are also random functions. Certain realization of $y_i(t)$ $(y_i = r_1, \psi_1, \theta_1)$ corresponds to the each realization of $A_i(t)$. Detailed definition of the behaviour of the solution $y_i(t)$ at each realization is difficult, and as general rule, is not of interest, because it is important to discover some probabalistic properties of solutions $y_i(t)$ based on known probabalistic characteristics of the process $A_i(t)$.

We can show, that the system (6) is equivalent to the next one:

$$\ddot{r}_1 + \omega_1^2 r_1 = \phi(t), \qquad (7a)$$

$$\dot{\psi}_1 = -2\frac{\psi_o}{r_o^3}r_1 + \frac{1}{r_o^2(t)}\int_0^t r_o(\tau)A_\psi(\tau)d\tau, \qquad (7b)$$

$$\ddot{x} + u_o'/r_o = A_x(t) \qquad (7c)$$

where $\omega_1^2(t) = 3\dot{\psi}_o^2 + u_o''$, $\phi(t) = A_r + \frac{2}{r_o^2}\int_0^t r_o(\tau)A_\psi(\tau)d\tau$,

$x(t) = r_o \cdot \theta_1$.

This system of equations is more convenient for investigation, because it reduces to analysis of equations (7a) and (7c). The solution of equation (7b) reduces to quadrature at the known $r_1(t)$. We are interested in the case, where unperturbed orbits are finite. In case of infinite unperturbed orbits, the retention time of the test particles in the system (the crossing time) is small, and the irregular forces will not exert essential influence on the evolution of such particles, at least within not very dense systems: they will leave the system without any essential change of their characteristics.

Circular Unperturbed Orbits: At first, we let's consider the case, when unperturbed orbit is circular $r_o(t) = const, \psi_o = \omega_o t + \psi_{oo}$, where $\omega_o^2 = (1/r_o)\, du/dr|_{r_o \psi_{oo}}$ = const. initial phase. In the dimensionless variable $x_1 = r_1/r_o$ the equations are the following:

$$\ddot{x}_1 + \omega_1^2 x_1 = f(t),$$

$$\psi_1 = -2\omega_o x_1 + \int_0^t A'_\psi(\tau)d\tau, \tag{8}$$

$$\ddot{\theta}_1 + \omega_o^2 \theta_1 = A_\theta(t)$$

where $$\omega_1^2 = 3\omega_o^2 - u_o'' \tag{9}$$

$$f(t) = \int d\tau . A'_\psi(\tau) + A'_r(t), \tag{10}$$

$$A'_i = A_i/r_o.$$

In the following we shall omit the primes for short. The "formal" solutions of the system of equations, corresponding to zero initial condition are the following:

$$\dot{x}_1(t) = \int_0^t f(\tau)\cos\omega_1(t-\tau)d\tau, \tag{11}$$

$$x_1(t) = \frac{1}{\omega_1}\int_0^t f(\tau)\sin\omega_1(t-\tau)d\tau, \tag{12}$$

$$\dot{\psi}_1(t) = -2\frac{\omega_o}{\omega_1}\int_0^t f(\tau)\sin\omega_1(t-\tau)d\tau + \int_0^t A_\psi(\tau)d\tau, \tag{13}$$

$$\psi_1(t) = -2\frac{\omega_o}{\omega_1^2}\int_0^t f(\tau)\cos\omega_1(t-\tau)d\tau + \int_0^t d\tau_1 \int_0^t d\tau_2 \times A_\psi(\tau_2), \tag{14}$$

$$\dot{\theta}_1(t) = \int_0^t A_\theta(\tau)\cos\omega_o(t-\tau)d\tau, \tag{15}$$

$$\theta_1(t) = \frac{1}{\omega_o}\int_0^t A_\theta(\tau)\sin\omega_o(t-\tau)d\tau. \tag{16}$$

Among the statistical characteristics of motion which can be received basing on of the "formal" solutions (11)-(16) we are interested of the second central moments, i.e. mean square values of coordinates and velocities. An averaging will be carried out for on ensemble of realization of a random force, that is equivalent to the averaging on ensemble of particles, moving along the unperturbed orbit.

The determination of those or other probabalistic proerties of solutions by known (given) probabalistic characteristics of a random process $A_i(t)$ is really difficult. In a general case of arbitrary probablistic properties $A_i(t)$, this problem is of course unresolvable and therefore the choice of the model of random influences acquires is of the great importance.

As to statistical properties of random force we shall make the next assumption: The random force is determined by their two second moments by the average value and by the correlation function. The higher moments of a random force are expressed through the first and second moments, i.e. $A_i(t)$ is a Gaussian random process.

Delta-correlated Random Force : Let us consider the random force $A_i(t)$ with the next correlation function:

$$< A_i(t_1)\, A_j(t_2) > \;= \delta_{ij}\, a^2 \delta(t_1 - t_2) \qquad (17)$$

where i,j-indexes of spherical co-ordinates, δ_{ij}-Kronekker's symbol. $\delta(t)$-Dirac's delta-function, a^2-parameter which characterized the intensity of the random influence.

