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RADII AND THE SAUSAGE CONJECTURE

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ABSTRACT. In 1975, L. Fejes Tóth conjectured that in E^d, $d \geq 5$, the sausage arrangement is denser than any other packing of n unit balls. This has been known if the convex hull C_n of the centers has low dimension. In this paper, we settle the case when the inner m-radius of C_n is at least $O(\ln d/m)$. In addition, we consider the extremal properties of finite ballpackings with respect to various intrinsic volumes.

1. **Introduction.** In 1975, L. Fejes Tóth conjectured (*cf.* [9]) that the densest packing of $n \geq 2$ unit balls in $\mathbf{E}^d$, $d \geq 5$, is the sausage arrangement; namely the centers are collinear. The conjecture is still open in any dimensions, $d \geq 5$, but numerous partial results have been obtained.

Let B^d the unit ball in $\mathbf{E}^d$ with volume κ_d. Its boundary is the unit sphere S^{d-1}. The symbol C_n always denotes the convex hull of the centers of n unit balls which form a packing. Let $\mathcal{F}_n^d$ be the family of these C_n in $\mathbf{E}^d$. Observe that the segment S_n of length $2(n-1)$ is in $\mathcal{F}_n^d$. The above mentioned Sausage Conjecture states that for $d \geq 5$ and $C_n \in \mathcal{F}_n^d$,

$$V(S_n + B^d) \leq V(C_n + B^d) \tag{1}$$

with equality if and only if $C_n = S_n$.

Denote by $\mathcal{K}^d$ the family of convex, compact sets. Let $K \in \mathcal{K}^d$ and $1 \leq m \leq d$. The inner m-radius $r_m(K)$ of K is the radius of the largest m-dimensional ball contained in K. The outer m-radius $R_m(K)$ of K is the minimal circumradius of a $(d-m+1)$-dimensional projection of K. Note that $r(K) = r_d(K)$ is the inradius, $r_1(K)$ is half of the diameter, $R(K) = R_1(K)$ is the circumradius and $R_d(K)$ is half of the width of K. In Section 2, we review the basic properties of the inner and outer radii.

Define the function

$$\psi(d) = \begin{cases} \min\{d, 10\} & \text{if } 3 \leq d \leq 18 \\ [\frac{7}{12}(d-1)] + 1 & \text{if } d \geq 19. \end{cases}$$

(1) has been verified if either C_n is not too far from being a segment (*cf.* [3], [4] and [16]); namely,

$$\dim C_n < \psi(d) \quad \text{and} \quad R_2(C_n) \leq \sqrt{\frac{d-1}{d-1+2\pi}}$$

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or if C_n is almost a ball (*cf.* [8]); namely,

$$\frac{R(C_n)}{\sqrt{2}} + \sqrt{2} - 1 \leq r(C_n)$$

for large d.

Let $d \geq 5$ and define $\varphi(d) = \psi(d)$ if $d \neq 18$, and $\varphi(18) = 11$. Note that $r_{\varphi(d)}(C_n) = 0$ implies the Sausage Inequality. We verify the Sausage Conjecture if C_n is not too close to a segment; namely, if

$$r_m(C_n) \geq \alpha(d, m)$$

where $\alpha(d, m)$ is finite and decreasing in m for $m \geq \min\{\varphi(d), 5 \ln d\}$. We prove this statement in Section 3 (see Theorem 3.1 and its corollary). Actually, for $d = 18$, $r_{10}(C_n) = 0$ already yields (1) but our method is not able to handle the case $(d, m) = (18, 10)$.

The intrinsic volumes $V_i(K)$ of $K \in \mathcal{K}^d$, $i = 0, \ldots, d$, are defined via the formula

$$V(K + \lambda B^d) = \sum_{i=0}^{d} \lambda^{d-i} \kappa_{d-i} V_i(K)$$

of Steiner where $\lambda \geq 0$. Note that $V_d(K) = V(K)$ and $2V_{d-1}(K)$ is the surface area of K. In Section 4, we investigate the problem whether some 'sausage properties' hold for specific intrinsic volumes; that is, what conditions on $\dim C_n$ ensure that $V_i(S_n + B^d) \leq V_i(C_n + B^d)$ for $C_n \in \mathcal{F}_n^d$.

