

FUNDAMENTAL SYSTEMS AND SOLUTIONS OF NONHOMOGENEOUS EQUATIONS FOR A PAIR OF MIXED LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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(Received 17 January 1989)

Communicated by A. J. Pryde

Abstract

Two different ordinary differential operators L_1 and L_2 (not of the same order) defined on two adjacent intervals I_1 and I_2 , respectively, with certain mixed conditions at the interface are considered. These problems are encountered in the study of ‘acoustic waveguides in ocean’, ‘transverse vibrations in nonhomogeneous strings’, etc. A complete set of physical conditions on the system give rise to three types of (selfadjoint) boundary value problems associated with the pair (L_1, L_2) . In a series of papers, a systematic study of these new classes of problems is being developed. In the present paper, we construct the fundamental systems and exhibit the forms of solutions of nonhomogeneous problems associated with the pair (L_1, L_2) .

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 Revision): 34 A 99.

0. Introduction

In the studies of ‘acoustic waveguides in ocean’, ‘optical fiber transmission’, ‘transverse vibrations of nonhomogeneous strings’, ‘one dimensional heat conduction in composite rods’, etc., we encounter a new class of problems of the type

$$L_1 x = \frac{1}{r_1} \sum_{k=0}^n P_k \frac{d^k x}{dt^k} = \lambda x$$

defined on an interval I_1 and

$$L_2 y = \frac{1}{r_2} \sum_{k=0}^m Q_k \frac{d^k y}{dt^k} = \lambda y$$

defined on the adjacent interval I_2 where I_1 and I_2 have a common boundary point $t = b$, λ is an unknown constant (eigenvalue), and the functions x, y are required to satisfy certain mixed conditions at the interface $t = b$. In most of the cases, the complete set of physical conditions on the system give rise to selfadjoint boundary value problems associated with the pair (L_1, L_2) . We may broadly classify these boundary value problems into three types, namely (i) where the values of x and y at the interface point $t = b$ are not explicitly related to each other, (ii) where x and y satisfy continuity conditions at $t = b$, and (iii) where x and y satisfy certain matching conditions at $t = b$. In the literature, to the authors' knowledge, there does not exist a systematic mathematical study of these problems. In the present work, we shall construct fundamental systems, and shall establish the forms of solutions of nonhomogeneous equations associated with (L_1, L_2) for each one of the three types of problems mentioned above.

In our earlier papers, we have studied the solutions of initial value problems [6], and characterized all selfadjoint boundary value problems [7] associated with the pair (L_1, L_2) . The next fundamental aspect of this new class of problems is the study of Green's functions and generalized Green's functions associated with (L_1, L_2) which in turn helps the study of resolvents of selfadjoint boundary value problems associated with (L_1, L_2) . But before embarking upon these studies, it is essential to know about the fundamental systems, and the form of solutions of nonhomogeneous equations for the pair (L_1, L_2) , which we shall consider here. To know the physical situations where these new classes of problems arise, we may refer to [1, 3, 4, 5].

Before indicating the sectionwise splitup of the work we shall introduce a few notations and make some assumptions. For an interval I of R and for any positive integer k , let $AC^k(I)$ denote the space of all complex valued functions f which have $(k-1)$ derivatives in I and the $(k-1)$ th derivative is absolutely continuous over each compact subinterval of I . For a function f , let $f^{(k)}$ denote the k th derivative of f , if it exists. Let $I_1 = [a, b]$ or $(a, b]$ and $I_2 = [b, c]$ or $[b, c)$, $-\infty \leq a < b < c \leq +\infty$. For a function $f \in AC^n(I_1)$, let $\tilde{f}(t) = \text{column}(f(t), \dots, f^{(n-1)}(t))$, $t \in I_1$, and for a function $f \in AC^m(I_2)$, let $\tilde{f}(t) = \text{column}(f(t), \dots, f^{(m-1)}(t))$, $t \in I_2$. Let $I = I_1 \cup I_2$. For a function f defined on I , let f/I_i denote the restriction of f to I_i , $i = 1, 2$. Let \mathbb{C}^k denote the k -dimensional complex space

whose elements we take to be column vectors. If $\alpha \in \mathbb{C}^{k_1}$ and $\beta \in \mathbb{C}^{k_2}$, then the column (α, β) denotes the $(k_1 + k_2)$ column vector formed by the components of α followed by the components of β taken in that order.