The assumption (17) is very strong, and it express the thought that the field stars, colliding with the probe stars and inducting the fluctuating force behave independently from each other except the case when they act simultaneously. The presence of Dirac's delta-function means physically, that we have the finite correlations of a random force, concentrated on time interval τ_c which is shorter than the others characteristic times. In case of delta-correlated random force the statistical characteristics of solutions (11)-(16) has the form [4]:

$$< r_1^2(t) >_T = b_1\, a^2 t, \qquad (18)$$

$$< \dot{r}_1^2(t) >_T = b_2\, a^2 t, \qquad (19)$$

$$< \psi_1^2(t) >_T = b_3\, a^2 t^3 + b_4 a^2 t, \qquad (20)$$

$$< \dot{\psi}_1^2(t) >_T = b_5\, a^2 t, \qquad (21)$$

$$< \theta_1^2(t) >_T = b_6 a^2 t, \quad (22)$$

$$< \dot{\theta}_1^2(t) >_T = b_7 a^2 t. \quad (23)$$

Index "T" means the additional time averaging exceeding the several times the crossing time of the system. From this ratios it is seen that under irregular forces the dispersions squared of all co-ordinates and velocities (excluding $<\psi_1^2(t)>$ increases proportionally to time t, like as at ususal diffusion. However, for all this the coefficients of proportionality, having the sense of diffusion coefficients in corresponding directions differ from each other. Consequently, the diffusion in the case under consideration has essentially nonisotropic character, both in the real space and in the space of the velocity. Then, the diffusion coefficients depend not only on the intensity of random influences, but also on concrete form of the regular force potential. Finally, the square of increment $< \psi_1^2(t)>$ contains a term proportional to time cubed, i.e., the evolution rate in this direction is considerably higher. This can be obviously imagined in a such way: the initially compact cloud of particles of ensemble after the sufficient time, run into a tore with center in origin of coordinates.

Finite Correlation Time of Random Force: The ratio (17) is an expression of the assumption that correlation time of random force is equal to zero. But delta-correlated processes are idealized, because all real physical processes in nature have finite correlation time. Because of long-range character of gravitational interaction, the presence of regular field and the finiteness of motion of both a probe stars and field stars, the correlations in gravitating systems can be expanded for large distances, up to those of the system, and can continue up to order to several crossing times. For example, wide binaries and multiply systems, stars streams etc.observed in galactic can apparently be considered, as an evidence of a space-time correlation with characteristical times, exceding the crossing time of galactic. Owing to this fact the field stars, interacting with probe stars and exciting the fluctuating force, influence on the later not independently, but more coordinated. The correlation function with finite correlation time is approximated as

$$< A_i(t_1).A_j(t_2) > = (a^2/2\tau_c)\exp(-|t-t|/\tau_c) \quad (24)$$

When $\tau_c \to 0$, the function $k(t) = (1/2\tau_c)\exp(-|t|/\tau_c)$ tends to delta-function. In this case, when the correlation function of a random force has form (18), the ratios for second moments of solutions (11)-(16) are given in work [5]. Because of their inwieldy sight we omit them here. We will only notice that in the asymptotic case, corresponding to the limit case $\tau_c \to 0$ these ratios, naturally turn into values, found for delta-correlated process.

Noncircular Unperturbed Orbits: (a) Radial or near to them orbits. In this case it is shown from equation (7b) that when the particle comes closer to center of a system ($r_o \to 0$), $\dot{\psi}_1$ (t) and ψ_1 (t) increase unlimitedly (second term in right side of (7b)). This lead to the destruction of initially purely radial (ψ_o = const.), or near to them trajectory at time order to crossing time. Therefore, under the action of irregular force, initially very extended orbits evolve very quickly in the time scale in the order of dynamical time (crossing time). Similar result have been received by one of the authors with help of another method [6].

(b) Moderate elongated unperturbed orbits. To sol of the system of equations (7), it is necessary to know ratios for unperturbed motion r_o(t) and ψ_o(t) in the form of explicit functions of time. It is known, that solutions of problem of motion in spherical symmetric field, give co-ordinates in form of non-explicit time functions. The explicit dependence of co-ordinates from time may be represented in the form of series. For case of periodic unperturbed orbits, these series are Fourier series. In important limit case of Newtonian and Hookean regular potentials, unperturbed orbits are known to be closed and are ellipses with origin of co-ordinates in focus and in center, respe-tively. Let's consider the case, when eccentricities of this ellipses are small (e << 1) and Fourier series contain periodical terms of lowest frequency. Then the equations (7a) and (7c) will have form of nonhomogeneous Mathiev's equation

$$\ddot{\xi} \quad (\nu^2 + \xi \cos pt)\xi = G(t),$$

where G(t) = ϕ(t) or A_θ(t), ν,p - parameters, ξ = h.e., h-numerical constant. The stable solutions of homogeneous Mathiev's equation are so called Mathiev's functions of order of ν : $ce_\nu(t,\xi)$ and $se_\nu(t,\xi)$. When ξ = 0 later reduced to ordinary trigonometric functions cos νt and sin νt, and when ε << 1 they differ a little from these functions. The solution of equations differ a little from each other when ε = 0 and ε << 1 respectively.

Thus, if unperturbed orbits are ellipses with small eccentricities e << 1, then solution of the system (7) differs a little from solutions of system (8), when unperturbed orbits are circular. In assumption of delta-correlation of random force, the second central moments of coordinates and velocities will be of form similar to (18 - (22) with coefficients $\tilde{b}_k$, which are a slightly different from b_k.

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