Adv. Appl. Prob. **22**, 498–500 (1990)
Printed in N. Ireland

ON A GENERALIZATION OF A MODEL BY LINDLEY AND SINGPURWALLA

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Abstract

The flexibility of a notion considered by Lindley and Singpurwalla is pointed out. It is shown that their set-up can be generalized by looking at systems whose component life lengths are *a priori* dependent.

BIVARIATE EXPONENTIAL DISTRIBUTION; DEPENDENT FAILURES

1. Introduction and the proposed model

Lindley and Singpurwalla (1986) consider the behavior of a two-component system, with independent life lengths, in an environment which is harsher or gentler than the laboratory environment. However, there are many situations wherein the assumption of independence in the laboratory environment cannot be defended. A multivariate model, taking into account component dependencies, would be a good starting point to generalize the Lindley and Singpurwalla theme, and the object of this note is to point out one such generalization.

By way of motivation, consider a two-component system which operates in a test environment consisting of shocks that lead to Marshall and Olkin's (1967) bivariate exponential distribution, BVE $(\lambda_1, \lambda_2, \lambda_3)$, as distribution for the component lifelengths. Note here that the density of BVE $(\lambda_1, \lambda_2, \lambda_3)$ can be written in the following form:

$$f(x_1, x_2) = \begin{cases} \lambda_i \gamma_j \exp(-\{\lambda_i x_i + \gamma_j x_j\}) & \text{for } 0 < x_i < x_j \text{ with } 1 \leqq i \neq j \leqq 2, \\ \lambda_3 \exp(-\lambda, x) & \text{for } 0 < x_1 = x_2 = x. \end{cases}$$

Here $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ and $\gamma_i = \lambda_i + \lambda_3$; $i = 1, 2$.

The actual operating environments for the system often change over time, and are usually harsher (or gentler) than the test environment. Let us assume that the operating conditions remain the same over time, but the rate of occurrence of shocks is η times the rate of occurrence of shocks in the test environment, where η is an unknown quantity whose uncertainty is explained by the distribution function $G(\eta)$ as follows:

$$dG(\eta) = \frac{b^a}{\Gamma(a)} \exp(-b\eta)\eta^{a-1}\, d\eta; \qquad a, b > 0, \quad \eta > 0.$$

One would expect $\eta > 1$ for harsher operating conditions, and $\eta < 1$ for operating conditions gentler than testing conditions (accelerated testing case, for example). Now, given η, the component lifelengths will be conditionally distributed as BVE $(\eta\lambda_1, \eta\lambda_2, \eta\lambda_3)$ where λ_1, λ_2,

Received 28 February 1989; revision received 19 March 1990.

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Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under grant number AFOSR-89-0406. The US Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation thereon.

λ_3 are parameters of the joint distribution of lifetimes under test environment. Note that if $G \sim \text{BVE}(\eta\lambda_1, \eta\lambda_2, \eta\lambda_3)$ and $F \sim \text{BVE}(\lambda_1, \lambda_2, \lambda_3)$, then the survival functions are related as $\bar{G}(x_1, x_2) = [\bar{F}(x_1, x_2)]^\eta$. One can easily show that this condition occurs if and only if F and G have proportional bivariate hazard rates. So we are assuming that the random environment is changing the bivariate hazard rates of the component lifetimes in a proportionate manner, the constant of proportionality being random.

We can show in this case that the unconditional joint density of lifetimes is

$$(1.1) \qquad f(x_1, x_2) = \begin{cases} \dfrac{a(a+1)\theta_i\theta_j^*}{[1+\theta_i x_i + \theta_j^* x_j]^{(a+2)}} & \text{for } 0 < x_i < x_j \text{ with } 1 \leq i \neq j \leq 2, \\ \dfrac{a\theta_3}{[1+\theta x]^{(a+1)}} & \text{for } 0 < x_1 = x_2 = x, \end{cases}$$

where $\theta_i = \lambda_i/b$; $i = 1, 2, 3$; $\theta_j^* = \theta_j + \theta_3$; $j = 1, 2$ and $\theta = \theta_1 + \theta_2 + \theta_3$. We shall call this BVE-G distribution.