ASSUMPTION 1. Let $L_1 = \frac{1}{r_1} \sum_{k=0}^n P_k d^k / dt^k$ and $L_2 = \frac{1}{r_2} \sum_{k=0}^m Q_k d^k / dt^k$, where $r_1, P_k, k = 0, 1, \dots, n$, are continuous complex valued functions defined on I_1 with $r_1(t) > 0$ and $P_n(t) \neq 0$ for all $t \in I_1$, and $r_2, Q_k, k = 0, 1, \dots, m$, are continuous complex valued functions defined on I_2 with $r_2(t) > 0$ and $Q_m(t) \neq 0$ for all $t \in I_2$. For the sake of definiteness, we assume that $n \geq m$.

ASSUMPTION 2. Let g be a Lebesgue measurable complex valued function defined on I which is integrable over every compact subinterval of I . Let $g_i = g/I_i, i = 1, 2$.

ASSUMPTION 3. Let A and B be $(m \times n)$ and $(m \times m)$ matrices with complex entries, respectively, such that the range of A equals the range of B , and (hence) $\text{rank } A = \text{rank } B = d (\leq m)$.

Consider the homogeneous equation

$$(1) \quad (L_1, L_2)f = 0$$

and the nonhomogeneous equation

$$(2) \quad (L_1, L_2)f = g.$$

In Section 1, we shall construct three types of fundamental system (nonexplicitly mixed, continuous, matching) for equation (1). In Section 2, we shall establish the forms of three types of solutions (nonexplicitly mixed, continuous, matching) of equation (2).

1. Construction of fundamental systems for $(L_1, L_2)f = 0$

Let us recall the following definitions from [6].

DEFINITION 1. We call a complex valued function f , defined on the interval I , a *solution* (nonexplicitly mixed) of the differential equation (2) if

- (i) $f/I_1 = x \in AC^n(I_1)$,
- (ii) x satisfies the equation $L_1 x = g_1$ for almost all $t \in I_1$,
- (iii) $f/I_2 = y \in AC^m(I_2)$,

and

- (iv) y satisfies the equation $L_2 y = g_2$ for almost all $t \in I_2$.

DEFINITION 2. We call a complex valued function f defined on the interval I , a *continuous solution* of the differential equation (2) if

- (i) f is a solution of equation (2) in the sense of Definition 1,

and

(ii) the functions $f/I_1 = x$ and $f/I_2 = y$ satisfy continuity conditions at the interface point $t = b$, namely,

$$x^{(j)}(b-) = y^{(j)}(b+), \quad j = 0, 1, \dots, m - 1.$$

DEFINITION 3. We call a complex valued function f defined on the interval I , a *matching solution* of the differential equation (2) if

(i) f is a solution of equation (2) in the sense of Definition 1, and

(ii) the functions $f/I_1 = x$ and $f/I_2 = y$ satisfy the matching conditions at the interface point $t = b$, given by

$$A\tilde{x}(b-) = B\tilde{y}(b+).$$

REMARK 1. All the above definitions carry over to equation (1) also.

DEFINITION 4. We say that the nontrivial functions f_1, \dots, f_p defined on I (with possible jump discontinuities at $t = b$) are *linearly independent* if for any set of scalars c_1, \dots, c_p

$$\sum_{i=1}^p c_i f_i(t) = 0 \quad \text{for all } t \in I, \quad t \neq b,$$

and

$$\sum_{i=1}^p c_i f_i(b-) = \sum_{i=1}^p c_i f_i(b+) = 0$$

imply $c_1 = c_2 = \dots = c_p = 0$.

DEFINITION 5. By a *fundamental system* (FS) (nonexplicitly mixed, continuous, matching) for equation (1), we mean a set of linearly independent solutions (nonexplicitly mixed, continuous, matching) of equation (1) which span the solution (nonexplicitly mixed, continuous, matching) space of equation (1).

