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NOTES ON THE INVERSE MAPPING THEOREM IN LOCALLY CONVEX SPACES

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Several problems arising from a functional analytic study on Omori's inverse mapping theorem are considered arriving at an inverse mapping theorem in locally convex spaces.

In this note we shall consider some problems which arose in the course of our attempt to generalize Omori's inverse mapping theorem [*6*, Theorem 3.1.1, p. 41] to locally convex spaces.

The domains and ranges of the maps considered in this note are therefore in real locally convex spaces. The method we shall use for the study on these maps is the one which we have introduced in [*10*]. Although we have to refer to it for the details of the method, the notion of "calibration" should be explained here.

A calibration for a locally convex space E is a set Γ_E of semi-norms which induces the topology of E. When F is another locally convex space, it will have a calibration Γ_F. A calibration for the family $\{E, F\}$ is a set of "correspondences", which we have called *semi-norm maps* in [*10*], between Γ_E and Γ_F. In other words, each $p \in \Gamma$ has its *E-component* p_E and its *F-component* p_F, and, hence, each $p \in \Gamma$ determines semi-normed spaces (E, p) and (F, p). Therefore, when a calibration Γ for $\{E, F\}$ is given, we have a family of pairs of semi-normed spaces paired by all $p \in \Gamma$:

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$$\{(E,\ p),\ (F,\ p)\ :\ p \in \Gamma\}\ .$$

For a subset X of E, let (X, p) be the set X regarded as a subset of (E, p). If (X, p) is an open subset of (E, p) for each $p \in \Gamma$, then X is called a *Γ-open subset* of E. Then, a map $f : X \to F$ will be studied by its behaviour as a map of (X, p) into (F, p) for each $p \in \Gamma$. For instance, f is said to be *Γ-continuous on* X if $f : (X, p) \to (F, p)$ is continuous for each $p \in \Gamma$.

A similar method has been used by Fischer [2] and Gutknecht [4]. But, the most illustrious example to which this method is applicable will be a series of works by Omori (see, for example, [6]) on the group $\mathcal{D}$ of all C^∞-diffeomorphisms on a closed manifold and its various subgroups. The locally convex linear spaces which appear there as tangent spaces are Fréchet spaces defined by increasing sequences of norms. In order to state his inverse mapping theorem, we need to be more precise.

Let $\{E, E_k : k \geq 0\}$ be a Sobolev chain, that is,

(1) all E_k are Banach spaces with the norms $\|\cdot\|_k$;

(2) E_{k+1} is linearly and densely imbedded in E_k ;

(3) E is the intersection of all E_k and has the inverse limit topology defined by $\{E_k, \|\cdot\|_k\}$.

The following theorem has been proved on this space E .

THEOREM 3.1.1 ([6], p. 41). *Let U and U' be open neighbourhoods in E_0. Suppose a map $f : U \cap E \to U' \cap E$ with $f(0) = 0$ satisfies the following conditions:*

(1) *f can be extended to a C^∞-map of $U \cap E_k$ into $U' \cap E_k$ for every $k \geq 0$;*

(2) *for every $x \in U \cap E$ and $z \in E$,*

$$\|f'(x)(z)\|_k \leq C\{\|x\|_k\|z\|_0+\|z\|_k\} + P_k(\|x\|_{k-1})\|z\|_{k-1}\ ,$$

$$\|f''(x)(z_1,\ z_2)\|_k \leq C\{\|x\|_k\|z_1\|_0\|z_2\|_0+\|z_1\|_k\|z_2\|_0+\|z_1\|_0\|z_2\|_k\} + P_k(\|x\|_{k-1})\|z_1\|_{k-1}\|z_2\|_{k-1}\ ;$$

(3) $f'(0) : E_k \to E_k$ *is an isomorphism for every* $k \geq 0$;

(4) $\|f'(0)(z)\|_k \geq C'\|z\|_k - D_k\|z\|_{k-1}$,

where C, C', D_k *are positive constants and* C, C' *are independent of* k , *and* P_k *is a polynomial with positive coefficients depending on* k .

Then, there are neighbourhoods W, W' *of zero in* E_0 *such that* $f : W \cap E_k \to W' \cap E_k$ *is a* C^∞*-diffeomorphism for every* $k \geq 0$. *Moreover,* f^{-1} *satisfies the same inequalities as* (3) *and* (4).

This theorem was an essential tool in his works to develop a Lie-group theory on the group $\mathcal{D}$ and its various subgroups, and, therefore, it is one of few inverse mapping theorems on (non-normed) locally convex spaces which have genuine applications. The idea which lies at the basis of this theorem may be expressed as follows: for a locally convex space E with a calibration Γ , take a family $\{E[p] : p \in \Gamma\}$ of Banach spaces, where $E[p]$ is a p-completion of E , and, when a map f on E is given, extend it if possible to maps f_p on $E[p]$ for all $p \in \Gamma$; then study the map f via the family $\{f_p\}$. This idea is capable of being adopted in the theory of linear and non-linear maps in general locally convex spaces. In this note, following [9], we shall restrict our attention to the above inverse mapping theorem.

We start with an elementary discussion on the Γ-*completions* in order to fix terminologies and notations which will be used in the subsequent sections. Section 1 contains only known facts. In Section 2, we shall introduce a class of open subsets which are natural domains of completionally Γ-continuous maps. This class of maps has been introduced in [9]. It will be studied in more detail in Section 3, and we shall give some conditions for such maps to be a local homeomorphism. There are two basic notions in these conditions: S-resonance and T-resonance. In Section 4, conditions for a map to be S-resonant will be given. The notion of *gauge for* (E, Γ) will play an important rôle. In Section 5, conditions for a map to be T-resonant will be given. Here, the notion of *maps of Gårding type* is essential. In the last three sections, we shall consider problems related to this class of maps and arrive at a form of

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inverse mapping theorem with weaker assumptions than Omori's.

1. Γ-completions

Let Γ be a calibration for E. For $p \in \Gamma$, a sequence or a net (x_i) in E is called a *p-Cauchy sequence* or *net* respectively if $p(x_i - x_j) \to 0$, that is, if it is Cauchy in the semi-normed space (E, p). Two p-Cauchy sequences (x_i) and (y_i) are said to be *equivalent* if $p(x_i - y_i) \to 0$. Then we have an equivalence relation on the set of all p-Cauchy sequences in E. A class defined by this equivalence relation will be called a *p-class*, and the set of all p-classes defined in this way will be denoted by $E[p]$.

For $\underline{\underline{a}}, \underline{\underline{b}} \in E[p]$ and real numbers α, β, $\alpha\underline{\underline{a}} + \beta\underline{\underline{b}}$ is defined to be the p-class which contains a p-Cauchy sequence $(\alpha x_i + \beta y_i)$ for some $(x_i) \in \underline{\underline{a}}$ and $(y_i) \in \underline{\underline{b}}$. Then $E[p]$ is a real vector space.

We denote by $S_p(x)$ the p-class which contains the p-Cauchy sequence whose terms are all identical to x. Then the zero element of the vector space $E[p]$ is $S_p(0)$.

Now, for $\underline{\underline{a}} \in E[p]$, we set

$$p(\underline{\underline{a}}) = \lim_{i\to\infty} p(x_i)$$

for some $(x_i) \in \underline{\underline{a}}$. It is obvious that the value $p(\underline{\underline{a}})$ does not depend on the choice of p-Cauchy sequences in $\underline{\underline{a}}$. Then the following fact is obvious.

(1.1). *$p(\underline{\underline{a}})$ $(\underline{\underline{a}} \in E[p])$ defines a norm on $E[p]$ and $E[p]$ is a Banach space with respect to this norm.*

The family of Banach spaces $\{E[p] : p \in \Gamma\}$ defined in this way will be called the *Γ-completion of* E.

To obtain the completion of E with respect to a semi-norm p, it is usual to consider first the normed space $E/p^{-1}(0)$ (the quotient space) and then apply the completion process. This will give the same Banach space $E[p]$. Since we need only the Banach space $E[p]$ in this note, we

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do note take the extra step of constructing the quotient space.

Now let $\{E[p]\}$ be the Γ-completion of E . First we have a linear map

$$S_p : E \to E[p] : x \to S_p(x) ,$$

which satisfies the equality

$$p\big(S_p(x)\big) = p(x) \quad \text{for all} \quad x \in E .$$

Next, let $p, q \in \Gamma$ and $p \geq q$. Then p-Cauchy sequences are q-Cauchy sequences, and, for each $\underline{\underline{a}}_p \in E[p]$, there corresponds a unique $\underline{\underline{a}}_q \in E[q]$ such that $\underline{\underline{a}}_p \subset \underline{\underline{a}}_q$ as sets. Thus, we have the following map:

$$T_{p,q} : E[p] \to E[q] : \underline{\underline{a}}_p \to \underline{\underline{a}}_q .$$

It is obvious that $T_{p,q}$ is linear and

$$q\big(T_{p,q}(\underline{\underline{a}}_p)\big) \leq p\big(\underline{\underline{a}}_p\big) .$$

It is easy to see the following diagram commutes:

$$\begin{tikzcd} E[p] \arrow[rr, "T_{p,q}"] & & E[q] \\ & E \arrow[ul, "S_p"] \arrow[ur, "S_q"'] & \end{tikzcd}$$

EXAMPLE 1. Let K be a compact subset of R (the reals) and let $C^\infty(K)$ be the set of all real-valued C^∞-functions on some open neighbourhood of K . We define a calibration Γ for $E = C^\infty(K)$ as the set of countable norms $\{p_k : k = 0, 1, 2, \ldots\}$ defined by

$$p_k(x) = \sup\{|x^{(i)}(t)| : t \in K, 0 \leq i \leq k\} .$$

Then $E[p_k] = C^k(K)$, the set of all C^k-functions on K , and, in this case,

$$S_{p_k} : C^\infty(K) \to C^k(K) \quad \text{and} \quad T_{p_k,p_0} : C^k(K) \to C^0(K)$$

are imbeddings:

$$S_{p_k}(x) = x \quad \text{and} \quad T_{p_k, p_0}(x) = x .$$

EXAMPLE 2. Let $E = C^0(R)$ be the set of all real-valued continuous functions on R. We define a calibration Γ for this E as the set of countable semi-norms $\Gamma = \{p_k : k = 0, 1, 2, \ldots\}$ defined by

$$p_k(x) = \sup\{|x(t)| : |t| \le k+1\} .$$

For any p_k-Cauchy sequence in E, there corresponds a continuous function on $[-k-1, k+1]$, and, by the Weierstrass theorem, for any continuous function on $[-k-1, k+1]$, there exists a sequence of polynomials which converges to this function uniformly on $[-k-1, k+1]$. Hence, as $E[p_k]$ we can take the Banach space $C^0(-k-1, k+1)$ of all continuous functions on $[-k-1, k+1]$. Then,

$$S_{p_k} : E \to E[p_k] : x \to \chi_k x ,$$

where χ_k is the characteristic function of $[-k-1, k+1]$, and

$$T_{p_k, p_0} : E[p_k] \to E[p_0] : x \to \chi_0 x .$$

2. Completionally Γ-open subsets

Let Γ be a calibration for E and $\{E[p]\}$ be the Γ-completion of E. A subset X of E is said to be *completionally* p*-open* for some $p \in \Gamma$ if there exists an open subset X_p of $E[p]$ such that $X = S_p^{-1}(X_p)$. Since $S_p : E \to E[p]$ is continuous, X is then an open subset of E. If X is completionally p-open for every $p \in \Gamma$, then X is said to be *completionally* Γ*-open*.

For a completionally p-open subset X, the open subset X_p of $E[p]$ is called the p*-cover* of X. When X is completionally Γ-open, the family $\{X_p : p \in \Gamma\}$ is called a Γ*-cover of* X. In fact, X_p covers X in the following sense.

