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On the Number of Distinct Terms in the Expansion of Symmetric and Skew Determinants

By A. C. Aitken.

1. The expansion of the symmetric determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc - af^2 - bg^2 - ch^2 + 2fgh$$

is familiar. It is of six terms; but one term, fgh, is duplicated, and so there are five distinct terms. If we ascertain the number u_n of distinct terms in the expansions of symmetric determinants of increasing order, we find the following sequence of values, for $n = 0, 1, 2, \ldots$,

$$1, 1, 2, 5, 17, 73, 388, 2461, 18155, \ldots,$$

where by convention $u_0 = 1$.

To Cayley (1874, *Coll. Papers* IX., pp. 185-190) is due the recurrence relation

$$u_n = nu_{n-1} - \tfrac{1}{2}(n-1)(n-2)u_{n-3} \tag{1}$$

and the theorem that u_n is the coefficient of $t^n/n!$ in the expansion of the generating function

$$g(t) = e^{\frac{1}{2}t + \frac{1}{4}t^2}(1-t)^{-\frac{1}{2}}, \tag{2}$$

the latter satisfying the differential equation

$$2(1-t)\frac{dg}{dt} = (2-t^2)g. \tag{3}$$

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Cayley's proof of (1) was indirect, based on the prior ascertainment of (3). Sylvester (*Amer. Journ. Math.* **2**, 1879, pp. 89-96, 214-222) and others have direct proofs. An account of these matters is given in Vol. III. of Muir's *History of the Theory of Determinants,* pp. 111, 112, 120, 122, 281, 282. In this present note we give simple proofs of three cognate recurrence relations of the above kind.

2. Any term in the expansion of a general determinant $|a_{11}a_{22}\,..\,a_{nn}|$ or $|A|$ may be specified by the permutation of column suffixes exhibited by its elements, when the factors are arranged so that the row suffixes are in natural order. Again, each such permutation is completely specified by the cyclic permutations or cycles contained in it. Cycles of one index only, or of two, are self-conjugate and on transposition, that is, interchange of row and column suffixes, refer to the same elements as before. For example the cycle (45) refers to $a_{45}a_{54}$, which on transposition becomes $a_{54}a_{45}$. On the other hand cycles of 3 or more indices give on transposition a different set of elements. For example $a_{12}a_{23}a_{31}$ becomes on transposition $a_{21}a_{32}a_{13}$. In fact transposition reverses the order of indices in a cycle, so that cycle (ijk) yields cycle (kji). In symmetric determinants, however, since $a_{ji}=a_{ij}$, a transposed cycle of 3 or more indices refers to a product of the same elements as form the original product, and so such products occur *twice* in association with any other fixed factors in a term of $|A|$. The consequence is immediate:

In the expansion of a symmetric determinant a term corresponding to a permutation containing s cycles of 3 or more indices receives duplication for each such cycle, and so appears with coefficient 2^s.

3. We may now proceed, by consideration of cycles, to the enumeration of distinct terms.

Let a symmetric determinant of order $n-1$ be considered. Its terms correspond to permutations of the first $n-1$ natural numbers, and each such permutation may be supposed to be written out in the form of its constituent cycles (ab), (cde) and the like. Let the additional index n be introduced into a permutation by the following unambiguous and reversible rule. Place n before or after any of the other $n-1$ indices, putting it always

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in the same cycle as the index which it *precedes*; if it is placed last it forms a cycle (n) of one index. In this way each former permutation yields n distinct new ones, corresponding to terms in the enlarged determinant. Thus u_n would be equal to nu_{n-1} but for one consideration. Certain cycles of two indices, by inclusion of n, have become cycles of 3 indices, so that terms with coefficient 2^s have given rise to terms with coefficient 2^{s+1}. We have to allow for this duplication by subtracting half the number of such terms. They evidently correspond to cycles of type (nab), where a and b are chosen in order from any of the remaining $n-1$ indices. There are thus $(n-1)(n-2)$ such cycles, and the remaining $n-3$ indices give rise in each case to u_{n-3} distinct terms. Hence we have Cayley's recurrence relation

$$u_n = nu_{n-1} - \tfrac{1}{2}(n-1)(n-2)\,u_{n-3}.$$

It is interesting to see in a table the enumeration of the cycle-types and the corresponding distinct terms. For example, the table for $n = 5$ is as follows:—

Cycle-type	—	1^5	$1^3 2$	$1^2 3$	14	12^2	23	5	Total
No. of Cycles	—	1	10	20	30	15	20	24	120
Distinct terms	—	1	10	10	15	15	10	12	73