(I) Nonexplicitly mixed fundamental system for $(L_1, L_2)f = 0$

Let $\alpha_1, \dots, \alpha_n$ be a basis for \mathbb{C}^n and β_1, \dots, β_m be a basis for \mathbb{C}^m . Let $t_i \in I_i, i = 1, 2$. Let $x_i \in AC^n(I_1)$ be the unique solution of $L_1 x_i = 0, \tilde{x}_i(t_1) = \alpha_i, i = 1, \dots, n$, and let $y_i \in AC^m(I_2)$ be the unique solution of $L_2 y_i = 0, \tilde{y}_i(t_2) = \beta_i, i = 1, \dots, m$. Clearly, x_1, x_2, \dots, x_n form a FS for $L_1 x = 0$, and y_1, \dots, y_m form a FS for $L_2 y = 0$. Define

$$f_i(t) = \begin{cases} x_i(t), & t \in I_1, \\ 0, & t \in I_2, t \neq b, i = 1, \dots, n, \end{cases}$$

$$f_{n+i}(t) = \begin{cases} 0, & t \in I_1, t \neq b, \\ y_i(t), & t \in I_2, i = 1, 2, \dots, m. \end{cases}$$

Then as shown in [6, Theorem 4], it can be verified that $f_1, \dots, f_n, f_{n+1}, \dots, f_{n+m}$ form a nonexplicitly mixed fundamental system for $(L_1, L_2)f = 0$.

NOTE 1. We note that the Wronskian of x_1, \dots, x_n , namely,

$$W(x_1, \dots, x_n)(t) \neq 0$$

for all $t \in I_1$, and the Wronskian of y_1, \dots, y_m , namely, $W(y_1, \dots, y_m)(t) = W(f_{n+1}, \dots, f_{n+m})(t) \neq 0$ for all $t \in I_2$. Let $W_i(f_1, \dots, f_n)(t)$ denote the determinant obtained from $W(f_1, \dots, f_n)(t)$ by replacing the i th column by $(0, 0, \dots, 0, 1) \in \mathbb{C}^n$, $i = 1, \dots, n$, and let $W_i(f_{n+1}, \dots, f_{n+m})(t)$ denote the determinant obtained from $W(f_{n+1}, \dots, f_{n+m})(t)$ by replacing the i th column by $(0, \dots, 0, 1) \in \mathbb{C}^m$, $i = 1, \dots, m$.

(II) Continuous fundamental system for $(L_1, L_2)f = 0$

Let β_1, \dots, β_m be a basis for \mathbb{C}^m and let $\gamma_1, \dots, \gamma_{n-m}$ be a basis for \mathbb{C}^{n-m} . Then the vectors $\alpha_1, \dots, \alpha_n$ defined by

$$\alpha_i = \begin{cases} \text{column}(\beta_i, 0), & 0 \in \mathbb{C}^{n-m}, i = 1, \dots, m, \\ \text{column}(0, \gamma_{i-m}), & 0 \in \mathbb{C}^m, i = m + 1, \dots, n, \end{cases}$$

form a basis for \mathbb{C}^n . Let $x_i \in AC^n(I_1)$ be the unique solution of $L_1 x_i = 0$, $\tilde{x}_i(b - 1) = \alpha_i$, $i = 1, \dots, n$. Let $y_i \in AC^m(I_2)$ be the unique solution of $L_2 y_i = 0$, $\tilde{y}_i(b +) = \beta_i$, $i = 1, \dots, m$. Clearly, x_1, \dots, x_n form a FS for $L_1 x = 0$, and y_1, \dots, y_m form a FS for $L_2 y = 0$. Define

$$\begin{aligned} f_i(t) &= \begin{cases} x_i(t), & t \in I_1, t \neq b, \\ y_i(t), & t \in I_2, i = 1, \dots, m, \end{cases} \\ &= \begin{cases} x_i(t), & t \in I_1, \\ 0, & t \in I_2, t \neq b, i = m + 1, \dots, n. \end{cases} \end{aligned}$$

Then as shown in [6, Theorem 5], it can be verified that $f_1, \dots, f_m, f_{m+1}, \dots, f_n$ form a continuous fundamental system for $(L_1, L_2)f = 0$.