(2.1). $\overline{S_p(X)} = \overline{X}_p$, *where the upper bar denotes the closure in* $E[p]$.

Proof. Since $S_p(X) \subset X_p$, we only need to show that $\overline{X}_p \subset \overline{S_p(X)}$. Let $\underline{\underline{a}} \in X_p$ and $(x_i) \in \underline{\underline{a}}$. Then, since

$$\lim_{i\to\infty} p\left(S_p(x_i)-\underline{\underline{a}}\right) = 0$$

and X_p is open, $S_p(x_i) \in X_p$ and, hence, $x_i \in X$ for large i. Therefore, $\underline{\underline{a}} \in \overline{S_p(X)}$, which shows $X_p \subset \overline{S_p(X)}$.

When Γ has the smallest element p_0, there is a simple way to construct a Γ-cover. For a general calibration Γ, if we take an arbitrary $p_0 \in \Gamma$ and consider the set $\{p \in \Gamma : p \geq p_0\}$, then it is again a calibration and p_0 is its smallest element.

(2.2). *Assume that* Γ *has the smallest element* p_0 *and* X_0 *is an open subset of* $E[p_0]$. *Then* $X = S_{p_0}^{-1}(X_0)$ *is non-empty and completionally* Γ-*open*.

Proof. By (2.1), X is non-empty. Now we denote the map

$$T_{p,p_0} : E[p] \to E[p_0]$$

by T_p. This notation will be used throughout this note when p_0 is the smallest element of Γ. We then put

$$X_p = T_p^{-1}(X_0) .$$

Then X_p is an open subset of $E[p]$. We shall show that

$$X = S_p^{-1}(X_p) \quad \text{for all} \quad p \in \Gamma .$$

Assume that $x \in X$; then $S_{p_0}(x) \in X_0$, or $\left(T_p \circ S_p\right)(x) \in X_0$, which implies

$$S_p(x) \in T_p^{-1}(X_0) = X_p ,$$

and, hence, $x \in S_p^{-1}(X_p)$. Conversely, let $x \in S_p^{-1}(X_p)$; then

$S_p(x) \in T_p^{-1}(X_0)$, or, $S_{p_0}(x) \in X_0$. Hence, $x \in X$.

Assume that $\underline{\underline{a}} \in E[p_0]$, where p_0 is the smallest element of Γ . If $\underline{\underline{a}} = S_{p_0}(x)$ for some $x \in E$, then $T_p^{-1}(\underline{\underline{a}}) = S_p(x)$ and, hence, the sequence (x) belongs to $T_p^{-1}(\underline{\underline{a}})$ for every $p \in \Gamma$. As can be seen easily, the converse is equivalent to the sequential completeness of E .

(2.3). *Assume that* Γ *has the smallest element* p_0 . *Then* E *is sequentially complete if and only if the following condition is satisfied: if* $\bigcap_{p\in\Gamma} T_p^{-1}(\underline{\underline{a}}) \neq \emptyset$ *and* $\underline{\underline{a}} \in E[p_0]$, *then* $\underline{\underline{a}} = S_{p_0}(x)$ *for some* $x \in E$.

The linear maps S_p and T_p are obviously not surjective. Each S_p is injective if p is a norm. In the case of T_p , the situation is a little less simple. A calibration with the smallest element p_0 is said to be *pairwise coordinated*, following [3], if it consists of norm-maps and the following condition is satisfied: a p_0-convergent and p-Cauchy sequence is p-convergent. Spaces determined by a Sobolev chain, as the one in Example 1 in Section 1, are pairwise coordinated.

(2.4). $T_p : E[p] \to E[p_0]$ *are injective for all* $p \in \Gamma$ *if and only if* Γ *is pairwise coordinated.*

Proof. Assume that all T_p are injective. To prove that p_0 is a norm on E , assume that $p_0(x) = 0$. Then, for every $p \in \Gamma$,

$$T_p \circ S_p(x) = S_{p_0}(x) = 0 ,$$

which implies $S_p(x) = 0$, or $p(x) = 0$, for all $p \in \Gamma$. Hence $x = 0$.

Next assume that (x_i) is a p-Cauchy sequence such that $p_0(x_i) \to 0$. Let $(x_i) \in \underline{\underline{a}} \in E[p]$. Then

$$p(S_p(x_i)-\underline{\underline{a}}) \to 0 .$$

Since T_p is continuous,

$$p_0\left(S_{p_0}(x_i)-T_p\underline{\underline{a}}\right) \to 0 ,$$

which implies $p_0\left(T_p\underline{\underline{a}}\right) = 0$. Since p_0 is a norm, $T_p\underline{\underline{a}} = 0$, and, since T_p is injective, $\underline{\underline{a}} = 0$. Therefore, $p\left(S_p(x_i)\right) \to 0$, or equivalently, $p(x_i) \to 0$.

Conversely, assume that Γ is pairwise coordinated and $T_p\underline{\underline{a}} = 0$ for some $\underline{\underline{a}} \in E[p]$. Then, for $(x_i) \in \underline{\underline{a}}$, $p_0\left(S_{p_0}(x_i)\right) \to 0$, that is, $p_0(x_i) \to 0$. Since (x_i) is p-Cauchy, we have $p(x_i) \to 0$. Hence $\underline{\underline{a}} = 0$.

The following fact will be used later.

(2.5). *Assume that* Γ *has the smallest element* p_0 *, it is pairwise coordinated,* $p \le q$ *for* $p, q \in \Gamma$ *and* $T_p\underline{\underline{a}}_p = T_q\underline{\underline{a}}_q$ *for some* $\underline{\underline{a}}_p \in E[p]$ *and* $\underline{\underline{a}}_q \in E[q]$ *. Then*

(1) *all* q*-Cauchy sequences in* $\underline{\underline{a}}_q$ *belong to* $\underline{\underline{a}}_p$ *, and*

(2) $p\left(S_p(x)-\underline{\underline{a}}_p\right) \le q\left(S_q(x)-\underline{\underline{a}}_q\right)$ *for all* $x \in E$ *.*

Proof. (1) Let $(x_i) \in \underline{\underline{a}}_q$; then, since $p \le q$, (x_i) is a p-Cauchy sequence. Let $\underline{\underline{b}}_p$ be the p-class which contains (x_i) . Then $T_p\underline{\underline{b}}_p$ is the p_0-class which contains (x_i) . Therefore

$$T_p\underline{\underline{b}}_p = T_q\underline{\underline{a}}_q = T_p\underline{\underline{a}}_p ,$$

which implies $\underline{\underline{b}}_p = \underline{\underline{a}}_p$. Hence $(x_i) \in \underline{\underline{a}}_p$.

(2) Let $(x_i) \in \underline{\underline{a}}_q$; then $(x_i) \in \underline{\underline{a}}_p$ and, for all $x \in E$,

$$\begin{aligned} p\left(S_p(x)-\underline{\underline{a}}_p\right) &= \lim_{i\to\infty} p\left(x-x_i\right) \\ &\le \lim_{i\to\infty} q\left(x-x_i\right) = q\left(S_q(x)-\underline{\underline{a}}_q\right) . \end{aligned}$$

3. Completionally Γ-continuous maps

Let F be a Γ-family and $E, F \in \mathrm{F}$. Let X be an open subset of

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E . Then a map $f : X \to F$ is said to be *completionally* Γ*-continuous on* X or a CC^0_Γ*-map on* X if the following condition is satisfied: for each $p \in \Gamma$, if (x_i) and (y_i) are p-Cauchy sequences contained in X such that $p_E(x_i - y_i) \to 0$, then $p_F(f(x_i) - f(y_i)) \to 0$. This notion was introduced in [9], where it was shown that a CC^0_Γ-map on X transforms a p-Cauchy sequence in X into a p-Cauchy sequence in F (see (5.2) in [9]). However, the essence of the completional Γ-continuity lies in the following fact.

(3.1). *Let* X *be a completionally* Γ*-open subset of* E *and* $\{X_p\}$ *be its* Γ*-cover. Then* $f : X \to F$ *is a* CC^0_Γ*-map if and only if, for each* $p \in \Gamma$ *, there exists a continuous map*

$$f_p : \overline{X}_p \to F[p]$$

such that the following diagram commutes:

$$\begin{tikzcd} X_p \arrow[r, "f_p"] & F[p] \\ X \arrow[u, "S_p"] \arrow[r, "f"] & F \arrow[u, "S_p"'] \end{tikzcd} .$$

In other words, f is a CC^0_Γ-map if and only if it has continuous "extensions" over X_p for all $p \in \Gamma$.

Proof. Let $f : X \to F$ be a CC^0_Γ-map and $p \in \Gamma$. For each $\underline{\underline{a}}_p \in X_p$, let $f_p(\underline{\underline{a}}_p)$ be the p-class which contains the sequence $(f(x_i))$ for some $(x_i) \in \underline{\underline{a}}_p$. This definition is meaningful because of the following two reasons. First, since f is a CC^0_Γ-map, $(f(x_i))$ is always a p-Cauchy sequence. Secondly, the definition of the completional Γ-continuity gives the independence of $f_p(\underline{\underline{a}}_p)$ on the choice of $(x_i) \in \underline{\underline{a}}_p$.

Now we prove the continuity of f_p . Assume that

$$p_E(\underline{\underline{a}}_n-\underline{\underline{a}}) \to 0 , \quad \underline{\underline{a}}_n, \underline{\underline{a}} \in \overline{X}_p .$$

By (2.1), there exist $(x_{n,i}) \in \underline{\underline{a}}_n$ and $(x_i) \in \underline{\underline{a}}$ such that $x_{n,i}, x_i \in X$. Let us assume that there exists a positive number α such that

$$\overline{\lim_{n\to\infty}} p_F[f_p(\underline{\underline{a}}_n)-f_p(\underline{\underline{a}})] > \alpha ,$$

which means that

$$\overline{\lim_{n\to\infty}} \lim_{i\to\infty} p_F[f(x_{n,i})-f(x_i)] > \alpha .$$

Then there exist sequences (n_k) and (i_k) such that

$$p_F[f(x_{n_k,i_k})-f(x_{i_k})] > \alpha$$

and

$$p_E(x_{n_k,i_k}-x_{i_k}) < 1/k ,$$

which is a contradiction.

The commutativity of the diagram is obvious.

Conversely, suppose we have continuous maps

$$f_p : \overline{X}_p \to F[p]$$

for all $p \in \Gamma$ such that the diagram commutes. Let (x_i) and (y_i) be p-Cauchy sequences contained in X such that $p_E(x_i-y_i) \to 0$. Let $\underline{\underline{a}}$ be the p-class which contains (x_i) and, therefore, (y_i) . Then $\underline{\underline{a}} \in \overline{X}_p$ and

$$\lim_{i\to\infty} p_E(S_p(x_i)-\underline{\underline{a}}) = \lim_{i\to\infty} p_E(S_p(y_i)-\underline{\underline{a}}) = 0 .$$

Hence

$$\lim_{i\to\infty} p_F[f_p(S_p(x_i))-f_p(\underline{\underline{a}})] = \lim_{i\to\infty} p_F[f_p(S_p(y_i))-f_p(\underline{\underline{a}})] = 0 ,$$

which implies

$$\lim_{i\to\infty} p_F[f_p(S_p(x_i))-f_p(S_p(y_i))] = 0 .$$

Then, since $f_p \circ S_p = S_p \circ f$, we have

$$\lim_{i\to\infty} p_F[S_p(f(x_i)-f(y_i))] = 0 ,$$

and, hence,

$$\lim_{i\to\infty} p_F[f(x_i)-f(y_i)] = 0 .$$

In particular, when Γ has the smallest element p_0 and X is a completionally Γ-open subset of E , the following diagram commutes:

$$\begin{tikzcd}
X_{p_0} \arrow[rrr, "f_{p_0}"] & & & F[p_0] \\
& X_p \arrow[ul, "T_p"'] \arrow[r, "f_p"] & F[p] \arrow[ur, "T_p"] & \\
X \arrow[uu, "S_{p_0}"] \arrow[ur, "S_p"] \arrow[rrr, "f"] & & & F \arrow[uu, "S_{p_0}"'] \arrow[ul, "S_p"']
\end{tikzcd} .$$

Throughout the remainder of this section, we assume that $\mathcal{F}$ is a Γ-family, $E, F \in \mathcal{F}$, X and Y are completionally Γ-open subsets of E and F respectively and $\{X_p\}$, $\{Y_p\}$ are their Γ-covers.