4. By a similar method we may establish the recurrence relation asserted by Sylvester (*loc. cit.* p. 93, Muir, *loc. cit.* p. 122) in regard to a symmetric determinant with principal elements a_{ii} all equal to zero. The vanishing elements are those that correspond to cycles of one index, and so no such cycles are admitted in our enumeration. We introduce the additional index n just as in §3, but we cannot put it last as a cycle of one index, and so we treat it in $n-1$ ways instead of the former n. Further, the cycles in which n appears under this process are of order 3 or more, and so we have still to add the cases in which n is associated with any of the remaining $n-1$ indices in cycles of type (na); and in each such case the rest of the indices form u_{n-2} distinct terms. The compensation for duplicated terms is exactly as before, and so we have

$$u_n = (n-1)(u_{n-1} + u_{n-2}) - \tfrac{1}{2}(n-1)(n-2)\,u_{n-3},$$

the first several values of u_n being 1, 0, 1, 1, 6, 22, 130, 822, 6202, ...

5. We can also enumerate the distinct terms in the expansion of a strictly skew determinant with principal elements zero. This result was also asserted by Sylvester (*loc. cit.*) in the form

$$u_n = (n-1)^2 u_{n-2} - \tfrac{1}{2}(n-1)(n-2)(n-3)\, u_{n-4}.$$

In a skew determinant, since $a_{ji} = -a_{ij}$, any cycle of an *odd* number of indices connotes a product of elements yielding on transposition a product equal in value but opposite in sign. Thus terms corresponding to permutations containing any cycles of odd order annihilate one another, and so only cycles of even order are admissible. Of these the only self-conjugate cycles are those of two indices.

There are various rules by which we may set up an unambiguous and reversible correspondence between distinct terms of $n-2$ elements and derived distinct terms of n elements, where n is even, equal to $2m$. We choose the following. Write out as before the cycles, corresponding to each distinct term of $n-2$ elements. Introduce the two new indices, $n-1 = p$, and n. Place n anywhere among the $n-2$ indices, in the manner of §3. This can be done in $n-1$ ways. Then introduce p anywhere, except that, if p falls in the same cycle as n, do not let it occupy first place in that cycle. (This is to prevent unnecessary duplication, since, for example, cycles $(panb)$ and $(anbp)$ are equivalent.) This can also be done in $n-1$ ways. Now take the index following n in cyclic order in its cycle and place it after p. We thus have a derived pattern of even cycles only, all patterns so derived are distinct, and the correspondence is reversible. Thus $u_n = (n-1)^2 u_{n-2}$, subject to the allowance for duplicated terms. Cycles that were previously of two indices give rise, under the rule, to cycles of 4 indices of the two types (i) $(nabc)$, of which there are $(n-2)(n-3)(n-4)$, and (ii) $(napb)$, of which, since we choose a and b in order and then may insert p in 3 ways, there are $3(n-2)(n-3)$. There are thus $(n-1)(n-2)(n-3)$ new cycles on 4 indices, and in each case the remaining $n-4$ indices correspond to u_{n-4} distinct terms. Hence we have the recurrence relation

$$u_n = (n-1)^2 u_{n-2} - \tfrac{1}{2}(n-1)(n-2)(n-3)\, u_{n-4},$$

the first several values of u_n being 1, 1, 6, 120, 5250, 395010, ...

Sylvester suggests that consideration of cycles may give a

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simple proof of the theorem that a skew symmetric determinant of even order is a perfect square. This can be done by examination of the way in which cycles of two indices may be agglomerated in cycles of more indices, but it would hardly seem to be so simple as Sylvester believed. One can, however, easily enumerate the terms in the square root, the Pfaffian. For the squared terms in the skew determinant correspond exclusively to permutations containing cycles of two indices only, since (ij) connotes $-a_{ij}\,a_{ji}$, or a_{ij}^2. Thus we have to find in how many ways $2m$ indices may be put into m such cycles. For first cycle take 1 and any a from the remaining $2m-1$ indices; for second cycle take the next surviving index in natural order and any b from the remaining $2m-3$; and so proceed. The number of terms in the Pfaffian is thus $(2m-1)(2m-3)(2m-5)\ \ldots\ 5.3.1$, a factorial composed of odd numbers. This is a well-known result.

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On the Newton-Raphson method of Approximation

By H. W. RICHMOND, F.R.S.

1. The Method.—An equation $F(x) = O$ has a root $x = r$, not known exactly. From a first approximation to r, $x = a$, a second approximation, $x = b$, is obtained from the formula

$$b = a - F(a)/F'(a) \tag{i}$$

From b a third approximation, $x = c$, is obtained by the same formula, and so on. The method is pointless unless the successive

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Recently I have had occasion to read the accounts of this method given in various books; among them

(a) Whittaker & Robinson, Calculus of Observations.

(b) Weber, Algebra.

(c) Todhunter, Theory of Equations.

The last, a forgotten text-book of 1880, contains the fullest account of the method known to me. I venture to offer some comments and criticisms.