NOTE 2. We note that $W(f_1, \dots, f_n)(t) \neq 0$ for all $t \in I_1$, and

$$W(f_1, \dots, f_m)(t) \neq 0$$

for all $t \in I_2$. Let $W_i(f_1, \dots, f_n)(t)$ denote the determinant obtained from $W(f_1, \dots, f_n)(t)$ by replacing the i th column by $(0, \dots, 0, 1) \in \mathbb{C}^n$, $i = 1, \dots, n$, and let $W_i(f_1, \dots, f_m)(t)$ denote the determinant obtained from

$$W(f_1, \dots, f_m)(t)$$

by replacing the i th column by $(0, \dots, 0, 1) \in \mathbb{C}^m$, $i = 1, \dots, m$.

(III) Matching fundamental system for $(L_1, L_2)f = 0$

Since $\text{rank } A = \text{rank } B = d (\leq m)$, there exist bases $\alpha_1, \dots, \alpha_n$ for \mathbb{C}^n and β_1, \dots, β_m for \mathbb{C}^m such that $\alpha_1, \dots, \alpha_{n-d}$ belong to the null space

of A and $\beta_1, \dots, \beta_{m-d}$ belong to the null space of B . Also since the range of A equals the range of B ,

$$(3) \quad \text{span}\{A\alpha_i\}_{i=n-d+1}^n = \text{span}\{B\beta_i\}_{i=m-d+1}^m.$$

Let $x_i \in AC^n(I_1)$ be the unique solution of $Lx_i = 0$, $\tilde{x}_i(b-) = \alpha_i$, $i = 1, \dots, n$, and let $y_i \in AC^m(I_2)$ be the unique solution of $L_2y_i = 0$, $\tilde{y}_i(b+) = \beta_i$, $i = 1, \dots, m$. By relation (3), we get that $A\alpha_i = \sum_{j=m-d+1}^m \theta_j^i B\beta_j$, $i = n-d+1, \dots, n$, where θ_j^i are some suitable scalars. Clearly, x_1, \dots, x_n form a FS for $L_1x = 0$. Define

$$f_i(t) = \begin{cases} x_i(t), & t \in I_1, \\ 0, & t \in I_2, t \neq b, i = 1, \dots, n-d, \\ x_i(t), & t \in I_1, t \neq b, \\ \sum_{j=m-d+1}^m \theta_j^i y_j(t), & t \in I_2, i = n-d+1, \dots, n, \end{cases}$$

and

$$f_{n+i}(t) = \begin{cases} 0, & t \in I_1, t \neq b, \\ y_i(t), & t \in I_2, i = 1, \dots, m-d. \end{cases}$$

It was shown in [6, Theorem 6] that $f_1, \dots, f_n, f_{n+1}, \dots, f_{n+m-d}$ form a matchingly mixed fundamental system for $(L_1, L_2)f = 0$.

CLAIM. $y_1, \dots, y_{m-d}, \sum_{j=m-d+1}^m \theta_j^{n-d+1} y_j, \dots, \sum_{j=m-d+1}^m \theta_j^n y_j$ form a fundamental system for $L_2y = 0$.

It is enough to verify that the above functions are linearly independent, so let c_1, \dots, c_m be scalars such that

$$(4) \quad c_1y_1 + \dots + c_{m-d}y_{m-d} + c_{m-d+1} \sum_{j=m-d+1}^m \theta_j^{n-d+1} y_j + \dots + c_m \sum_{j=m-d+1}^m \theta_j^n y_j = 0$$

Upon differentiation, relation (4) implies that

$$(5) \quad c_1\tilde{y}(b+) + \dots + c_{m-d}\tilde{y}_{m-d}(b+) + c_{m-d+1} \sum_{j=m-d+1}^m \theta_j^{n-d+1} \tilde{y}_j(b+) + \dots + c_m \sum_{j=m-d+1}^m \theta_j^n \tilde{y}_j(b+) = 0.$$

Operating the matrix B on both sides of equation (5), we get that

$$(6) \quad c_{m-d+1}B \left(\sum_{j=m-d+1}^m \theta_j^{n-d+1} \tilde{y}_j(b+) \right) + \dots + c_mB \left(\sum_{j=m-d+1}^m \theta_j^n \tilde{y}_j(b+) \right) = 0.$$