(3.2). *If a map* $f : X \to Y$ *is a* CC^0_Γ*-homeomorphism (that is, a bijective* CC^0_Γ*-map whose inverse is also a* CC^0_Γ*-map), then all the "extensions"*

$$f_p : \overline{X}_p \to \overline{Y}_p$$

are homeomorphisms.

Proof. First, we prove that $f_p(\overline{X}_p) \subset \overline{Y}_p$. Since $f_p \circ S_p = S_p \circ f$ and $f(X) = Y$, it is easy to see that

$$f_p(S_p(X)) = S_p(Y) .$$

Now let $\underline{\underline{a}} \in \overline{X}_p$; then there is $(x_i) \in \underline{\underline{a}}$ such that $x_i \in X$ and

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$$p_E\left(S_p(x_i)-\underline{\underline{a}}\right) \to 0 .$$

Since f_p is continuous,

$$p_F\left[f_p\left(S_p(x_i)\right)-f_p(\underline{\underline{a}})\right] \to 0 .$$

Hence, by (2.1),

$$f_p(\underline{\underline{a}}) \in \overline{f_p\left[S_p(X)\right]} = \overline{S_p(Y)} = \overline{Y}_p .$$

Next we prove that f_p is injective. Assume that there are $\underline{\underline{a}}, \underline{\underline{b}} \in \overline{X}_p$ such that $f_p(\underline{\underline{a}}) = f_p(\underline{\underline{b}})$. Take $(x_i) \in \underline{\underline{a}}$ and $(y_i) \in \underline{\underline{b}}$ such that $x_i, y_i \in X$. Then $p_F\left(f(x_i)-f(y_i)\right) \to 0$. Since f^{-1} is a CC^0_Γ-map, we have $p_E\left(x_i-y_i\right) \to 0$, which means that $\underline{\underline{a}} = \underline{\underline{b}}$.

Finally, to show that f_p is surjective, let $\underline{\underline{b}} \in \overline{Y}_p$ and take $(y_i) \in \underline{\underline{b}}$ such that $y_i \in Y$. Then there exist $x_i \in X$ such that $f(x_i) = y_i$. Since f^{-1} is a CC^0_Γ-map, (x_i) is also a p-Cauchy sequence. Let $\underline{\underline{a}}$ be the p-class which contains (x_i). Then $\underline{\underline{a}} \in \overline{X}_p$ and $f_p(\underline{\underline{a}})$ is the p-class which contains $\left(f(x_i)\right)$, that is, $f_p(\underline{\underline{a}}) = \underline{\underline{b}}$.

From the commuting diagram, it is obvious that $\left(f_p\right)^{-1} = \left(f^{-1}\right)_p$, and, hence, $\left(f_p\right)^{-1}$ is continuous.

The converse of this statement does not seem to hold in general. A map $f : X \to Y$ is said to be *S-resonant* if the following condition is satisfied: if, for some $y \in Y$, there exist $\underline{\underline{a}}_p \in X_p$ such that

$$f_p\left(\underline{\underline{a}}_p\right) = S_p(y) \quad \text{for all } p \in \Gamma ,$$

then there exists $x \in X$ such that $\underline{\underline{a}}_p = S_p(x)$ for every $p \in \Gamma$.

It is obvious that f is S-resonant if it is a bijection.

(3.3). *Assume that $f_p : X_p \to Y_p$ are homeomorphisms for all $p \in \Gamma$. If f is S-resonant, then $f : X \to Y$ is a CC^0_Γ-homeomorphism.*

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Proof. We only need to show that f is a bijection. To show that it is an injection, assume that $f(x) = f(y)$ for $x, y \in X$. Then, for each $p \in \Gamma$,

$$f_p\big(S_p(x)\big) = S_p\big(f(x)\big) = S_p\big(f(y)\big) = f_p\big(S_p(y)\big) .$$

Since $S_p(x), S_p(y) \in X_p$ and f_p is injective, we have $S_p(x) = S_p(y)$, which is equivalent to $p(x-y) = 0$. Since this holds for all $p \in \Gamma$, we have $x = y$.

To show that f is surjective, let $y \in Y$. Then there exists $\underline{\underline{a}}_p \in X_p$ such that $f_p(\underline{\underline{a}}) = S_p(y)$ for every $p \in \Gamma$. Since f is S-resonant, $\underline{\underline{a}}_p = S_p(x)$ for some $x \in X$. Thus $f_p\big(S_p(x)\big) = S_p(y)$ for every $p \in \Gamma$, which implies $f(x) = y$.

Therefore we have the following conclusion.

(3.4). *Let* Γ *be a calibration for* $\{E, F\}$. *Let* X *and* Y *be completionally* Γ*-open subsets of* E *and* F *respectively, and* $\{X_p\}$ *and* $\{Y_p\}$ *be their* Γ*-covers. Let* $f : X \to Y$ *be a* CC^0_Γ*-map. Then it is a* CC^0_Γ*-homeomorphism if and only if it is* S*-resonant and each* $f_p : X_p \to Y_p$ *is a homeomorphism.*

It will be seen in the next section that, when the spaces E and F are complete metric linear spaces which are pairwise coordinated, the S-resonance can be removed from the assumptions of (3.3). Here we shall show that the "completional" Γ-openess, instead of mere Γ-openess, is an essential requirement.

Let us consider the map

$$f : x \to \exp \circ x .$$

Eells [1] has shown that this map can never be a local bijection at zero if it is regarded as a map of the space $C^0(R)$ of Example 2 in Section 1 into itself (see also [7], p. 68). However, $f : C^0(R) \to C^0(R)$ is S-resonant and, for each $k \geq 0$, there are open neighbourhoods U_k of zero and V_k of 1 in $C^0(-k-1, k+1)$ such that $f : U_k \to V_k$ is a homeomorphism. In

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this case, these families, $\{U_k\}$ and $\{V_k\}$, of open subsets, determined by the map f, cannot be Γ-covers of any open subset of $C^0(R)$.

We shall now introduce another kind of resonance. Assume that Γ has the smallest element p_0. Then, a map $f : X \to Y$ is said to be *T-resonant* if the following condition is satisfied: if there exist $\underline{\underline{a}}_{p_0} \in X_{p_0}$ and $\underline{\underline{b}}_p \in Y_p$ such that $f_{p_0}(\underline{\underline{a}}_{p_0}) = T_p\underline{\underline{b}}_p$, then $\underline{\underline{a}}_{p_0} = T_p\underline{\underline{a}}_p$ for some $\underline{\underline{a}}_p \in X_p$.

(3.5). *Assume that* Γ *has the smallest element* p_0 *and it is pairwise coordinated. Assume also that* $f : X \to Y$ *is a* CC^0_Γ*-map and* $f_{p_0} : X_{p_0} \to Y_{p_0}$ *is a bijection. Then* f *is* T*-resonant if and only if* $f_p : X_p \to Y_p$ *is a bijection for every* $p \in \Gamma$.

Proof. Assume that f is T-resonant. To prove that f_p is injective, suppose that $f_p(\underline{\underline{a}}_1) = f_p(\underline{\underline{a}}_2)$ for some $\underline{\underline{a}}_1, \underline{\underline{a}}_2 \in X_p$. Then

$$f_{p_0}(T_p\underline{\underline{a}}_1) = T_p(f_p(\underline{\underline{a}}_1)) = T_p(f_p(\underline{\underline{a}}_2)) = f_{p_0}(T_p\underline{\underline{a}}_2) .$$

Hence $T_p\underline{\underline{a}}_1 = T_p\underline{\underline{a}}_2$ and, by (2.4), $\underline{\underline{a}}_1 = \underline{\underline{a}}_2$.

To prove that f_p is surjective, take an arbitrary $\underline{\underline{b}} \in Y_p$. Then there exists $\underline{\underline{a}}_{p_0} \in X_{p_0}$ such that $f_{p_0}(\underline{\underline{a}}_{p_0}) = T_p\underline{\underline{b}}$. Hence, by the T-resonance, there exists $\underline{\underline{a}}_p \in X_p$ such that $\underline{\underline{a}}_{p_0} = T_p\underline{\underline{a}}_p$. Then

$$T_p(f_p(\underline{\underline{a}}_p)) = f_{p_0}(T_p\underline{\underline{a}}_p) = T_p\underline{\underline{b}} ,$$

which implies $f_p(\underline{\underline{a}}_p) = \underline{\underline{b}}$.

Conversely, assume that $f_{p_0}(\underline{\underline{a}}_{p_0}) = T_p\underline{\underline{b}}$. Then, by the assumption, there exists $\underline{\underline{a}}_p \in X_p$ such that $f_p(\underline{\underline{a}}_p) = \underline{\underline{b}}$, and

$$f_{p_0}(T_p\underline{\underline{a}}_p) = T_p(f_p(\underline{\underline{a}}_p)) = T_p\underline{\underline{b}} .$$

Hence $\underline{\underline{a}}_{p_0} = T_p \underline{\underline{a}}_p$ because f_{p_0} is injective.

Thus we have the following conclusion.

(3.6). *Suppose that* Γ *is a calibration for* $\{E, F\}$ *with the smallest element* p_0 *and it is pairwise coordinated. Let* X *and* Y *be completionally* Γ*-open subsets of* E *and* F *respectively and* $\{X_p\}$ *and* $\{Y_p\}$ *be their* Γ*-covers. Let* $f : X \to Y$ *be a* CC^0_Γ*-map. Then it is a* CC^0_Γ*-homeomorphism if and only if it is* S*-resonant,* T*-resonant and* $f_{p_0} : X_{p_0} \to Y_{p_0}$ *is a homeomorphism.*

Obviously, the calibration Γ for $C^\infty(K)$ in Section 1 has the smallest element p_0 and it is pairwise coordinated. Assume that we have a CC^0_Γ-map $f : C^\infty(K) \to C^\infty(K)$ and assume that we have open subsets X_0 and Y_0 of $C^0(K)$ such that $f_{p_0} : X_0 \to Y_0$ is a homeomorphism. Then, if f is S-resonant and T-resonant, $f : X \to Y$ is a CC^0_Γ-homeomorphism, where $X = X_0 \cap C^\infty(K)$ and $Y = Y_0 \cap C^\infty(K)$. For instance, the map $f : x \to \exp \circ x$ is a local CC^0_Γ-homeomorphism at zero in $C^\infty(K)$.

4. Gauged calibrations and S-resonance

Let $\mathcal{F}$ be a Γ-family and $E \in \mathcal{F}$. Assume that the E-component Γ_E of Γ is a directed set. In other words, we assume that, for each pair $p, q \in \Gamma$, there exists $r \in \Gamma$ such that $r_E \geq p_E \cup q_E$, or equivalently,

$$r_E(x) \geq \max\left(p_E(x), q_E(x)\right) \quad \text{for all} \quad x \in E .$$

Then we can consider a net on this directed set Γ_E . If there is a net

$$\gamma_E : \Gamma_E \to R \quad \text{(the reals)}$$

such that $\gamma_E(p) \to 0$, then the calibration Γ_E is said to be *gauged* and the net γ_E is called a gauge on Γ_E .

When Γ_E consists of countable elements p_k $(k = 0, 1, 2, \ldots)$ such that $p_{k+1} \geq p_k$ for all k, then it is obviously gauged with $\gamma_E(p_k) = 1/(k+1)$.

