A Reduction Formula for Indefinite Integrals.

By George A. Gibson.

On p. 403 of Greenhill's *Calculus* (2nd Ed.) the following sentence occurs :—" By differentiation of the integral

$$\int \frac{\mathrm{H}x+\mathrm{K}}{\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C}} \quad \frac{dx}{\sqrt{(ax^2+2bx+c)}}$$

with respect to A, B, or C we can deduce the results of

$$\int \frac{f(x)}{(\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C})^n} \quad \frac{dx}{\sqrt{(ax^2+2bx+c)}} \text{"}$$

For the evaluation of the typical form in which $f(x)$ is a linear function, especially when A, B, etc., are given numbers, the method of differentiation does not seem very suitable; be that as it may, it may perhaps be of some interest to investigate a formula of reduction analogous to those in use for the integrals in which $\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C}$ is replaced by a linear function and $f(x)$ is a constant. I had occasion lately, in class work, to use such a reduction formula; but so far as I can find none of our text-books contains the reduction, except Robert's *Integral Calculus*, § 50. I propose to apply the method so beautifully used by Hermite in his *Cours* (4ième Éd., Léçon IV.) to effect the reduction in the case mentioned. It should be stated, however, that every method seems likely to be very laborious in practice, even when the constants are small numbers. At the same time the reduction is of great theoretical interest if the use of complex constants is to be avoided in a course on the *Integral Calculus*.

Let $\qquad \mathrm{R}=ax^2+2bx+c, \quad \mathrm{S}=\mathrm{A}x^2+2\mathrm{B}x+\mathrm{C}$

$$\mathrm{S}'=\frac{d\mathrm{S}}{dx}=2(\mathrm{A}x+\mathrm{B}), \qquad \mathrm{T}=\tfrac{1}{2}\mathrm{RS}'$$

S is supposed to have only complex factors.

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If T be prime to S and to $Hx+K$, then by the theory of partial fractions

$$\frac{Hx+K}{ST}=\frac{Px+Q}{S}+\frac{Lx^2-Mx-N}{T}$$

where obviously $L=-aP$, and the constants have to be determined in the usual way. Hence

$$Hx+K=(Px+Q)T-(aPx^2+Mx+N)S.$$

Substituting this value of $Hx+K$ in the integral we get

$$\int\frac{Hx+K}{S^n\sqrt{R}}dx=\tfrac{1}{2}\int\frac{(Px+Q)\sqrt{R}}{S^n}S'dx-\int\frac{aPx^2+Mx+N}{S^{n-1}\sqrt{R}}dx.$$

Integrating by parts the first integral on the right we have

$$\int\frac{Hx+K}{S^n\sqrt{R}}dx=\frac{-1}{2n-2}\ \frac{(Px+Q)\sqrt{R}}{S^{n-1}}-\frac{1}{2n-2}\int\frac{F(x)dx}{S^{n-1}\sqrt{R}}$$

where

$$F(x)=(2n-4)aPx^2+\{(2n-2)M-3bP-aQ\}x+(2n-2)N-bQ.$$

If $n=2$, $F(x)$ is linear and the formula of reduction has been obtained; but if $n\neq 2$, we may write

$$F(x)=(2n-4)\frac{aP}{A}S+H'x+K'$$

and the given integral is connected with two integrals of the same type in which n is less by 1 and by 2 respectively.

We know that values of P, Q, M, N exist, but these are very complicated expressions. Let us use the following notations:—

$$\beta=Ac-Ca,\quad \gamma=Ab-Ba,\quad \delta=(Ac+Ca-2Bb)^2-4(AC-B^2)(ac-b^2)$$

then

$$\delta(AC-B^2)P=\beta A(BH-AK)-2\gamma A(CH-BK)$$
$$\delta(AC-B^2)Q=2\gamma C(BH-AK)+(\beta A-4\gamma B)(CH-BK)$$
$$AM=aAQ+2bAP-aBP$$
$$CN=cBQ-K$$

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If it should happen that $Hx+K$ is not prime to T, then (i.) $Hx+K=k(Ax+B)$, (ii.) $Hx+K$ is a factor of R, say $R=(Hx+K)(a'x+b')$. In case (i.) we have

$$\frac{k}{SR}=\frac{px+q}{S}+\frac{lx+m}{R}$$

or
$$k=\ (px+q)R\quad+(lx+m)S$$

and
$$Hx+K=\tfrac{1}{2}(px+q)RS'+(lx+m)(Ax+B)S$$

and we proceed as before. In case (ii.)

$$\frac{1}{S(Ax+B)(a'x+b')}=\frac{(px+q)}{S}+\frac{lx+m}{(Ax+B)(a'x+b')}$$

and
$$Hx+K=\tfrac{1}{2}(px+q)RS'+(lx+m)(Hx+K)S$$

If R were not prime to S, this would mean $S=kR$ and the integral then belongs to a well-known form.

In applying the method to a numerical example the first step is to find P, Q; we may take a very simple case by way of illustration.

Let
$$u=\int\frac{2x+1}{(x^2+2x+4)^n}\quad\frac{dx}{\sqrt{(x^2-2x+3)}}$$

By decomposing
$$\frac{2x+1}{(x^2+2x+4)(x+1)(x^2-2x+3)}$$

we get
$$57(2x+1)=(9x+11)(x+1)R-(9x^2-16x-6)S$$

so that

$$57u=\frac{-(9x+11)\sqrt{R}}{(2n-2)S^{n-1}}-\int\frac{(2n-4)9x^2-(2n-3)16x-12n-4}{(2n-2)S^{n-1}\sqrt{R}}dx$$

$$=\frac{-(9x+11)\sqrt{R}}{(2n-2)S^{n-1}}+\int\frac{(34n-60)x+42n-70}{(n-1)S^{n-1}\sqrt{R}}dx$$

$$-\frac{9(n-2)}{n-1}\int\frac{dx}{S^{n-2}\sqrt{R}}$$

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If $n = 2$, this gives

$$57u = \frac{-(9x+11)\sqrt{R}}{2S} + \int \frac{8x+14}{S\sqrt{R}} dx$$

To integrate $(8x+14)/S\sqrt{R}$ we may take Greenhill's method ; or we may use the bilinear substitution $x = (py+q)/(y+1)$ and choose p, q (which can always be done) so as to make the integral take the form

$$\int \frac{Hx+K}{Ax^2+C} \frac{dx}{\sqrt{(ax^2+c)}}$$

i.e.

$$H\int \frac{xdx}{(Ax^2+C)\sqrt{(ax^2+c)}} + K\frac{dx}{(Ax^2+C)\sqrt{(ax^2+c)}}$$

The first of these is integrated by putting $ax^2 + c = z^2$ and the second by putting $a + cx^{-2} = z^2$.

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