2. Some properties of the model

The joint survival function, the marginal survival function and density function of X_j, $j = 1, 2$ are as follows:

$$(2.1) \qquad \bar{F}(x_1, x_2) = [1 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 \max(x_1, x_2)]^{-a}; \qquad x_1, x_2 \geqq 0.$$

$$(2.2) \qquad \bar{H}_j(x) = \Pr[X_j > x] = [1 + \theta_j^* x]^{-a}; \qquad x \geqq 0.$$

$$(2.3) \qquad h_j(x) = -\frac{d}{dx}\bar{H}_j(x) = a\theta_j^*[1 + \theta_j^* x]^{-(a+1)}; \qquad x \geqq 0.$$

Note that (2.2) is the univariate Pareto type 2 distribution. Min $\{X_1, X_2\}$ is also a univariate Pareto type 2 distribution. This result can be useful in calculating reliability of a series system. It is observed that if $\{(X_{1i}, X_{2i}); 1 \leqq i \leqq n\}$ is a random sample from distribution (2.1), then $(\min\{X_{11}, \dots, X_{1n}\}, \min\{X_{21}, \dots, X_{2n}\})$ is also from distribution (2.1) with parameters $(a' = na, \theta_1, \theta_2, \theta_3)$. The distribution (2.1) can be shown to be bivariate decreasing failure rate (BDFR) distribution and if (X_1, X_2) follows (2.1), then (X_1, X_2) is positively quadrant dependent (PQD).

The conditional density of X_2 given $X_1 = x_1$ and the regression of X_2 on X_1 are as follows:

$$f_{X_2|X_1=x_1}(x_2) = \begin{cases} \dfrac{(a+1)\theta_1\theta_2^*[1+\theta_1 x_1 + \theta_2^* x_2]^{-(a+2)}}{\theta_1^*[1+\theta_1^* x_1]^{-(a+1)}} & \text{for } 0 < x_1 < x_2 \\ \dfrac{(a+1)\theta_2[1+\theta_2 x_2 + \theta_1^* x_1]^{-(a+2)}}{[1+\theta_1^* x_1]^{-(a+1)}} & \text{for } 0 < x_2 < x_1 \\ \dfrac{\theta_3(1+\theta_x)^{-(a+1)}}{\theta_1^*[1+\theta_1^* x]^{-(a+1)}} & \text{for } 0 < x_1 = x_2 = x, \end{cases}$$

$$E[X_2 \mid X_1 = x_1] = \frac{1}{a\theta_1^*}(1 + \theta_1^* x_1) - \theta_3\left[\frac{x_1}{\theta_1^*} + \frac{\theta(1+\theta x_1)}{a\theta_2\theta_1^*\theta_2^*}\right]\left[\frac{1+\theta x_1}{1+\theta_1^* x_1}\right]^{-(a+1)}.$$

Hence the regression of X_2 on X_1 is not linear unless $\theta_3 = 0$ which imply structural independence of X_1 and X_2. Simple but tedious calculations will show the following:

$$E(X_j) = [(a-1)\theta_j^*]^{-1}; \qquad \text{var}(X_j) = a[(a-1)^2(a-2)\theta_j^{*2}]^{-1}; \qquad j = 1, 2,$$

$$E(X_1 \,.\, X_2) = \left[\frac{1}{\theta_1^*} + \frac{1}{\theta_2^*}\right][\theta(a-1)(a-2)]^{-1}; \qquad \rho = \text{cor}(X_1, X_2) = \frac{\theta_1 + \theta_2 + a\theta_3}{a\theta}.$$

Since one needs $a > 2$ for finite second moments, it can be shown that $0 < \rho < (\theta + \theta_3)/2\theta$. Assuming $\theta_3 = 0$ one gets

$$E(X_j) = [(a-1)\theta_j]^{-1}; \qquad \text{var}\,(X_j) = a[(a-1)^2(a-2)\theta_j^2]^{-1}; \qquad j = 1, 2.$$

$$E(X_1 \,.\, X_2) = [(a-1)(a-2)\theta_1\theta_2]^{-1}; \qquad \text{cov}\,(X_1, X_2) = [(a-1)^2(a-2)\theta_1\theta_2]^{-1}$$

$$\text{cor}\,(X_1, X_2) = a^{-1}.$$

These results completely agree with Lindley and Singpurwalla's (1986) results.

References

LINDLEY, D. V. AND SINGPURWALLA, N. D. (1986) Multivariate distributions for the life lengths of a system sharing a common environment. *J. Appl. Prob.* **23,** 418–431.

MARSHALL, A. W. AND OLKIN, I. (1967) A multivariate exponential distribution. *J. Amer. Stat. Assoc.* **62,** 30–44.

https://doi.org/10.2307/1427552 Published online by Cambridge University Press