In the case of the space E of all C^∞-maps with compact supports in finite-dimensional spaces, we have a calibration $\Gamma_E = \{p_{\alpha,m}\}$, where $\alpha = (\alpha_k)$ and $m = (m_k)$ are increasing sequences of positive numbers and integers respectively (see [10], p. 13 or [11], §6). Then

$$\gamma_E(p_{\alpha,m}) = 1/(\alpha_1+m_1)$$

defines a gauge on this Γ_E.

The existence of a gauge makes it possible to choose a "diagonal" net from a system of sequences.

(4.1). *Let E be a complete locally convex linear space and Γ be a gauged calibration for E. Assume that there is a family of sequences*

$$\{(x_{p,i}) : p \in \Gamma,\ i = 1, 2, \ldots\}$$

such that, for some $x \in E$ and for all $p \in \Gamma$,

$$\lim_{i\to\infty} p(x_{p,i}-x) = 0 .$$

Then there is a positive-integer-valued net $\{i_p : p \in \Gamma\}$ such that $x_{p,i_p} \to x$ in E.

The proof is similar to that of the following fact.

(4.2). *Let E be a complete locally convex linear space, Γ be a gauged calibration for E with the smallest element p_0 and be pairwise coordinated. If there are $\underline{\underline{a}}_p \in E[p]$ and $\underline{\underline{a}} \in E[p_0]$ such that $T_p\underline{\underline{a}}_p = \underline{\underline{a}}$ for all $p \in \Gamma$, then there exists $x \in E$ such that $\underline{\underline{a}}_p = S_p(x)$ for all $p \in \Gamma$.*

Proof. Let $(x_{p,i}) \in \underline{\underline{a}}_p$; then

$$p\left(S_p(x_{p,i})-\underline{\underline{a}}_p\right) \to 0 \quad \text{as} \quad i \to \infty .$$

Therefore, for a gauge γ on Γ, we have $\{i_p\}$ such that

$$p\left(S_p\left(x_{p,i_p}\right)-\underline{\underline{a}}_p\right) < \gamma(p) \quad \text{for all} \quad p \in \Gamma .$$

Let $r \in \Gamma$; then, if $p, q \geq r$, (2.5) (2) implies

$$\begin{aligned} r\left(x_{p,i_p}-x_{q,i_q}\right) &\leq r\left(S_r\left(x_{p,i_p}\right)-\underline{\underline{a}}_r\right) + r\left(S_r\left(x_{q,i_q}\right)-\underline{\underline{a}}_r\right) \\ &\leq p\left(S_p\left(x_{p,i_p}\right)-\underline{\underline{a}}_p\right) + q\left(S_q\left(x_{q,i_q}\right)-\underline{\underline{a}}_q\right) \\ &\leq \gamma(p) + \gamma(q) , \end{aligned}$$

which shows that $\left(x_{p,i_p}\right)$ is a Cauchy net in E. Since E is complete, it converges to an element $x \in E$. Then,

$$\begin{aligned} p_0\left(S_{p_0}(x)-\underline{\underline{a}}\right) &\leq p_0\left(S_{p_0}(x)-S_{p_0}\left(x_{p,i_p}\right)\right) + p_0\left(S_{p_0}\left(x_{p,i_p}\right)-T_p\underline{\underline{a}}_p\right) \\ &\leq p_0\left(x-x_{p,i_p}\right) + p\left(S_p\left(x_{p,i_p}\right)-\underline{\underline{a}}_p\right) \to 0 . \end{aligned}$$

Hence $S_{p_0}(x) = \underline{\underline{a}}$, that is, $T_pS_p(x) = T_p\underline{\underline{a}}_p$ for all $p \in \Gamma$. Therefore $S_p(x) = \underline{\underline{a}}_p$ for all $p \in \Gamma$.

Now we can give a sufficient condition for the S-resonance.

(4.3). *Let* F *be a* Γ*-family,* $E, F \in \mathrm{F}$ *and* X, Y *be completionally* Γ*-open subsets of* E, F *respectively. Let* $f : X \to Y$ *be a* CC^0_Γ*-map. Assume that*

1. E *is complete and* Γ_E *is gauged;*
2. Γ *has the smallest element* p_0 *and it is pairwise coordinated;*
3. $f_{p_0} : X_{p_0} \to Y_{p_0}$ *is bijective.*

Then f *is* S*-resonant.*

Proof. Assume that $y \in Y$, $\underline{\underline{a}}_p \in X_p$ and

$$f_p\left(\underline{\underline{a}}_p\right) = S_p(y) \quad \text{for all} \quad p \in \Gamma .$$

Then we have

$$f_{p_0}\left(T_p \underset{=}{a}_p\right) = S_{p_0}(y) \quad \text{for all} \quad p \in \Gamma .$$

Since f_{p_0} is injective,

$$T_p \underset{=}{a}_p = \underset{=}{a}_{p_0} \quad \text{for all} \quad p \in \Gamma .$$

Then, by (4.2), there exists $x \in E$ such that $\underset{=}{a}_p = S_p(x)$ for all $p \in \Gamma$. Furthermore, since

$$f_{p_0}\left(S_{p_0}(x)\right) = f_{p_0}\left(T_p \underset{=}{a}_p\right) = S_{p_0}(y)$$

and f_{p_0} is a bijection, we have $S_{p_0}(x) \in X_{p_0}$ and, hence, $x \in X$.

Hence the following fact is a consequence of (3.4).

(4.4). *Let E, F be locally convex linear spaces, E be complete and Γ be a calibration for $\{E, F\}$. Assume that Γ_E is gauged, Γ has the smallest element p_0 and it is pairwise coordinated. Let X and Y be completionally Γ-open subsets of E and F respectively and $f : X \to Y$ be a CC^0_Γ-map. Then f is a CC^0_Γ-homeomorphism if and only if each $f_p : X_p \to Y_p$ is a homeomorphism.*

5. T-resonance and maps of Gårding type

Throughout this section, let F be a Γ-family and we assume that Γ has the smallest element p_0 and it is pairwise coordinated. We also assume that $E, F \in F$ and X, Y are completionally Γ-open subsets of E, F respectively with Γ-covers $\{X_p\}$, $\{Y_p\}$.

We start with the following simple fact.

(5.1). *If $f : X \to Y$ is a CC^0_Γ-map and*

$$f_{p_0} : X_{p_0} \to Y_{p_0}$$

is an injection, then, for each $p \in \Gamma$,

$$f_p : X_p \to Y_p$$

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is an injection.

Proof. This follows from $T_p \circ f_p = f_{p_0} \circ T_p$ and the injectivity of T_p.

A map $f : X \to F$ is said to be *of Gårding type* if, for each $p \in \Gamma$, the following condition is satisfied: if, for a sequence $(\underline{a}_{p,n}) \in X_p$ $(n \geq 1)$,

(1) $(T_p\underline{a}_{p,n})$ is convergent in $E[p_0]$, and

(2) $(f_p(\underline{a}_{p,n}))$ is convergent in $F[p]$,

then $(\underline{a}_{p,n})$ is convergent in $E[p]$.

A map $f : X \to F$ is obviously of Gårding type if, for each $p \in \Gamma$, there are $\alpha_p > 0$ and $\beta_p > 0$ such that, for all $x_1, x_2 \in X$,

$$p(f(x_1)-f(x_2)) \geq \alpha_p p(x_1-x_2) - \beta_p p_0(x_1-x_2) .$$

If $f = u$, a linear map, and

$$p(u(x)) \geq \alpha_p p(x) - \beta_p p_0(x) ,$$

then u is of Gårding type. When $E = C^\infty(K)$ and Γ consists of an increasing sequence of Sobolev-norms, this is an inequality of Gårding type for some elliptic differential operators.

In general, if X is convex and $f : X \to E$ is a $C^1_{B\Gamma}$-map such that there exists $\alpha \in (0, 1)$ for which

$$\|f'(x)-1\|_\Gamma < \alpha$$

is satisfied for all $x \in X$, then

$$p(f(x_1)-f(x_2)) \geq (1-\alpha)p(x_1-x_2) \quad \text{for} \quad x_1, x_2 \in X$$

(see [8]). Hence such a $C^1_{B\Gamma}$-map is of Gårding type. A more general form of this fact will be presented in the subsequent three sections.

(5.2). *Assume that* $f : X \to Y$ *is a* CC^0_Γ*-map of Gårding type and*

$f_{p_0} : X_{p_0} \to Y_{p_0}$ *is a homeomorphism. Then, for each* $p \in \Gamma$, $f_p(X_p)$ *is closed in* Y_p .

Proof. Assume that there are $\underline{\underline{a}}_n \in X_p$ $(n \geq 1)$ and $\underline{\underline{b}} \in Y_p$ such that $f_p(\underline{\underline{a}}_n) \to \underline{\underline{b}}$ in $F[p]$. Then

$$f_{p_0}(T_p\underline{\underline{a}}_n) = T_p f_p(\underline{\underline{a}}_n) \to T_p\underline{\underline{b}} .$$

Since f_{p_0} is a homeomorphism, $(T_p\underline{\underline{a}}_n)$ converges to some $\underline{\underline{a}}_0 \in X_{p_0}$. Since f is of Gårding type, $(\underline{\underline{a}}_n)$ is convergent to some $\underline{\underline{a}}$ in $E[p]$. Then $\underline{\underline{b}} = f_p(\underline{\underline{a}})$. Furthermore

$$f_{p_0}(\underline{\underline{a}}_0) = T_p\underline{\underline{b}} = T_p f_p(\underline{\underline{a}}) = f_{p_0}(T_p\underline{\underline{a}}) ,$$

which implies $\underline{\underline{a}}_0 = T_p\underline{\underline{a}}$, or $\underline{\underline{a}} \in T_p^{-1}(X_{p_0}) = X_p$. Hence $f_p(X_p)$ is closed in Y_p .

Hence the following statement is obvious. Note that Y_p are connected if Y_{p_0} is connected.

(5.3). *Assume that* Y_{p_0} *is connected and* $f : X \to Y$ *is a* CC^0_Γ*-map of Gårding type. If*

(1) $f_{p_0} : X_{p_0} \to Y_{p_0}$ *is a homeomorphism, and*

(2) $f_p(X_p)$ *is open for every* $p \in \Gamma$,

then f *is* T*-resonant.*

We shall consider condition (2) in (5.3). It holds if f_p , which is a map between Banach spaces, is of class C^1 and $(f_p)'(X_p) \subset \mathrm{GL}(E[p], F[p])$, where $\mathrm{GL}(E[p], F[p])$ is the set of all invertible elements of the space $L(E[p], F[p])$ of all continuous linear maps of $E[p]$ into $F[p]$. On the other hand, if $f : X \to Y$ is Γ-differentiable on X , then, for each $x \in X$, $f'(x)$ is a Γ-continuous linear map of E into F. Therefore it is a CC_Γ^0-map and has extensions

$$f'(x)_p : E[p] \to F[p]$$

for every $p \in \Gamma$. For any subset A of E, we set

$$f'(A)_p = \{f'(x)_p : x \in A\} ,$$

which is a subset of $L(E[p], F[p])$. Let

$$B_p(\beta) = \{x \in E; \ p(x) < \beta\}$$

for $p \in \Gamma$ and $\beta > 0$.