Since f_1, \dots, f_{n+m-d} form a matching fundamental system for $(L_1, L_2)f = 0$, we get from relation (6) that

$$c_{m-d+1}A\tilde{x}_{n-d+1}(b-) + \dots + c_m A\tilde{x}_n(b-) = 0,$$

that is,

$$A(x_{m-d+1}\tilde{x}_{n-d+1}(b-) + \dots + c_m\tilde{x}_n(b-)) = 0,$$

that is,

$$A(c_{m-d+1}\alpha_{n-d+1} + \dots + c_m\alpha_n) = 0,$$

which implies that $c_{m-d+1}\alpha_{n-d+1} + \dots + c_m\alpha_n$ belongs to the null space of A and this is true only if $c_{m-d+1} = \dots = c_m = 0$. Now, relation (4) reduces to $c_1y_1 + \dots + c_{m-d}y_{m-d} = 0$, and since y_1, \dots, y_{m-d} are linearly independent, we get that $c_1 = \dots = c_{m-d} = 0$. This proves the claim.

NOTE 3. We note that $W(f_1, \dots, f_n)(t) \neq 0$ for all $t \in I_1$, and

$$W(f_{n-d+1}, \dots, f_n, f_{n+1}, \dots, f_{n+m-d})(t) \neq 0$$

for all $t \in I_2$. Let $W_i(f_1, \dots, f_n)(t)$ denote the determinant obtained from $W(f_1, \dots, f_n)(t)$ by replacing the i th column by $(0, \dots, 0, 1) \in \mathbb{C}^n$, $i = 1, 2, \dots, n$, and let $W_i(f_{n-d+1}, \dots, f_n, f_{n+1}, \dots, f_{n+m-d})(t)$ denote the determinant obtained from $W(f_{n-d+1}, \dots, f_{n+m-d})(t)$ by replacing the i th column by $(0, \dots, 0, 1) \in \mathbb{C}^m$, $i = 1, \dots, m$.

2. Forms of solutions of $(L_1, L_2)f = g$

THEOREM 1 (Variation of constants method for nonexplicitly mixed solutions of $(L_1, L_2)f = g$). *Let f_1, \dots, f_{n+m} be the nonexplicitly mixed fundamental system for $(L_1, L_2)f = 0$ as defined in (I). Then all nonexplicitly mixed solutions f of $(L_1, L_2)f = g$ are of the form $f(t) = \sum_{i=1}^{n+m} c_i(t)f_i(t)$, $t \in I$, where*

$$c_i(t) = \begin{cases} a_i + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} r_1(s)g_1(s)ds \right), & t \in I_1, \\ 0, & t \in I_2, t \neq b, i = 1, \dots, n, \end{cases}$$

$$c_{n+i}(t) = \begin{cases} 0, & t \in I_1, t \neq b, \\ b_i + \int_{t_2}^t \left(\frac{W_i(f_{n+1}, \dots, f_{n+m})(s)}{Q_m(s)W(f_{n+1}, \dots, f_{n+m})(s)} \right) r_2(s)g_2(s)ds, & t \in I_2, i = 1, \dots, m, \end{cases}$$

and a_i ($i = 1, \dots, n$), b_i ($i = 1, \dots, m$) are scalars, and $t_i \in I_i$, $i = 1, 2$.

PROOF. By [6, Theorem 1], there exist nontrivial nonexplicitly mixed solutions of equation (2). Let us assume a nonexplicitly mixed solution f of equation (2) to be of the form

$$f(t) = \sum_{i=1}^{n+m} c_i(t)f_i(t), \quad t \in I,$$

where the c_i 's are to be determined by the variation of constants method.

Case (1): for $t \in I_1$, $f(t) = \sum_{i=1}^n c_i(t)f_i(t) = \sum_{i=1}^n c_i(t)x_i(t)$ satisfies the equation $L_1f = g_1$.