2. **The outer and inner radii.** The inner and outer radii have been long considered by approximation theorists under the names Berstein and Kolgomorov widths, respectively (see [15]). Recently they have attracted more attention from the point of view of convex geometry (see [2] and [11]). The work [13] is a systematic study of various notions of radii. One of the classical results about the inradius is due to Steinhagen (*cf.* [7]).

THEOREM 2.1 (STEINHAGEN). *For $d \geq 2$ and $K \in \mathcal{K}^d$,*

$$R_d(K) \leq \begin{cases} \frac{d+1}{\sqrt{d+2}} r_d(K) & \text{if } d \text{ is even} \\ \sqrt{d}\, r_d(K) & \text{if } d \text{ is odd.} \end{cases}$$

The paper [17] of S. V. Pukhov contains the following fundamental theorem:

THEOREM 2.2 (PUKHOV). *Let $m = 1, \ldots, d$ and $K \in \mathcal{K}^d$. Then*

i) $r_m(K) \leq R_m(K) \leq (m+1) r_m(K)$;

2) $R_m(K) \leq \sqrt{e}\sqrt{\min\{m, d-m+1\}}\, r_m(K)$ *if* $K = -K$.

Actually instead of ii), he proved that if $K = -K$ then $R_m(K) \leq \sqrt{e}\sqrt{m}\, r_m(K)$. This inequality yields ii) since (*cf.* [11]) if $\operatorname{int} K \neq \emptyset$ then

$$R_{d-m+1}(K^0) \cdot r_m(K) = 1 \tag{2}$$

where K^0 is the polar of K.

The given estimates of this theorem are almost best possible. Consider the regular crosspolytope of $O^d = \mathrm{conv}\{\pm e_1, \ldots, \pm e_d\}$, where $e_1, \ldots, e_d$ form an orthonormal basis for $\mathbf{E}^d$. Then it is well-known (*cf.* [17]) that

$$r_m(O^d) = \frac{1}{\sqrt{m}} \quad \text{and} \quad R_m(O^d) = \sqrt{\frac{d-m+1}{d}}. \tag{3}$$

This shows that the inequality ii) of Theorem 2.2 can not be improved up to a constant. Let us remark that the known proofs of (3) use lengthy arguments which yield the radii of some more general crosspolytopes. At the end of Section 1 we give a shorter elementary proof.

To show that also inequality i) is amost sharp, extend our basis into an orthonormal basis $e_1, \ldots, e_{d+1}$ of $\mathbf{E}^{d+1}$. Pukhov deduced for the regular simplex $T^d = \mathrm{conv}\{e_q, \ldots, d_{d+1}\}$ that

$$R_m(T^d) \geq R_{m+1}(O^{d+1}) = \sqrt{\frac{d-m+1}{d+1}}. \tag{4}$$

Now, assume that $d+1 = m \cdot n$ for an integer $n \geq 2$. Let $A = \mathrm{aff}\{x_1, \ldots, x_m\}$, where

$$x_i = \frac{e_{(i-1)n+1} + \cdots + e_{in}}{n}$$

for $i = 1, \ldots, m$. Then A and its orthogonal complement A' in $\mathrm{aff}\{e_1, \ldots, e_{d+1}\}$, respectively, show that we have inequality in (4) for $R_{d-m+2}(T^d)$ and $R_m(T^d)$. In other words, equality holds in (4) if either m or $d-m+2$ divides $d+1$. With respect to the inner radii, K. Ball has established quite recently (*cf.* [2]) that

$$r_m(T^d) = r_m(T^m) = \frac{1}{\sqrt{m(m+1)}}. \tag{5}$$

Combining (4) and (5) shows that the estimate of inequality i) is best possible up to a constant if $m \leq \gamma \cdot d$ for a fixed $\gamma \in (0,1)$.

PROPOSITION 2.3. *For $1 \leq m \leq d$ we have*

$$r_m(O^d) = \frac{1}{\sqrt{m}} \quad \textit{and} \quad R_m(O^d) = \sqrt{\frac{d-m+1}{d}}.$$

PROOF. We assume $d \geq 2$. Let L be an arbitrary $(d-m+1)$-dimensional linear subspace with orthonormal basis $u_1, \ldots, u_{d-m+1}$. For a subset $P \subset \mathbf{E}^d$ let $\rho(P)$ be the orthogonal projection of P onto L.