(5.4). *Assume that* $f : X \to Y$ *is a* CC_Γ^1*-map,* $p \in \Gamma$ *and*

$$\overline{f'\left(X \cap B_p(\beta)\right)_p} \subset \mathrm{GL}(E[p], F[p]) \quad \textit{for any} \quad \beta > 0 .$$

Then $f_p\left(X_p\right)$ *is open.*

Proof. First we prove that $f_p : X_p \to F[p]$ is of class C^1. By the definition of the CC_Γ^1-maps (see [9]), f is a CC_Γ^0-map, Γ-differentiable and $f' : X \to L_\Gamma(E, F)$ is also a CC_Γ^0-map. (For the definition of the space $L_\Gamma(E, F)$, see [10], p. 5 or the remark after (6.2).) Hence, for each $p \in \Gamma$, f' has the continuous extensions

$$(f')_p : X_p \to L_\Gamma(E, F)[p] ,$$

where $L_\Gamma(E, F)[p]$ is the p-completion of $L_\Gamma(E, F)$. Hence, if $A \in L_\Gamma(E, F)[p]$, there exists $\left(u_i\right) \subset L_\Gamma(E, F)$ such that

$$p\left(S_p\left(u_i\right) - A\right) \to 0 .$$

Then, for $\underline{\underline{a}}_p \in E[p]$ and $\left(x_i\right) \in \underline{\underline{a}}_p$, the sequence $\left(u_i\left(x_i\right)\right)$ is a p-Cauchy sequence in F. Denote by $\bar{A}\left(\underline{\underline{a}}_p\right)$ the p-class containing $\left(u_i\left(x_i\right)\right)$. Then we have the map

$$L_p : L_\Gamma(E, F)[p] \to L(E[p], F[p]) : A \mapsto \bar{A} .$$

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It is easy to see that L_p is continuous; in fact,

$$p\left[L_p(A)\right] \leq p(A) \ .$$

Furthermore f_p is differentiable and $\left(f_p\right)'\left(\underline{\underline{a}}_p\right) = L_p\left((f')_p\right)\left(\underline{\underline{a}}_p\right)$. Since L_p is continoous, f_p is a C^1-map. It remains to show that $\left(f_p\right)'\left(X_p\right) \subset \mathrm{GL}(E[p], F[p])$.

Let $\underline{\underline{a}}_p \in X_p$ and $\left(x_i\right) \in \underline{\underline{a}}_p$. Then $\left(x_i\right)$ is p-bounded,

$$\left(f_p\right)'\left(S_p\left(x_i\right)\right) = L_p\left[(f')_p\left(S_p\left(x_i\right)\right)\right] = f'\left(x_i\right)_p$$

and

$$\left(f_p\right)'\left(S_p\left(x_i\right)\right) \to \left(f_p\right)'\left(\underline{\underline{a}}_p\right) \quad \text{in} \quad L(E[p], F[p]) \ .$$

Hence

$$\left(f_p\right)'\left(\underline{\underline{a}}_p\right) \in \overline{f'(X)_p} \subset \mathrm{GL}(E[p], F[p]) \ .$$

6. δ-extensions and their Omori semi-norms

In the remainder of this note, we shall be concerned about the maps of Gårding type. In Section 8, it will be shown that there is a locally convex algebra consisting of some continuous linear maps of a locally convex space into itself which has a neighbourhood of the identity consisting of linear maps of Gårding type. This fact will be used to derive an inverse mapping theorem of the same type as Omori's.

As the preparation, we shall construct such a locally convex algebra in a general manner. The fundamental notion of this construction is that of Omori semi-norms, which has been introduced by Omori in [6], p. 140, in the case of Gelfand spaces (see [9], p. 337).

Let Γ be a calibration for a family F . A *δ-extension of* Γ is a pair $\left(\Gamma_\delta, \delta\right)$, where Γ_δ is a calibration for F such that $\Gamma \subset \Gamma_\delta$ and δ is a family of maps of Γ_δ into Γ_δ :

$$\delta = \left\{\delta_\lambda : \lambda \in \Lambda\right\} , \quad \delta_\lambda : \Gamma_\delta \to \Gamma_\delta ,$$

such that the following conditions are satisfied:

(δ.1) for each $p \in \Gamma$ and $\lambda \in \Lambda$ there exists $\lambda(1, p) \geq 0$

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such that $p \leq \delta_\lambda(p) \leq \lambda(1, p)p$;

(δ.2) for $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ such that

$$\delta_{\lambda_1} \circ \delta_{\lambda_2}(p) \leq \delta_\lambda(p) \quad \text{and} \quad \delta_{\lambda_2} \circ \delta_{\lambda_1}(p) \leq \delta_\lambda(p)$$

for all $p \in \Gamma$;

(δ.3) if there exist $E, F \in \mathsf{F}$ and $x \in E$, $y \in F$ such that

$$p_E(x) \leq \delta_\lambda(p)_F(y) \quad \text{for all} \quad p \in \Gamma ,$$

then, for every $\lambda_1 \in \Lambda$,

$$\delta_{\lambda_1}(p)_E(x) \leq \left(\delta_\lambda \circ \delta_{\lambda_1}\right)(p)_F(y) \quad \text{for all} \quad p \in \Gamma .$$

An immediate consequence of the definition is the following.

(6.1). *For* $\lambda_1, \lambda_2 \in \Lambda$ *there exists* $\lambda \in \Lambda$ *such that*

$$\delta_{\lambda_1}(p) \leq \delta_\lambda(p) \quad \textit{and} \quad \delta_{\lambda_2}(p) \leq \delta_\lambda(p) \quad \textit{for all} \quad p \in \Gamma .$$

Proof. From (δ.1), we have

$$p \leq \delta_{\lambda_1}(p) \quad \text{and} \quad p \leq \delta_{\lambda_2}(p) \quad \text{for all} \quad p \in \Gamma .$$

Hence, from (δ.3) with $x = y$, we have

$$\delta_{\lambda_2}(p) \leq \left(\delta_{\lambda_1} \circ \delta_{\lambda_2}\right)(p) \quad \text{and} \quad \delta_{\lambda_1}(p) \leq \left(\delta_{\lambda_2} \circ \delta_{\lambda_1}\right)(p)$$

for all $p \in \Gamma$. Therefore the λ determined by (δ.2) gives the required inequalities.

When $\Gamma = \Gamma_\delta$ and δ consists of the identity map, the δ-extension is said to be *trivial*. A non-trivial example of the δ-extension is found in the setting which Omori has adopted in [6] for his study on groups of C^∞-diffeomorphisms on closed manifolds. In this case, Γ consists of an increasing sequence of norms

$$\Gamma = \{p_k : k = 0, 1, 2, \ldots\} .$$

Let Γ_δ be the set of all finite linear combinations of elements of Γ with non-negative coefficients

$$\Gamma_\delta = \{\alpha p_i+\beta p_j : \alpha \geq 0,\ \beta \geq 0,\ i,\ j = 0, 1, 2, \ldots\} .$$

Let Λ be the set of all sequences of non-negative numbers

$$\Lambda = \{\lambda = (\lambda_k) : \lambda_0 = 0\} ,$$

and, for each $\lambda \in \Lambda$, we define a map δ_λ by

$$\delta_\lambda(p_0) = p_0 ,\quad \delta_\lambda(p_k) = p_k + \lambda_k p_{k-1} \quad \text{for} \quad k \geq 1$$

and

$$\delta_\lambda(\alpha p_i+\beta p_j) = \alpha\delta_\lambda(p_i) + \beta\delta_\lambda(p_j) .$$

Then (Γ_δ, δ), thus defined, is a δ-extension of Γ, which we shall call the *Omori* δ-*extension*.

Now let Γ be a calibration for F and (Γ_δ, δ) be a δ-extension of Γ. For $E, F \in \mathsf{F}$ and $\lambda \in \Lambda$, we define a calibration Γ_λ for $\{E, F\}$ by

$$\Gamma_\lambda = \{(\delta_\lambda(p)_E, p_F) : p \in \Gamma\} .$$

Namely, Γ_λ is a family of semi-norm maps p_λ such that

$$(p_\lambda)_E = \delta_\lambda(p)_E \quad \text{and} \quad (p_\lambda)_F = p_F \quad \text{for} \quad p \in \Gamma .$$

The condition (δ.1) ensures that Γ_λ is also a calibration for $\{E, F\}$. Then, as in [*10*], p. 5, we can consider the space $L_{B\Gamma_\lambda}(E, F)$ of all $B\Gamma_\lambda$-continuous linear maps of E into F. By definition, a linear map $u : E \to F$ belongs to $L_{B\Gamma_\lambda}(E, F)$ if and only if

$$\|u\|_{\Gamma_\lambda} = \sup_{p\in\Gamma} \sup\{p_F[u(x)] : \delta_\lambda(p)_E(x) \leq 1\} < +\infty ,$$

and $\|u\|_{\Gamma_\lambda}$ thus defined makes $L_{B\Gamma_\lambda}(E, F)$ into a normed space, and a Banach space if F is sequentially complete. For the sake of simplicity, we denote $L_{B\Gamma_\lambda}(E, F)$ by $L_\lambda(E, F)$ and $\|\cdot\|_{\Gamma_\lambda}$ by $\|\cdot\|_\lambda$.

Now we set

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$$L_\delta(E, F) = \cup\{L_\lambda(E, F) : \lambda \in \Lambda\} .$$

Hence $L_\delta(E, F)$ is a union of normed spaces and it is a linear space. In fact, let $u, v \in L_\delta(E, F)$; then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $u \in L_{\lambda_1}(E, F)$ and $v \in L_{\lambda_2}(E, F)$. Let λ be the index determined by $(\delta. 1)$. Then, for all $p \in \Gamma$ and $x \in E$,

$$p_F[u(x)] \le \|u\|_{\lambda_1}\delta_{\lambda_1}(p)_E(x) \le \|u\|_{\lambda_1}\delta_\lambda(p)_E(x)$$

and

$$p_F[v(x)] \le \|v\|_{\lambda_2}\delta_{\lambda_2}(p)_E(x) \le \|v\|_{\lambda_2}\delta_\lambda(p)_E(x) ,$$

which imply $u, v \in L_\lambda(E, F)$.

When (Γ_δ, δ) is trivial, we have $L_\delta(E, F) = L_{B\Gamma}(E, F)$, that is, $L_\delta(E, F)$ in this case is a normed linear space.

(6.2). *(i) For every* $\lambda \in \Lambda$, $L_{B\Gamma}(E, F) \subset L_\lambda(E, F)$ *and*

$$\|u\|_\lambda \le \|u\|_\Gamma \quad \textit{for} \quad u \in L_{B\Gamma}(E, F) .$$

(ii) $L_\delta(E, F) \subset L_\Gamma(E, F)$ *and*

$$p(u) \le \lambda(1, p)\|u\|_\lambda \quad \textit{if} \quad u \in L_\lambda(E, F) \quad \textit{and} \quad p \in \Gamma .$$

Proof. *(i)* is an immediate consequence of $(\delta.1)$. Note that $u \in L_{B\Gamma}(E, F)$ if and only if

$$\|u\|_\Gamma = \sup\{p_F[u(x)] : p_E(x) \le 1, p \in \Gamma\} < +\infty .$$

To prove *(ii)*, let $u \in L_\lambda(E, F)$. By $(\delta.1)$ we always have

$$\delta_\lambda(p)_E(x) \le \lambda(1, p)p_E(x) \quad \text{for} \quad p \in \Gamma \quad \text{and} \quad x \in E .$$

Hence, for all $p \in \Gamma$ and $x \in E$,

$$p_F[u(x)] \le \|u\|_\lambda\delta_\lambda(p)_E(x) \le \|u\|_\lambda\lambda(1, p)p_E(x) ,$$

which implies $u \in L_\Gamma(E, F)$ and $p(u) \le \|u\|_\lambda\lambda(1, p)$.

REMARK. By definition ([10], p. 5), the space $L_\Gamma(E, F)$ consists of all Γ-continuous linear maps of E into F, namely, linear maps $u : E \to F$ such that

$$p_{(E,F)}(u) = \sup\{p_F[u(x)] : p_E(x) \leq 1\} < +\infty$$

for each $p \in \Gamma$. It is regarded as a locally convex space belonging to F with the following set as its component of Γ:

$$\Gamma_{(E,F)} = \{p_{(E,F)} : p \in \Gamma\} .$$

In this note we denote $p_{(E,F)}$ simply by p.