Therefore, by the variation of constants method we get that

$$c_i(t) = a_i(t) + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds, \quad i = 1, \dots, n,$$

where the a_i 's are scalars and $t_1 \in I_1$ (see [2, Theorem 3.6.4]). Extend the functions c_i to the interval I_2 by defining $c_i(t) = 0$, $t \in I_2$, $t \neq b$, $i = 1, 2, \dots, n$.

Case (2): for $t \in I_2$,

$$f(t) = \sum_{i=n+1}^{n+m} c_i(t)f_i(t) = \sum_{i=1}^m c_{n+i}(t)y_i(t)$$

satisfies the equation $L_2f = g_2$.

Therefore by the variation of constants method, we get that

$$c_{n+i}(t) = b_i + \int_{t_2}^t \left(\frac{W_i(f_{n+1}, \dots, f_{n+m})(s)}{Q_m(s)W(f_{n+1}, \dots, f_{n+m})(s)} \right) r_2(s)g_2(s)ds, \quad i = 1, 2, \dots, m,$$

where the b_i 's are scalars and $t_2 \in I_2$. Extend the functions c_{n+i} to the interval I_1 by defining $c_{n+i}(t) = 0$, $t \in I_1$, $t \neq b$, $i = 1, \dots, m$. Finally, recalling the definitions of $f_1, \dots, f_n, \dots, f_{n+m}$, we see that the proof is complete.

THEOREM 2 (Variation of constants method for continuous solutions of $(L_1, L_2)f = g$). *Let f_1, \dots, f_n be the continuous fundamental system for $(L_1, L_2)f = 0$ as defined in (II). Then all continuous solutions of $(L_1, L_2)f = g$ are of the form*

$$f(t) = \sum_{i=1}^n c_i(t)f_i(t), \quad t \in I,$$

where

$$c_i(t) = \begin{cases} a_i + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds, & t \in I_1, \\ a_i + \int_{t_1}^b \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds \\ + \int_b^t \left(\frac{W_i(f_1, \dots, f_m)(s)}{Q_m(s)W(f_1, \dots, f_m)(s)} \right) r_2(s)g_2(s)ds, & t \in I_2, i = 1, \dots, m, \\ 0, & t \in I_2, t \neq b, i = m + 1, \dots, n, \end{cases}$$

and $a_i, i = 1, \dots, n$, are scalars and $t_1 \in I_1$.

PROOF. By [6, Theorem 2], there exist nontrivial continuous solutions of equation (2). Let us assume a continuous solution f of equation (2) to be of the form

$$(7) \quad f(t) = \sum_{i=1}^n c_i(t)f_i(t), \quad t \in I,$$

where the c_i 's are to be determined by the variation of constants method.

Case (1): for $t \in I_1$, $f(t) = \sum_{i=1}^n c_i(t)f_i(t) = \sum_{i=1}^n c_i(t)x_i(t)$ satisfies the equation $L_1 f = g$.

Therefore, by the variation of constants method we get that

$$(8) \quad c_i(t) = a_i + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds, \quad i = 1, \dots, n,$$

where the a_i 's are scalars and $t_1 \in I_1$. We notice that the c_i 's satisfy the relation

$$(9) \quad \sum_{i=1}^n c_i^{(1)}(t)f_i^{(j)}(t) = 0, \quad t \in I_1, j = 0, 1, \dots, n - 2.$$

Case (2): for $t \in I_2$, $f(t) = \sum_{i=1}^m c_i(t)f_i(t) = \sum_{i=1}^m c_i(t)y_i(t)$ satisfies the equation $L_2 f = g_2$.

Therefore, by the variation of constants method we get that

$$(10) \quad c_i(t) = b_i + \int_b^t \left(\frac{W_i(f_1, \dots, f_m)(s)}{Q_m(s)W(f_1, \dots, f_m)(s)} \right) r_2(s)g_2(s)ds, \quad i = 1, \dots, m,$$

where the b_i 's are scalars. We notice that c_1, \dots, c_m satisfy the relation

$$(11) \quad \sum_{i=1}^m c_i^{(1)}(t)f_i^{(j)}(t) = 0, \quad t \in I_2, j = 0, \dots, m - 2.$$