Obviously, we have $R\big(\rho(O^d)\big) = \max\{|\rho(e_i)| : 1 \leq i \leq d\}$ and $\rho(e_i) = \sum_{j=1}^{d-m+1} \langle e_i, u_j \rangle u_j$. Thus

$$\sum_{i=1}^{d} |\rho(e_i)|^2 = \sum_{i=1}^{d} \sum_{j=1}^{d-m+1} \langle e_i, u_j \rangle^2 = \sum_{j=1}^{d-m+1} |u_j|^2 = d-m+1, \tag{6}$$

which yields $R\big(\rho(O^d)\big) \geq \sqrt{(d-m+1)/d}$. By the arbitrariness of L we get $R_m(O^d) \geq \sqrt{(d-m+1)/d}$.

Now assume $R_m(O^d) = R\big(\rho(O^d)\big) \doteq |\rho(e_1)| > \sqrt{(d-m+1)/d}$ and $|\rho(e_1)| > |\rho(e_2)|$. For $\alpha \in [0, \pi/2]$ consider the crosspolytopes with vertices $\pm e_1(\alpha)$, $\pm e_2(\alpha)$, $\pm e_3, \ldots, \pm e_d$ with $e_1(\alpha) = e_1 \cos\alpha + e_2 \sin\alpha$ and $e_2(\alpha) = -e_1 \sin\alpha + e_2 \cos\alpha$. Note that $\big|\rho\big(e_1(\alpha)\big)\big|^2 + \big|\rho\big(e_2(\alpha)\big)\big|^2 = |\rho(e_1)|^2 + |\rho(e_2)|^2$ and the function $g(\alpha) = \big|\rho\big(e_1(\alpha)\big)\big| - \big|\rho\big(e_2(\alpha)\big)\big|$ is continuous with $g(0) > 0$ and $g(\pi/2) < 0$. Thus there exists a β such that $\big|\rho\big(e_1(\beta)\big)\big| = \big|\rho\big(e_2(\beta)\big)\big| < |\rho(e_1)|$, and this contradicts the minimal property of the plane L. Thus $R_m(O^d) = \sqrt{(d-m+1)/d}$.

Let $a_1, \ldots, a_{2^d}$ be the vertices of the polar cube W^d of O^d. Again, we have $R\big(\rho(W^d)\big) = \max\{|\rho(a_i)| : 1 \leq i \leq 2^d\}$. Observe, that any vertex a_i is in the form $(\pm e_1, \ldots, e_d)$. With the same method as above we deduce $R\big(\rho(W^d)\big) \geq \sqrt{d-m+1}$ and so $R_m(W^d) \geq \sqrt{d-m+1}$. On the other hand, W^d contains a $(d-m+1)$-dimensional cube with edge length 2 and thus $R_m(W^d) = \sqrt{d-m+1}$. Together with (2), we get $r_m(O^d) = 1/\sqrt{m}$. ∎

3. **The fatness of the sausage.** Recall from the introduction that

$$\phi(d) = \begin{cases} \min\{d, 10\} & \text{if } 5 \leq d \leq 17 \\ 11 & \text{if } d = 18 \\ [\frac{7}{12}(d-1)] + 1 & \text{if } d \geq 19. \end{cases}$$

THEOREM 3.1. *Let $d \geq 5$, $\min\{\varphi(d), 5\ln d\} \leq m \leq d$ and $C_n \in \mathcal{F}_n^d$. Then there exists a function $\alpha(d,m)$ such that $r_m(C_n) \geq \alpha(d,m)$ implies $V(S_n + B^d) < V(C_n + B^d)$. For $d \geq 31$, one may choose*

$$\alpha(d,m) = \frac{3(\sqrt{2}+1)}{\sqrt{2}}\Big(1 + 33\frac{\ln d}{m}\Big)\frac{\ln d}{m}.$$

In order to prove Theorem 3.1, we need some preparation. We frequently use the estimate (*cf.* [4])

$$\sqrt{\frac{d}{2\pi}} < \frac{\kappa_{d-1}}{\kappa_d} < \sqrt{\frac{d+1}{2\pi}}. \tag{7}$$

LEMMA 3.2. *For $C \in \mathcal{K}^d$ and $1 \leq m \leq d$,*

$$V(C + \sqrt{2}B^d) \leq \left(\frac{r_m(C) + \sqrt{2}}{r_m(C) + 1}\right)^m \sqrt{2}^{d-m} V(C + B^d).$$