If $\delta_{\lambda_1}(p) \geq \delta_{\lambda_2}(p)$ for all $p \in \Gamma$, then $L_{\lambda_1}(E, F) \supset L_{\lambda_2}(E, F)$ and $\|u\|_{\lambda_1} \leq \|u\|_{\lambda_2}$ for all $u \in L_{\lambda_2}(E, F)$. Hence, by (6.1), the space $L_\delta(E, F)$ is the union of an increasing sequence of normed linear spaces. In this setting we can define the *Omori semi-norm* on $L_\delta(E, F)$ by

$$|u| = \inf_{\lambda \in \Lambda} \{\|u\|_\lambda : u \in L_\lambda(E, F)\} .$$

In order to see that it is a semi-norm on $L_\delta(E, F)$, let α be an arbitrary positive number such that

$$\alpha > |u| + |v| , \quad u, v \in L_\delta(E, F) ,$$

and choose α_1 and α_2 such that $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 > |u|$ and $\alpha_2 > |v|$. Then there exist $\lambda_1, \lambda_2 \in \Lambda'$ such that, for all $p \in \Gamma$ and $x \in E$,

$$p_F[u(x)] < \alpha_1\delta_{\lambda_1}(p)_E(x) \quad \text{and} \quad p_F[v(x)] < \alpha_2\delta_{\lambda_2}(p)_E(x) .$$

Take λ determined by (δ.1) from λ_1 and λ_2; then, for all $p \in \Gamma$ and $x \in E$,

$$\begin{aligned} p_F[u(x)+v(x)] &< \alpha_1\delta_{\lambda_1}(p)_E(x) + \alpha_2\delta_{\lambda_2}(p)_E(x) \\ &\leq \alpha\delta_\lambda(p)_E(x) , \end{aligned}$$

which means $u + v \in L_\lambda(E, F)$ and $|u+v| \leq \|u+v\|_\lambda \leq \alpha$.

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We recall that, whenever we write $L_\lambda(F, G)$ for $F, G \in \mathsf{F}$, the first space F has the calibration $\delta_\lambda(\Gamma)_F$ and the second space G has the calibration Γ_G.

(6.3). *If* $u \in L_\delta(E, F)$ *and* $v \in L_\delta(F, G)$ *for* $E, F, G \in \mathsf{F}$, *then* $v \circ u \in L_\delta(E, G)$ *and* $|v \circ u| \le |u||v|$.

Proof. Let α be an arbitrary number such that $\alpha > |u||v|$ and choose α_1 and α_2 such that $\alpha = \alpha_1\alpha_2$, $\alpha_1 > |u|$ and $\alpha_2 > |v|$. Then there exist $\lambda_1, \lambda_2 \in \Lambda$ such that

$$p_F[u(x)] < \alpha_1\delta_{\lambda_1}(p)_E(x) \quad\text{and}\quad p_G[v(y)] < \alpha_2\delta_{\lambda_2}(p)_F(y) .$$

By (δ.3) we have

$$\delta_{\lambda_2}(p)_F[u(x)] \le \alpha_1\left(\delta_{\lambda_1} \circ \delta_{\lambda_2}\right)(p)_E(x) .$$

Hence, for all $x \in E$,

$$p_G[(v \circ u)(x)] < \alpha_2\delta_{\lambda_2}(p)_F[u(x)] \le \alpha_1\alpha_2\left(\delta_{\lambda_1} \circ \delta_{\lambda_2}\right)(p)_E(x) .$$

Then, for the $\lambda \in \Lambda$ determined by (δ.2), we have $v \circ u \in L_\lambda(E, G)$ and $\|v \circ u\|_\lambda \le \alpha$, that is, $|v \circ u| \le |u||v|$.

As we have seen in (6.2) *(ii)*, $L_\delta(E, F)$ is a linear subspace of $L_\Gamma(E, F)$. Hence, if $u \in L_\delta(E, F)$, then

$$p(u) = \sup\{p_F[u(x)] : p_E(x) \le 1\} < +\infty .$$

We use this fact to define the $L_\delta(E, F)$-component of each $p \in \Gamma$ by

$$|p| = \max\{p, |\cdot|\} .$$

Therefore the locally convex topology on $L_\delta(E, F)$ is stronger than the relative topology induced from $L_\Gamma(E, F)$, and $L_\delta(E, F)$ is closed in $L_\Gamma(E, F)$ in the sense described in (6.4) which holds when the δ-extension is diagonalizable. A δ-extension of Γ is said to be *diagonalizable* if the following condition is satisfied: *for any map* $\phi : \Gamma \to \Lambda$ *there exists*

$\lambda_\phi \in \Lambda$ *such that* $\delta_{\lambda_\phi}(p) = \delta_{\phi(p)}(p)$ *for all* $p \in \Gamma$.

Let (Γ_δ, δ) be the Omori δ-extension. Then, for $\phi : \Gamma \to \Lambda$, set $\phi(p_k) = \lambda^{(k)}$ for $k \ge 0$. If we take $\lambda_\phi = \left\{\lambda_k^{(k)}\right\}$, the diagonal sequence, then

$$\delta_{\lambda_\phi}(p_k) = p_k + \lambda_k^{(k)} p_{k-1} = \delta_{\lambda^{(k)}}(p_k) = \delta_{\phi(p_k)}(p_k) .$$

Hence the Omori δ-extension is diagonalizable.

(6.4). *Assume that* (Γ_δ, δ) *is diagonalizable. Let* $u_n \in L_\delta(E, F)$, $u_n \to u$ *in* $L_\Gamma(E, F)$ *and there exists a number* α *such that* $\alpha > |u_n|$ *for all* $n \ge 1$. *Then* $u \in L_\delta(E, F)$ *and* $|u| \le \alpha$.

Proof. Let ε be an arbitrary positive number. The assumption that $u_n \to u$ in $L_\Gamma(E, F)$ means

$$\lim_{n\to\infty} p(u_n-u) = 0 \quad \text{for each} \quad p \in \Gamma .$$

Hence we have $\{n_p : p \in \Gamma\}$ such that $p(u_{n_p}-u) < \varepsilon$. On the other hand, since $|u_n| < \alpha$, there exist $\lambda_n \in \Lambda$ such that, for all $p \in \Gamma$ and $x \in E$,

$$p_F[u_n(x)] < \alpha\delta_{\lambda_n}(p)_E(x) .$$

Let $\phi : \Gamma \to \Lambda$ be defined by $\phi(p) = \lambda_{n_p}$. Then we have $\lambda_\phi \in \Lambda$ such that, for all $p \in \Gamma$,

$$\delta_{\lambda_{n_p}}(p) = \delta_{\phi(p)}(p) = \delta_{\lambda_\phi}(p) .$$

Therefore, for all $p \in \Gamma$ and $x \in E$,

$$\begin{aligned} p_F[u(x)] &\le p_F[u_{n_p}(x)] + p_F[(u_{n_p}-u)(x)] \\ &< \alpha\delta_{\lambda_\phi}(p)_E(x) + \varepsilon p_E(x) \\ &\le (\alpha+\varepsilon)\delta_{\lambda_\phi}(p)_E(x) , \end{aligned}$$

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which means that $u \in L_{\lambda_\phi}(E, F)$ and $\|u\|_{\lambda_\phi} \leq \alpha + \varepsilon$. Hence $u \in L_\delta(E, F)$ and $|u| \leq \alpha$.

Now we can prove the completeness of $L_\delta(E, F)$.

(6.5). *If* (Γ_δ, δ) *is diagonalizable and* F *is sequentially complete,* $L_\delta(E, F)$ *is sequentially complete.*

Proof. Let (u_n) be a Cauchy sequence in $L_\delta(E, F)$ and ε be an arbitrary positive number. Then there exists n_0 such that

$$|u_n - u_m| < \varepsilon \quad \text{if} \quad m, n \geq n_0 .$$

Since (u_n) is also a Cauchy sequence in $L_\Gamma(E, F)$, which is sequentially complete ([*10*], p. 6), there exists $u \in L_\Gamma(E, F)$ such that $u_n \to u$ in $L_\Gamma(E, F)$. By (6.4), $u \in L_\delta(E, F)$. We need to show that $|u_n - u| < \varepsilon$ if $n \geq n_0$.

Now, from the fact that $p(u_n - u) \to 0$ for each $p \in \Gamma$, we have $\{n_p : p \in \Gamma\}$ such that $n_p \geq n_0$ and

$$p\left(u_{n_p} - u\right) < \varepsilon/2 \quad \text{for all} \quad p \in \Gamma .$$

Then, if $n \geq n_0$, since $n, n_p \geq n_0$, we have $|u_n - u_{n_p}| < \varepsilon$. Hence there exist $\lambda_{n,p} \in \Lambda$ such that

$$p_F\left[\left(u_n - u_{n_p}\right)(x)\right] < (\varepsilon/2)\delta_{\lambda_{n,p}}(p)_E(x) \quad \text{for all} \quad p \in \Gamma \quad \text{and} \quad x \in E .$$

For the map

$$\phi_n : p \mapsto \lambda_{n,p}$$

choose $\lambda_{\phi_n} \in \Lambda$. Then, for every $p \in \Gamma$,

$$\begin{aligned} p_F[(u_n-u)(x)] &\le p_F[(u_n-u_{n_p})(x)] + p_F[(u_{n_p}-u)(x)] \\ &< (\varepsilon/2)\delta_{\lambda_{\phi_n}}(p)_E(x) + (\varepsilon/2)p_E(x) \\ &\le \varepsilon\delta_{\lambda_{\phi_n}}(p)_E(x) , \end{aligned}$$

which means

$$u_n - u \in L_{\lambda_{\phi_n}}(E, F) \quad \text{and} \quad \|u_n-u\|_{\lambda_{\phi_n}} \le \varepsilon .$$

Hence, in particular, $|u_n-u| \le \varepsilon$. Thus $u_n \to u$ in $L_\delta(E, F)$.

From now on, we shall assume that E and F are the same member of F , that is, $E = F$ as locally convex spaces equipped with the same component of Γ . We shall denote $L_\Gamma(E, E)$ by $L_\Gamma(E)$, which therefore consists of all linear maps $u : E \to E$ such that

$$p(u) = \sup\{p_E[u(x)] : p_E(x) \le 1\} < +\infty \quad \text{for each} \quad p \in \Gamma .$$

We shall omit the letter E in p_E . The space $L_\lambda(E, E)$ consists of all linear maps $u : E \to E$ such that

$$\|u\|_\lambda = \sup_{p\in\Gamma} \sup\{p[u(x)] : \delta_\lambda(p)(x) \le 1\} < +\infty .$$

We set

$$L_\delta(E) = L_\delta(E, E) = \cup\{L_\lambda(E, E) : \lambda \in \Lambda\} .$$

Then, as we have seen above, $L_\delta(E)$ is a locally convex space equipped with the calibration $\{|p| : p \in \Gamma\}$.

(6.6). $L_\delta(E)$ *is a locally convex algebra and, for each* $p \in \Gamma$,

$$|p|(v \circ u) \le |p|(u)\,|p|(v) \quad \textit{for all} \quad u, v \in L_\delta(E) .$$

Proof. When $u, v \in L_\delta(E)$, (6.3) implies $v \circ u \in L_\delta(E)$ and $|v \circ u| \le |u||v|$. Furthermore, for each $p \in \Gamma$ and $x \in E$,

$$p[(v \circ u)(x)] \le p(v)p[u(x)] \le p(v)p(u)p(x) .$$

Since $|p| = \max\{p, |\cdot|\}$, we have the required inequality.