Since f is a continuous solution of equation (2), we have that $f^{(j)}(b-) = f^{(j)}(b+)$, $j = 0, 1, \dots, m - 1$. Therefore, from relation (7), using relations (9) and (11), we get that

$$\sum_{i=1}^n c_i(b-)f_i^{(j)}(b-) = \sum_{i=1}^m c_i(b+)f_i^{(j)}(b+), \quad j = 0, 1, \dots, m - 1,$$

which implies that

$$\sum_{i=1}^m c_i(b-) \text{column}(f_i(b-), \dots, f_i^{(m-1)}(b-)) = \sum_{i=1}^m c_i(b+) \tilde{f}_i(b+),$$

that is, (see (II)),

$$(12) \quad \sum_{i=1}^m c_i(b-) \beta_i = \sum_{i=1}^m c_i(b+) \beta_i.$$

Now, since β_1, \dots, β_m are linearly independent, we get from relation (12) that $c_i(b-) = c_i(b+)$, $i = 1, \dots, m$. Hence we choose the b_i 's such that

$$(13) \quad b_i = a_i + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds, \quad i = 1, \dots, m.$$

Substituting the values of b_i 's from relation (13) into relation (10), and extending c_{m+1}, \dots, c_n to the interval I_2 by defining

$$(14) \quad c_i(t) = 0, \quad t \in I_2, t \neq b, i = m + 1, \dots, n,$$

we see in view of relations (8), (10), (13), (14) and the definitions of f_1, \dots, f_n that the proof is complete.

THEOREM 3 (Variation of constants method for matching solutions of $(L_1, L_2)f = g$). *Let f_1, \dots, f_{n+m-d} be the matching fundamental system for $(L_1, L_2)f = 0$ as defined in (III). Then all matching solutions f of $(L_1, L_2)f = g$ are of the form*

$$f(t) = \sum_{i=1}^{n+m-d} c_i(t)f_i(t), \quad t \in I,$$

where

$$c_i(t) = \begin{cases} a_i + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds, & t \in I_1, i = 1, \dots, n, \\ 0, & t \in I_2, t \neq b, i = 1, \dots, n-d, \\ a_i + \int_{t_1}^b \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds \\ + \int_b^t \left(\frac{W_i(f_{n-d+1}, \dots, f_{n+m-d})(s)}{Q_m(s)W(f_{n-d+1}, \dots, f_{n+m-d})(s)} \right) r_2(s)g_2(s)ds, & t \in I_2, i = n-d+1, \dots, n, \end{cases}$$

$$c_{n-d+i}(t) = \begin{cases} 0, & t \in I_1, t \neq b, \\ b_i + \int_b^t \left(\frac{W_i(f_{n-d+1}, \dots, f_{n+m-d})(s)}{Q_m(s)W(f_{n-d+1}, \dots, f_{n+m-d})(s)} \right) r_2(s)g_2(s)ds, & t \in I_2, i = d+1, \dots, m, \end{cases}$$

and a_i ($i = 1, \dots, n$), b_i ($i = 1, \dots, m$) are scalars, and $t_1 \in I_1$.

PROOF. By [6, Theorem 3], there exist nontrivial matching solutions of equation (2). Let us assume a matching solution f of equation (2) to be of the form

$$(15) \quad f(t) = \sum_{i=1}^{n+m-d} c_i(t)f_i(t), \quad t \in I,$$

where the c_i 's are to be determined by the variation of constants method.

Case (1): for $t \in I_1$,

$$f(t) = \sum_{i=1}^{n+m-d} c_i(t)f_i(t) = \sum_{i=1}^n c_i(t)f_i(t) = \sum_{i=1}^n c_i(t)x_i(t)$$

satisfies the equation $L_1 f = 0$. Therefore, by the variation of constants method we get that

$$(16) \quad c_i(t) = a_i + \int_{t_1}^t \left(\frac{W_i(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds, \quad i = 1, \dots, n,$$

where the a_i 's are scalars and $t_1 \in I_1$. We notice that the c_i 's satisfy the relation

$$(17) \quad \sum_{i=1}^n c_i^{(1)}(t)f_i^{(j)}(t) = 0, \quad t \in I_1, j = 0, 1, \dots, n-2.$$

