A PROOF THAT SOUSLIN SOUSLIN H $\subset$ SOUSLIN H

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Introduction. We write ω for the set of natural numbers (including zero) and $\mathcal{F}$ for the set of all finite sequences of natural numbers. If $n \in \omega$ we write $\overline{n} = \{m: m \in \omega, 0 \leq m \leq n\}$. If x is a function which takes its values in ω and whose domain of definition contains $\overline{n}$ then we write $x|\overline{n}$ for the element $(x(0), \ldots, x(n))$ of $\mathcal{F}$.

If H is a family of sets and $f \in H^{\mathcal{F}}$ (i.e., f is a function from $\mathcal{F}$ to H) we write

$$A_f = \bigcup_{x \in \omega^\omega} \bigcap_{n \in \omega} f(x|\overline{n}) .$$

We define

$$\text{Souslin } H = \{A_f : f \in H^{\mathcal{F}}\} .$$

It is known [(1), Theorem 4.3 and (2)] that Souslin Souslin H = Souslin H. The nontrivial part is to show the inclusion $\subset$. In view of the set-theoretic intuition needed to follow both of the above proofs, we have felt it worthwhile presenting this proof, which requires only the ability to manipulate quantifiers and no intuition at all. We give three easy combinatorial lemmas, after which the Theorem follows immediately. Our proof is also more direct, in the sense that we derive the statement $A = \cup \cap h(\cdot)$ without going through the intermediate stages of $A \subset \cup \cap h(\cdot)$ and $A \supset \cup \cap h(\cdot)$. If α and β are functions and the composition $\alpha(\beta(\cdot))$ is defined, we shall abbreviate this to $\alpha\beta$.

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1. <u>The function $\nu(\cdot,\cdot)$.</u> For our proof we shall need a fixed function $\nu:\omega\times\omega \to \omega$ which defines a (1, 1) correspondence between $\omega\times\omega$ and ω. We define $\varphi:\omega \to \omega$ and $\psi:\omega \to \omega$ by the relationship $(\varphi(\cdot), \psi(\cdot)) = \nu^{-1}(\cdot)$. We require also that

(1) $\quad \psi(p) \leq p \qquad$ for all $p \in \omega$

and

(2) $\quad \nu(q, \psi(p)) \leq p \qquad$ for all $p \in \omega$ and $q \in \omega$ such that $0 \leq q \leq \varphi(p)$

One such function is given by Sierpinski, (2), and can be written (after minor modifications) $\nu(m,n) = 2^m(2n+1) - 1$. In fact we can take for ν any (1, 1) map of $\omega\times\omega$ onto ω with the properties

$$\nu(\cdot, m):\omega \to \omega \text{ is increasing for each } m \in \omega$$

and

$$\nu(0, \cdot):\omega \to \omega \text{ is increasing.}$$

In this case $p = \nu(\varphi(p), \psi(p)) > \nu(\varphi(p)-1, \psi(p)) > \ldots >$ $\nu(1, \psi(p)) > \nu(0, \psi(p)) > \nu(0, \psi(p)-1) > \ldots > \nu(0,1) > \nu(0,0)$. Thus (2) is true; we observe also that the above series contains $\varphi(p) + \psi(p) + 1$ terms and so $\varphi(p) + \psi(p) \leq p$, which certainly implies (1). So we could also have taken for ν, e.g., the standard triangularwise enumeration of $\omega \times \omega$.

2. <u>Three combinatorial lemmas.</u>

LEMMA 1. <u>If</u> X <u>is a function on</u> $\omega \times \omega^\omega$ <u>to sets then</u>

$$\bigcap_{m \in \omega} \bigcup_{y \in \omega^\omega} X(m, y) = \bigcup_{y \in \omega^{\omega\times\omega}} \bigcap_{m \in \omega} X(m, y(\cdot, m)) .$$

<u>Proof.</u> Immediate. See Lemma 4.8 of (1).

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LEMMA 2. If X is a function on $\omega\times\omega$ to sets then

$$\bigcap_{m\in\omega}\ \bigcap_{n\in\omega} X(m,n) = \bigcap_{p\in\omega} X(\psi(p),\varphi(p)) .$$

Proof. Both $m=\psi(p)$ and $n=\varphi(p)$ when $p=\nu(n,m)$.

LEMMA 3. If X is a function on $\omega^{\omega}\times\omega^{\omega\times\omega}$ to sets then

$$\bigcup_{x\in\omega^{\omega}}\ \bigcup_{y\in\omega^{\omega\times\omega}} X(x,y) = \bigcup_{z\in\omega^{\omega}} X(\varphi z,\psi z\nu) .$$

Proof. Both $x=\varphi z$ and $y=\psi z\nu$ when $z(\cdot)=\nu(x(\cdot),y\nu^{-1}(\cdot))$.

3. Proof of the theorem. If $A\in$ Souslin Souslin H then there exists $f:\mathcal{F}\to$ Souslin H such that

$$A = \bigcup_{x\in\omega^{\omega}}\ \bigcap_{m\in\omega} f(x|\overline{m})$$

and, for each $t\in\omega^{\overline{m}}$, there exists $g(t;\cdot):\mathcal{F}\to H$ such that

$$f(t) = \bigcup_{y\in\omega^{\omega}}\ \bigcap_{n\in\omega} g(t;y|\overline{n}) .$$

Hence

$$A = \bigcup_{x\in\omega^{\omega}}\ \bigcap_{m\in\omega}\ \bigcup_{y\in\omega^{\omega}}\ \bigcap_{n\in\omega} g(x|\overline{m};y|\overline{n}) .$$

We now apply Lemmas 1, 2 and 3 in turn and derive that

$$A = \bigcup_{x\in\omega^{\omega}}\ \bigcup_{y\in\omega^{\omega\times\omega}}\ \bigcap_{m\in\omega}\ \bigcap_{n\in\omega} g(x|\overline{m};y(\cdot,m)|\overline{n})$$

$$= \bigcup_{x\in\omega^{\omega}}\ \bigcup_{y\in\omega^{\omega\times\omega}}\ \bigcap_{p\in\omega} g(x|\overline{\psi(p)};y(\cdot,\psi(p))|\overline{\varphi(p)})$$

$$= \bigcup_{z\in\omega^{\omega}}\ \bigcap_{p\in\omega} g(\varphi z|\overline{\psi(p)};\psi z\nu(\cdot,\psi(p))|\overline{\varphi(p)}) .$$

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For $t \in \omega^{\bar{p}}$ we write $h(t) = g(\varphi t|\overline{\psi(p)}; \psi t \nu(\cdot, \psi(p))|\overline{\varphi(p)}) \in H$, and we are permitted to do this since the numbers that appear as arguments of $t(\cdot)$ in the above expression are $0, 1, \ldots, \psi(p)$, $\nu(0, \psi(p)), \ldots, \nu(\varphi(p), \psi(p))$ and all of these numbers are $\leq p$ by our choice of $\nu(\cdot, \cdot)$. Thus

$$A = \bigcup_{z \in \omega^\omega} \bigcap_{p \in \omega} h(z|\bar{p}) \in \text{Souslin } H .$$

This completes the proof of the Theorem.

Remark. We have deliberately used the symbols ν, φ and ψ to enable our proof to be compared easily with that in (2).

What appear as k_n, c_n, $N_j(n)$, x' and x'' in (1), Theorem 4.3, correspond to $\varphi(n)$, $\psi(n)$, $\nu(j, n)$, φx and ψx, respectively, in our proof.

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