Thus $L_\delta(E)$ is a locally convex algebra with jointly continuous products. In fact, it is a locally m-convex algebra in the sense of [5]. We shall show in (6.7) that this is also a continuous inverse algebra if E is sequentially complete and (Γ_δ, δ) is diagonalizable and summable. A δ-extension (Γ_δ, δ) is said to be *summable* if the following condition is satisfied: *for all* $p \in \Gamma$, $\lambda \in \Lambda$ *and* $E \in F$,

$$\lambda(n, p) = \sup\left\{\delta_\lambda^n(p)_E(x) : p_E(x) \le 1\right\} < +\infty$$

and

$$\overline{\lim_{n\to\infty}} \lambda(n, p)^{1/n} < +\infty .$$

The value of this limit will be called the *summability constant* of δ .

When (Γ_δ, δ) is the Omori δ-extension, we have

$$\delta_\lambda^n\left(p_k\right)_E(x) \le n^k\left(1+\lambda_k\right)^k\left(p_k\right)_E(x) ,$$

which implies that the Omori δ-extension is summable.

Now let us denote by $\mathrm{GL}_\delta(E)$ the set of all $u \in L_\delta(E)$ such that the inverse u^{-1} exists and belongs to $L_\delta(E)$. More precisely, a linear map $u : E \to E$ belongs to $\mathrm{GL}_\delta(E)$ if it is a bijection and satisfies the following conditions: there exist $\alpha > 0$, $\beta > 0$ and $\lambda_1, \lambda_2 \in \Lambda$ such that, for all $p \in \Gamma$ and $x \in E$,

$$p[u(x)] \le \alpha\delta_{\lambda_1}(p)(x) \quad \text{and} \quad p\left[u^{-1}(x)\right] \le \beta\delta_{\lambda_2}(p)(x) .$$

It immediately follows from (δ.1) that $L_\delta(E)$ contains the identity map. If $u, v \in \mathrm{GL}_\delta(E)$, then, since $u^{-1}, v^{-1} \in \mathrm{GL}_\delta(E)$, (6.1) implies $(v \circ u)^{-1} = u^{-1} \circ v^{-1} \in \mathrm{GL}_\delta(E)$. Therefore $\mathrm{GL}_\delta(E)$ is a group.

(6.7). *Assume that* E *is sequentially complete and* (Γ_δ, δ) *is diagonalizable and summable. Then* $\mathrm{GL}_\delta(E)$ *is a* Γ*-open subset of* $L_\delta(E)$

and the inverse operation is Γ*-continuous on* $\mathrm{GL}_\delta(E)$.

Proof. Let us take an arbitrary $q \in \Gamma$ and consider the $|q|$-open subset U of $L_\delta(E)$ defined by

$$U = \{u \in L_\delta(E) : |q|(u) < 1/\alpha\} ,$$

where α is a number which is greater than the summability constant of δ and 1 . Then, for $u \in U$, since $|u| < 1/\alpha$, there exists $\lambda \in \Lambda$ such that

$$p[u(x)] < (1/\alpha)\delta_\lambda(p)(x) \quad \text{for all } p \in \Gamma \text{ and } x \in E .$$

The condition (δ.3) then implies that, for $i \geq 1$, we have

$$\delta_\lambda^i(p)[u(x)] \leq (1/\alpha)\delta_\lambda^{i+1}(p)(x) ,$$

and

$$p\left[u^n(x)\right] \leq (1/\alpha)\delta_\lambda(p)\left[u^{n-1}(x)\right] \leq \left(1/\alpha^2\right)\delta_\lambda^2(p)\left[u^{n-2}(x)\right] \leq \ldots$$
$$\leq \left(1/\alpha^n\right)\delta_\lambda^n(p)(x) .$$

Hence, since $\left(\Gamma_\delta, \delta\right)$ is summable, for every $p \in \Gamma$ and $x \in E$,

$$\sum_{n=0}^{\infty} p\left[u^n(x)\right] \leq \sum_{n=0}^{\infty} \left(1/\alpha^n\right)\delta_\lambda^n(p)(x) \leq \sum_{n=0}^{\infty} \left(1/\alpha^n\right)\lambda(n, p)p(x) < +\infty .$$

This implies

$$\sum_{n=0}^{\infty} p\left(u^n\right) < +\infty .$$

Furthermore, since $|u| < 1/\alpha$, we have

$$\sum_{n=0}^{\infty} \left|u^n\right| \leq \sum_{n=0}^{\infty} |u|^n < +\infty ,$$

which implies

$$\sum_{n=0}^{\infty} |p|\left(u^n\right) < +\infty .$$

Hence, by (6.5), the series

$$\sum_{n=0}^{\infty} (-1)^n u^n$$

converges in $L_\delta(E)$.

Thus we have shown that for every $u \in U$, $1 + u$ has the inverse in $L_\delta(E)$, or $1 + U \subset \mathrm{GL}_\delta(E)$.

Now assume that $u_0 \in \mathrm{GL}_\delta(E)$ and

$$|q|(u_0-u) < (1/\alpha)|q|\left(u_0^{-1}\right)^{-1} \text{ and } u \in L_\delta(E) .$$

Then, since

$$|q|\left(1-u_0^{-1}u\right) \le |q|\left(u_0^{-1}\right)|q|(u_0-u) < 1/\alpha ,$$

we have $u_0^{-1} \circ u \in \mathrm{GL}_\delta(E)$ and, hence, $u \in \mathrm{GL}_\delta(E)$. Thus $\mathrm{GL}_\delta(E)$ is $|q|$-open in $L_\delta(E)$ for every $q \in \Gamma$; that is, $\mathrm{GL}_\delta(E)$ is a Γ-open subset of $L_\delta(E)$.

Next we prove that the inverse operation is Γ-continuous on $\mathrm{GL}_\delta(E)$. Let us assume that $u_n \in \mathrm{GL}_\delta(E)$, $p \in \Gamma$ and $|p|(u_n-1) \to 0$. Then there exists n_0 such that

$$|p|(u_n-1) < 1/\alpha \text{ if } n \ge n_0 ,$$

and, as we have shown above,

$$p\left(u_n^{-1}\right) \le \sum_{m=0}^{\infty} \left(1/\alpha^m\right)\lambda(m, p) .$$

Hence

$$p\left(u_n^{-1}-1\right) \le p\left(u_n^{-1}\right)p\left(1-u_n\right) \to 0 .$$

Similarly, for $n \ge n_0$, since

$$u_n^{-1} = \sum_{m=0}^{\infty} (-1)^m(u_n-1)^m ,$$

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we have

$$\left|u_n^{-1}\right| \le \sum_{m=0}^{\infty} |u_n-1|^m \le \alpha/(\alpha-1) .$$

Hence

$$\left|u_n^{-1}-1\right| = \left|u_n^{-1}(1-u_n)\right| \le \left|u_n^{-1}\right| |1-u_n| \to 0 .$$

Therefore the inverse operation is Γ-continuous at the identity map. Since $u_n^{-1} - u^{-1} = u_n^{-1}(u-u_n)u^{-1}$, we have the Γ-continuity at every point of $GL_\delta(E)$.

7. C_δ^1-maps

Let Γ be a calibration for F and (Γ_δ, δ) be a δ-extension of Γ . Let $E, F \in \mathrm{F}$ and X be an open subset of E . Then a map $f : X \to F$ is said to be *of class* C_δ^1 *at* $a \in X$ if the following conditions are satisfied:

1. f is of class CC_Γ^1 on X and $f'(X) \subset L_\delta(E, F)$;
2. the map $f' : X \to L_\delta(E, F)$ is Γ-continuous at a .

If f is of class C_δ^1 at every point of X , it is called a C_δ^1-map on X .

When (Γ_δ, δ) is trivial and X is convex, any $C_{B\Gamma}^2$-map $f : X \to F$ such that $\sup\{\|f''(x)\|_\Gamma : x \in X\} < +\infty$ is a C_δ^1-map on X . For the definition of $C_{B\Gamma}^n$-maps, see [10], p. 23.

It is obvious that every element of $L_\delta(E, F)$ is a C_δ^1-map on E .

(7.1). *Let* X *and* Y *be open subsets of* E *and* F *respectively and* $G \in \mathrm{F}$. *If* $f : X \to Y$ *is of class* C_δ^1 *at* $a \in X$ *and* $g : Y \to G$ *is*

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of class C^1_δ *at* $f(a)$ *, then* $g \circ f$ *is of class* C^1_δ *at* a *.*

Proof. Obviously, $g \circ f$ is of class CC^0_Γ on X. By [*10*], (II.3.2), $g \circ f$ is of class C^1_Γ on X and
$(g \circ f)'(x) = g'(f(x)) \circ f'(x)$. Hence $g \circ f$ is of class CC^1_Γ on X and $(g \circ f)'(X) \subset L_\delta(E, F)$. Therefore, we only need to show that the map $(g \circ f)' : X \to L_\delta(E, G)$ is Γ-continuous at a.

Now assume that $p \in \Gamma$ and $p(x_n - a) \to 0$ and $x_n \in X$. Then, by (6.2) and [*10*], (I.2.2),

$$|p|[(g \circ f)'(x_n) - (g \circ f)'(a)] \le |p|[g'(f(x_n)) - g'(f(a))]|p|[f'(x_n)] + |p|[g'(f(a))]|p|[f'(x_n) - f'(a)] \to 0 .$$

(7.2). *Let* X *and* Y *be open subsets of* E *and* F *respectively and let* E *be sequentially complete. Assume that* (Γ_δ, δ) *is diagonalizable and summable. Then, if* $f : X \to Y$ *is a* CC^0_Γ*-homeomorphism of class* C^1_δ *and, for every* $x \in X$ *,* $f'(x)$ *has the inverse belonging to* $L_\delta(F, E)$ *, the inverse* f^{-1} *is of class* C^1_δ *on* Y *.*

Proof. Under these assumptions, it follows from [*10*], (III.3.1), that f^{-1} is Γ-differentiable at every point of Y and

$$(f^{-1})'(f(x)) = f'(x)^{-1} \text{ for every } x \in X .$$

Hence f^{-1} is of class CC^1_Γ on Y and $(f^{-1})'(Y) \subset L_\delta(F, E)$. Therefore we only need to show that the map $(f^{-1})' : Y \to L_\delta(F, E)$ is Γ-continuous.

Now let $a \in X$, $b = f(a)$, $p \in \Gamma$ and

$$p_F(y_n - b) \to 0 \text{ and } y_n \in Y .$$

Then there exist $x_n \in X$ such that $y_n = f(x_n)$ and, since f^{-1} is Γ-continuous, $p(x_n - a) \to 0$. Then, by (6.7),

$$|p|\left[\left(f^{-1}\right)'\left(y_n\right)-\left(f^{-1}\right)'(b)\right] = |p|\left[f'\left(x_n\right)^{-1}-f'(a)^{-1}\right] \to 0 .$$

8. δ-extensions of order p_0 and maps of Gårding type

Let Γ be a calibration for a locally convex space E. A δ-extension (Γ_δ, δ) of Γ is said to be *of order* p_0 if it is diagonalizable, has the smallest element p_0 and satisfies the following conditions:

1. Γ_δ contains all finite linear combinations of elements of Γ with non-negative coefficients, and each δ_λ is linear and monotone;
2. for each $(\lambda, p) \in \Lambda \times \Gamma$ there exists a non-negative number λ_p such that $\delta_\lambda(p) \le p + \lambda_p p_0$ and $\lambda_{p_0} = 0$.

The Omori δ-extension is of order p_0 if the following condition (#) is satisfied: for each $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $p_{k-1} \le \varepsilon p_k + C(\varepsilon)p_0$ (see [12], p. 176).

(8.1). *A δ-extension of order* p_0 *is summable and the summability constant is* 1 .

Proof. Since $\delta_\lambda^n(p) \le p + n\lambda_p p_0$, we have $\lambda(n, p) \le 1 + n\lambda_p$.