PROOF. We may assume that $\mathbf{E}^m$ is embedded into $\mathbf{E}^d$ and $rB^m \subset C$ for $r = r_m(C)$. If $d = m$ then

$$C + \sqrt{2}B^d = C + B^d + (\sqrt{2}-1)B^d \subset C + B^d + \frac{\sqrt{2}-1}{r+1}(C + B^d) = \frac{r+\sqrt{2}}{r+1}(C + B^d)$$

yields the lemma, and hence assume that $m \leq d-1$. Denote by L the linear $(d-m)$-space orthogonal to $\mathbf{E}^m$, and for $x \in \mathbf{E}^d$, define $T(x) = \frac{r+\sqrt{2}}{r+1}y + \sqrt{2}z$ where y and z, respectively, are the projections of x onto $\mathbf{E}^m$ and L. We prove that for any $p \in \mathbf{E}^m$,

$$V_{d-m}\big((p+L)\cap(C+\sqrt{2}B^d)\big) \leq V_{d-m}\big((p+L)\cap T(C+B^d)\big). \tag{8}$$

By Fubini's theorem this yields that

$$V(C+\sqrt{2}B^d) \leq V\big(T(C+B^d)\big) = \left(\frac{r_m(C)+\sqrt{2}}{r_m(C)+1}\right)^m \sqrt{2}^{d-m}\, V(C+B^d).$$

So consider a line l passing through $p \in \mathbf{E}^m$ and perpendicular to $\mathbf{E}^m$, and assume that l intersects $C+\sqrt{2}B^d$ in $\operatorname{conv}\{x_1,x_2\}$. There exists $y_i \in C$ such that $x_i \in y_i + B^d$, and denote by z_i the projection of y_i onto l.

Set $\sigma' = \frac{r+1}{r+\sqrt{2}} \cdot \sigma$ for any point or subset σ of $\mathbf{E}^d$ and observe that $T(l') = l$, $z_i' \in l'$ and $y_i' \in C$. Denote by u_i the point of $\operatorname{conv}\{y_i', z_i'\} \cap C$ with minimal distance to l'. The m-dimensional ball $y_i' + \frac{\sqrt{2}-1}{r+\sqrt{2}} rB^m$ is contained in C because rB^m and y_i are in C, and that $d(y_i', z_i') = \frac{r+1}{r+\sqrt{2}} d(y_i, z_i)$. Since

$$\frac{r+1}{r+\sqrt{2}}t - \frac{(\sqrt{2}-1)r}{r+\sqrt{2}} \leq \frac{1}{\sqrt{2}}t$$

for $0 < t \leq \sqrt{2}$, we have $d(u_i, z_i') \leq \frac{1}{\sqrt{2}} d(y_i, z_i) \leq 1$, and hence $u_i + B^d$ intersects l'. Let v_i be the point of $(u_i + bdB^d) \cap l'$ so that $z_1', z_2' \in \operatorname{conv}\{v_1, v_2\}$. As $d(v_i, z_i') \geq \frac{1}{\sqrt{2}} d(x_i, z_i)$ follows by $d(u_i, z_i') \leq \frac{1}{\sqrt{2}} d(y_i, z_i)$, and $d(z_1', z_2') = \frac{r+1}{r+\sqrt{2}} d(z_1, z_2) \geq \frac{1}{\sqrt{2}} d(z_1, z_2)$, we deduce that $d(x_1, x_2) \leq d\big(T(v_1,), T(v_2)\big)$. Finally, it also follows that

$$\max\{d(p,x_1), d(p,x_2)\} \leq \max\big\{d\big(p, T(v_1,)\big), d\big(p, T(v_2)\big)\big\}.$$