Case (2): for $t \in I_2$,

$$\begin{aligned} f(t) &= \sum_{i=1}^{n+m-d} c_i(t) f_i(t) = \sum_{i=n-d+1}^{n+m-d} c_i(t) f_i(t) \\ &= \sum_{i=n-d+1}^n c_i(t) \sum_{j=m-d+1}^m \theta_j^i y_i(t) + \sum_{i=1}^{m-d} c_{n+i}(t) y_i(t) \end{aligned}$$

satisfies the equation $L_2 f = g_2$. Therefore, by the variation of constants method we get that

$$(18) \quad c_{n-d+i}(t) = b_i + \int_b^t \left(\frac{W_i(f_{n-d+1}, \dots, f_{n+m-d})(s)}{Q_m(s)W(f_{n-d+1}, \dots, f_{n+m-d})(s)} \right) r_2(s) g_2(s) ds, \quad i = 1, 2, \dots, m,$$

where the b_i 's are scalars. We notice that $c_i, i = n - d + 1, \dots, n + m - d$ satisfy the relation

$$(19) \quad \sum_{i=n-d+1}^{n+m-d} c_i^{(1)} f_i^{(j)}(t) = 0, \quad t \in I_2, j = 0, \dots, m - 2.$$

Since f is a matching solution of equation (2), we have that $A\tilde{f}(b-) = B\tilde{f}(b+)$. Therefore, from relation (15), using relations (17) and (19), we get that

$$\begin{aligned} Af(b-) &= \sum_{i=1}^n c_i(b-) A\tilde{f}_i(b-) = \sum_{i=n-d+1}^n c_i(b-) A\tilde{f}_i(b-) = B\tilde{f}(b+) \\ &= \sum_{i=n-d+1}^{n+m-d} c_i(b+) B\tilde{f}_i(b-) = \sum_{i=n-d+1}^n c_i(b+) B\tilde{f}_i(b+), \end{aligned}$$

which implies that

$$\sum_{i=n-d+1}^n c_i(b-) A\tilde{f}_i(b-) = \sum_{i=n-d+1}^n c_i(b+) A\tilde{f}_i(b+) \quad (\text{since } A\tilde{f}_i(b-) = B\tilde{f}_i(b+)),$$

that is,

$$A \left(\sum_{i=n-d+1}^n (c_i(b-) - c_i(b+)) \tilde{f}_i(b-) \right) = 0,$$

that is,

$$(20) \quad A \left(\sum_{i=n-d+1}^n (c_i(b-) - c_i(b+)) \alpha_i \right) = 0 \quad (\text{see (III)}).$$

Now, relation (20) implies that $\sum_{i=n-d+1}^n (c_i(b-) - c_i(b+))\alpha_i$ belongs to the null space of A and this is the true only if $c_i(b-) = c_i(b+)$, $i = n - d + 1, \dots, n$. Hence we choose b_i ($i = 1, \dots, d$) such that

$$(21) \quad b_i = a_{n-d+i} + \int_{t_1}^b \left(\frac{W_{n-d+i}(f_1, \dots, f_n)(s)}{P_n(s)W(f_1, \dots, f_n)(s)} \right) r_1(s)g_1(s)ds.$$

Substituting these values of the b_i 's from relation (21) into relation (18), for $i = 1, \dots, d$, and extending c_1, \dots, c_{n-d} to the interval I_2 by defining $c_i(t) = 0$, $t \in I_2$, $t \neq b$, $i = 1, \dots, n - d$, and $c_{n+1}, \dots, c_{n+m-d}$ to the interval I_1 by defining $c_i(t) = 0$, $t \in I_1$, $t \neq b$, $i = n + 1, \dots, n + m - d$, we see in view of relations (6), (18), (21) and definitions of f_1, \dots, f_{n+m-d} that the proof is complete.

REMARK 2. In a subsequent paper, using the results developed here, we shall construct Green's functions and generalized Green's functions for the three types of boundary value problems associated with (L_1, L_2) .

Acknowledgements

The authors dedicate the work to the chancellor of the institute, Bhagavan Sri Sathya Sai Baba.

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