Now let (Γ_δ, δ) be a δ-extension of Γ of order p_0. Assume that $u \in L_\delta(E)$ and $|u| < \alpha$. Then there exists $\lambda \in \Lambda$ such that, for every $p \in \Gamma$,

$$p[u(x)] < \alpha\left(p(x)+\lambda_p p_0(x)\right) \quad \text{for all } x \in E .$$

We set

$$\mu_p(\alpha, u) = \sup\left\{\left(\alpha p_0(x)\right)^{-1}\left(p\left(u(x)\right)-\alpha p(x)\right) : p_0(x) \ne 0\right\} .$$

Then $\mu_p(\alpha, u) < +\infty$ and we always have

$$p\left(u(x)\right) \le \alpha\left(p(x)+\mu_p(\alpha, u)p_0(x)\right) \quad \text{for all } p \in \Gamma \text{ and } x \in E ,$$

whenever $|u| < \alpha$. Note that $\mu_p(\alpha, u)$ is defined only when $|u| < \alpha$.

(8.2). *If* $|u| < \alpha$ *and* $|v| < \beta$ *for* $u, v \in L_\delta(E)$ *, then for every* $\lambda \in \Lambda$ *and* $p \in \Gamma$ *,*

$$\mu_p(\alpha\beta, v \circ u) \le \mu_p(\alpha, u) + \mu_p(\beta, v) .$$

Proof.

$$\begin{aligned} p\big(v \circ u(x)\big) &\le \beta\big(p\big(u(x)\big)+\mu_p(\beta, v)p_0\big(u(x)\big)\big) \\ &\le \beta\big(\alpha p(x)+\alpha\mu_p(\alpha, u)p_0(x)+\mu_p(\beta, v)\alpha p_0(x)\big) \\ &\le \alpha\beta\big(p(x)+\big(\mu_p(\alpha, u)+\mu_p(\beta, v)\big)p_0(x)\big) . \end{aligned}$$

The following theorem implies that there is a Γ-open neighbourhood of the identity in $L_\delta(E)$ which consists of linear maps of Gårding type.

(8.3). *Let* E *be sequentially complete and* (Γ_δ, δ) *be of order* p_0 . *If* $|1-u| < \alpha < 1$ *for* $u \in L_\delta(E)$ *, then this* u *is of Gårding type, invertible and*

$$\mu_p\big(\alpha(1-\alpha)^{-1}, u^{-1}-1\big) \le (1-\alpha)^{-1}\mu_p(\alpha, 1-u) .$$

Proof. Since, for each $p \in \Gamma$,

$$p\big(x-u(x)\big) < \alpha\big(p(x)+\mu_p(\alpha, 1-u)p_0(x)\big) ,$$

we have

$$p\big(u(x)\big) > (1-\alpha)p(x) - \alpha\mu_p(\alpha, 1-u)p_0(x) ,$$

which shows that u is of Gårding type. Furthermore,

$$p\big((1-u)^n(x)\big) < \alpha^n p(x) + n\alpha^n\mu_p(\alpha, 1-u)p_0(x) .$$

Hence

$$\sum_{n=1}^{\infty} p\big((1-u)^n(x)\big) \le \alpha(1-\alpha)^{-1}p(x) + \left(\sum_{n=1}^{\infty} n\alpha^n\right)\mu_p(\alpha, 1-u)p_0(x) .$$

Therefore

$$p\big(\big(u^{-1}-1\big)(x)\big) \le \alpha(1-\alpha)^{-1}\big(p(x)+(1-\alpha)^{-1}\mu_p(\alpha, 1-u)p_0(x)\big) ,$$

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which implies

$$\mu_p\left(\alpha(1-\alpha)^{-1},\ u^{-1}-1\right) \le (1-\alpha)^{-1}\mu_p(\alpha,\ 1-u)\ .$$

Let U be a p_0-open subset of E. Then a map $f : U \to E$ is said to be *p_0-bounding on* U if, for any sequence $(x_i) \subset U$ and $p \in \Gamma$ such that $(f(x_i))$ is p-bounded (that is, $\sup\{p(f(x_i)) : i \ge 1\} < +\infty$), (x_i) is also p-bounded. It is straightforward to see that a linear map is of Gårding type if it is p_0-bounding on E.

Let again U be a p_0-open subset of E. Then a C^1_δ-map $f : U \to E$ is said to be *of order* p_0 *at* $a \in U$ if, for any $\varepsilon > 0$, there exists $\alpha > 0$ such that the following condition is satisfied: for any $\beta > 0$ and $p \in \Gamma$,

$$\sup\{\mu_p(\varepsilon,\ f'(a+x)-f'(a)) : p_0(x) < \alpha,\ p(x) < \beta\} < +\infty\ .$$

Now we are ready to state an inverse mapping theorem as a consequence of (4.4), (5.3) and (5.4). Under our assumptions on Γ, the inverse mapping theorem for $C^2_{B\Gamma}$-maps proved in [*10*], (III.5.2), corresponds to the case when $(\Gamma_\delta,\ \delta)$ is trivial in the following theorem. It can be easily seen that our assumptions are weaker than those of Omori's inverse mapping theorem stated in the beginning of this note when the condition (#) holds. Therefore the following theorem can be applied to various problems to which Omori's theorem has been applicable.

(8.4). *Let* Γ *be a pairwise coordinated calibration for a complete locally convex space* E *and let* $(\Gamma_\delta,\ \delta)$ *be a* δ*-extension of order* p_0. *Let* U *be a* p_0*-open neighbourhood of zero in* E *and* $f : U \to E$ *be a* C^1_δ*-map such that* $f(0) = 0$ *and* $f'(0) = 1$. *If* f *is* p_0*-bounding on* U *and of order* p_0 *at zero, then there are completionally* Γ*-open neighbourhoods* X *and* Y *of zero such that* $f : X \to Y$ *is a* C^1_δ*-diffeomorphism. Furthermore* $f^{-1} : Y \to X$ *is also* p_0*-bounding on* Y *and of order* p_0 *at zero.*

Proof. By (4.4), (5.3) and (5.4) we need to find completionally Γ-open neighbourhoods X and Y of zero such that

1. $f_{p_0} : X_{p_0} \to Y_{p_0}$ is a C^1-diffeomorphism;

2. $f : X \to Y$ is of Gårding type; and

3. $\overline{f'(X \cap B_p(\beta))_p} \subset \mathrm{GL}(E[p])$ for all $p \in \Gamma$ and $\beta > 0$.

Then f will be a CC^0_Γ-homeomorphism and hence, by (7.2), a C^1_δ-diffeomorphism of X onto Y .

Since f is a C^1_δ-map and U is p_0-open, for any $\varepsilon > 0$ such that $\varepsilon > 1$, there exists $\alpha > 0$ such that, for $V = \{x \in E : p_0(x) < 2\alpha\}$, we have $V \subset U$ and

$$|p_0|\big(1-f'(x)\big) < \varepsilon \quad \text{if} \quad x \in V .$$

Let

$$V_0 = \{\underline{a} \in E[p_0] : p_0(\underline{a}) < \alpha\} ;$$

then, since f is of class CC^1_Γ , there is a C^1-map

$$f_{p_0} : V_0 \to E[p_0]$$

such that $S_{p_0} \circ f = f_{p_0} \circ S_{p_0}$. Hence $f_{p_0}\big(S_{p_0}(0)\big) = S_{p_0}(0)$ and

$$\big(f_{p_0}\big)'\big(S_{p_0}(0)\big) = f'(0)_{p_0} = \text{the identity map on } E[p_0] .$$

Therefore, for

$$X_0 = V_0 \cap \big(f_{p_0}\big)^{-1}\big(Y_0\big) \quad \text{and} \quad Y_0 = \{\underline{a} \in E[p_0] : p_0(\underline{a}) < \alpha/2\} ,$$

the map $f_{p_0} : X_0 \to Y_0$ is a C^1-diffeomorphism, and, by (2.2), $X = S^{-1}_{p_0}\big(X_0\big)$ and $Y = S^{-1}_{p_0}\big(Y_0\big)$ are completionally Γ-open neighbourhoods of zero. We may assume that X is convex.

Next, to show that $f : X \to E$ is of Gårding type, we note that $X \subset U$ and, moreover,

$$p_0(x) < \alpha \quad \text{if} \quad x \in X$$

and

$$p_0(y) < \alpha/2 \quad \text{if} \quad y \in Y \; .$$

For each $p \in \Gamma$, let

$$\mu_p(x) = \mu_p\big(\varepsilon,\ 1-f'(x)\big)$$

and

$$\mu_p(\alpha,\ \beta) = \sup\{\mu_p(x) \; : \; p_0(x) < \alpha,\ p(x) < \beta\} \; .$$

Now let $(x_i) \subset X$ be p_0-Cauchy and $\big(f(x_i)\big)$ be p-Cauchy. Since f is p_0-bounding, (x_i) is p-bounded. Let $p(x_i) \le \beta$. Then, by the mean-value theorem,

$$\begin{aligned} p\big(x_i-x_j-\big(f(x_i)-f(x_j)\big)\big) &\le p\big((1-f)'\big(x_j+\theta(x_i-x_j)\big)(x_i-x_j)\big) \\ &< \varepsilon\big(p(x_i-x_j)+\mu_p(\alpha,\ \beta)p_0(x_i-x_j)\big) \; , \end{aligned}$$

which implies that (x_i) is p-Cauchy, and, hence, f is of Gårding type.

To prove that

$$\overline{f'\big(X \cap B_p(\beta)\big)_p} \subset \mathrm{GL}(E[p]) \quad \text{for all} \quad \beta > 0 \; ,$$

let $u \in \overline{f'\big(X \cap B_p(\beta)\big)_p}$ and choose $x_i \in X \cap B_p(\beta)$ such that $f'(x_i)_p \to u$ in $L(E[p])$. From

$$p\big(\big(1-f'(x_i)\big)(z)\big) < \varepsilon\big(p(z)+\mu_p(\alpha,\ \beta)p_0(z)\big) \; ,$$

we have

$$p\Big(\big(1-f'(x_i)\big)^n(z)\Big) < \varepsilon^n p(z) + n\varepsilon^n\mu_p(\alpha,\ \beta)p_0(z)$$

for all $z \in E$ and $n \ge 1$. Hence

$$p\big((1-u)^n(\underline{\underline{a}})\big) \le \varepsilon^n p(\underline{\underline{a}}) + n\varepsilon^n\mu_p(\alpha,\ \beta)p_0\big(T_p\underline{\underline{a}}\big)$$

for every $\underline{\underline{a}} \in E[p]$, which shows that $u \in \mathrm{GL}(E[p])$.

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Thus we have shown that $f : X \to Y$ is a C^1_δ-diffeomorphism. To show that $f^{-1} : Y \to X$ is p_0-bounding, let $(y_i) \subset Y$ be such that $\left(f^{-1}(y_i)\right)$ is p-bounded and choose $x_i \in X$ such that $y_i = f(x_i)$. Then $p_0(x_i) < \alpha$ and $p(x_i) < \beta$ for some β. Hence

$$\begin{aligned} p\left(f(x_i)-x_i\right) &\le p\left((1-f)'(\theta x_i)(x_i)\right) \\ &< \varepsilon\left(p(x_i)+\mu_p(\alpha, \beta)p_0(x_i)\right) , \end{aligned}$$

which implies

$$\begin{aligned} p(y_i) &< (1+\varepsilon)p(x_i) + \varepsilon\mu_p(\alpha, \beta)p_0(x_i) \\ &< (1+\varepsilon)\beta + \varepsilon\mu_p(\alpha, \beta)\alpha . \end{aligned}$$

Hence (y_i) is also p-bounded.

Finally, the fact that f^{-1} is of order p_0 at zero follows from (8.3), that is, for $y = f(x)$,

$$\begin{aligned} \mu_p\left(\varepsilon(1-\varepsilon)^{-1}, 1-\left(f^{-1}\right)'(y)\right) &= \mu_p\left(\varepsilon(1-\varepsilon)^{-1}, 1-f'(x)^{-1}\right) \\ &\le (1-\varepsilon)^{-1}\mu_p\left(\varepsilon, 1-f'(x)\right) . \end{aligned}$$

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