Now with the help of spherical coordinates, we may write

$$V_{d-m}\big((p+L)\cap(C+\sqrt{2}B^d)\big) = \frac{1}{2}\int_{S^{d-m-1}} \int_{\alpha_1(\omega)}^{\alpha_2(\omega)} |\rho|^{d-m-1}\, d\rho\, d\omega \quad \text{and}$$
$$V_{d-m}\big((p+L)\cap T(C+B^d)\big) = \frac{1}{2}\int_{S^{d-m-1}} \int_{\beta_1(\omega)}^{\beta_2(\omega)} |\rho|^{d-m-1}\, d\rho\, d\omega$$

for suitable $\alpha_i(\omega)$ and $\beta_i(\omega)$ such that $\alpha_1(\omega) \leq \alpha_2(\omega)$, $\alpha_1(-\omega) = -\alpha_2(\omega)$ and $\alpha_2(-\omega) = -\alpha_1(\omega)$, and similar properties hold for $\beta_i(\omega)$. Let $\omega \in S^{d-m-1}$. The considerations above show that $\alpha_2(\omega) - \alpha_1(\omega) \leq \beta_2(\omega) - \beta_1(\omega)$ and

$$\max\{|\alpha_2(\omega)|, |\alpha_1(\omega)|\} \leq \max\{|\beta_2(\omega)|, |\beta_1(\omega)|\}.$$

We deduce that

$$\int_{\alpha_1(\omega)}^{\alpha_2(\omega)} |\rho|^{d-m-1}\, d\rho \leq \int_{\beta_1(\omega)}^{\beta_2(\omega)} |\rho|^{d-m-1}\, d\rho,$$

which in turn yields (8). ■

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According to Blichfeldt's classical formula (see [12] p. 388 or [5]),

$$V(C_n+\sqrt{2}B^d)\geq \frac{2\kappa_d}{d+2}\sqrt{2}^d\cdot n.$$

Combining this with Lemma 3.2 yields

$$V(C_n+B^d)\geq\left(\frac{\sqrt{2}r_m(C_n)+\sqrt{2}}{r_m(C_n)+\sqrt{2}}\right)^m\frac{2\kappa_d}{d+2}\cdot n. \tag{9}$$

PROOF OF THEOREM 3.1. The sausage S_n+B^d is contained in a cylinder of height $2n$, and hence

$$V(S_n+B^d)<2\kappa_{d-1}\cdot n. \tag{10}$$

Let $d\geq 2$. Note that for any $\varepsilon>0$ there exists a $\rho(\varepsilon)$ such that if $r(C_n)>\rho(\varepsilon)$ then

$$V(C_n+B^d)>\frac{n\kappa_d}{\delta_d}-\varepsilon, \tag{11}$$

where δ_d is the packing density (see [5]).

Now let $d=5,6$, and hence $m=d$. Since $\frac{\kappa_d}{\delta_d}<2\kappa_{d-1}$ according to the table in [6], p. 15, (10) and (11) yield the existence of $\alpha(d,m)$.

So let $d\geq 7$ and $r=r_m(C_n)$. By (9) and (10), the Sausage Inequality holds if

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m\frac{2\kappa_d}{d+2}\cdot n>2\kappa_{d-1}\cdot n,$$

which is equivalent to

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m\geq\frac{\kappa_{d-1}}{\kappa_d}(d+2). \tag{12}$$

Note that the function $\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}$ is monotonically increasing, and (12) has a non-negative solution if and only if $\sqrt{2}^m>\frac{\kappa_{d-1}}{\kappa_d}(d+2)$. This inequality does not hold if $(d,m)=(5,5)$, $(6,6)$ or $(18,10)$.

Let $d=7,\ldots,30$. Then the condition on m becomes $m\geq\varphi(d)$ and (12) has some non-negative solution. Table 1 contains the minimal r satisfying (12) with $m=\varphi(d)$, and the corresponding lower bound for $R_{\varphi(d)}(C_n)$ which we calculated via Theorems 2.1 and 2.2. That minimal r can be chosen as $\alpha\big(d,\varphi(d)\big)=\cdots=\alpha(d,d)$. In the case $d=6$, an improvement on Blichfeldt's method by Rankin (*cf.* [18]) yields some lower bound for $r_6(C_n)$ which is also contained in Table 1.

Let $d\geq 31$, and hence $5\ln d<\varphi(d)$. The inequality (7) yields that $\frac{\kappa_{d-1}}{\kappa_d}(d+2)<d^{3/2}$, and so set r to be the solution of the equation

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m=d^{3/2}. \tag{13}$$

With the help of $\tau = \frac{(\sqrt{2}-1)r}{r+\sqrt{2}}$, (13) can be rewritten in the form

$$1+\tau = e^{\frac{3}{2}\cdot\frac{\ln d}{m}}. \tag{14}$$

Now $\ln d/m \le 1/5$ yields $1+\tau \le e^{3/10} < \sqrt{2}$, and so (13) has a positive solution

$$r = (2+\sqrt{2})\tau \frac{1}{1-(\sqrt{2}+1)\tau}. \tag{15}$$

In order to give an upper bound for r, we used the estimates

$$e^t - 1 \le \left(1 + \frac{e^\nu - \nu - 1}{\nu^2} t\right) t \quad \text{and} \quad \frac{1}{1-t} \le 1 + \frac{1}{1-\nu} t,$$

where $0 < t \le \nu < 1$. Since $\ln d/m \le 1/5$, via (14) and (15) we arrived at

$$r < \frac{3(\sqrt{2}+1)}{\sqrt{2}}\left(1 + 32.5305\frac{\ln d}{m}\right)\frac{\ln d}{m} \tag{16}$$

which completes the proof of the theorem. ∎

By Theorems 2.1, 2.2 and using the estimate (16) we deduce

COROLLARY 3.3. *Let $d \ge 5$, $\min\{\varphi(d), 5\ln d\} \le m \le d$ and $C_n \in \mathcal{F}_n^d$. Then there is a $\beta(d,m)$ such that $R_m(C_n) \ge \beta(d,m)$ yields $V(S_n + B^d) < V(C_n + B^d)$. For $d \ge 31$, we may choose*

i) $\beta(d,m) = \frac{3(\sqrt{2}+1)}{\sqrt{2}}\left(1 + 35\frac{\ln d}{m}\right)\ln d$;

ii) $\beta(d,m) = \frac{3(\sqrt{2}+1)\sqrt{e}}{\sqrt{2}}\left(1 + 33\frac{\ln d}{m}\right)\frac{\sqrt{\min\{d-m+1,m\}}}{m}\ln d$ *if* $C_n = -C_n$;

iii) $\beta(d,m) = \frac{3(\sqrt{2}+1)}{\sqrt{2}}\left(1 + 33\frac{\ln d}{m}\right)\frac{\ln d}{\sqrt{m}}$ *if* $\dim C_n = m \ge \varphi(d)$.

Observe that if m is at least, say $(\ln d)^3$, then the lower bounds in ii) and iii) of Corollary 3.3 approach zero as d tends to infinity.

4. **Some additional sausage properties.** Let $2 \le k \le i \le d$. We say that the *sausage property* $\mathrm{SP}(d,i,k)$ holds if

$$V_i(S_n + B^d) \le V_i(C_n + B^d)$$

for each $C_n \in \mathcal{F}_n^d$ with $\dim C_n \le k$. The Sausage Conjecture states that $\mathrm{SP}(d,d,d)$ holds for $d \ge 5$. This notion was introduced in [16], and here we add some new observations to the ones in [16]. First we give a complete description of the case $k = 2$. After this we prove that $\mathrm{SP}(d,i,i)$ does not hold for $2 \le i \le \frac{1}{2}d$ (see Theorem 4.2), and finally we deal with the case $i = d-1$. Note that

$$\kappa_{d-i} V_i(K + B^d) = \sum_{j=0}^{i} \binom{d-j}{d-i} \kappa_{d-j} V_j(K). \tag{17}$$

THEOREM 4.1. *For $d \geq 3$, $\mathrm{SP}(d,i,2)$ holds if and only if $i > \theta(d)$ where $\theta(d) = \frac{(d-1)\kappa_{d-1}}{\sqrt{3}\kappa_{d-2}} + 1$.*

REMARK. (7) yields that $\theta(d) \sim \sqrt{\frac{2\pi}{3}}\sqrt{d}$ and $\theta(d) > \sqrt{\frac{2\pi}{3}}\sqrt{d}$.

PROOF. Let $C_n \in \mathcal{F}_n^d$ with $\dim C_n \leq 2$. According to the inequality of Oler (*cf.* [14] and [10]),

$$1 + \frac{1}{2}V_1(C_n) + \frac{1}{2\sqrt{3}}V_2(C_n) \geq n. \tag{18}$$

On the other hand, with the help of (17), one can write $V_i(S_n + B^d) \leq V_i(C_n + B^d)$ as

$$1 + \frac{1}{2}V_1(C_n) + \frac{(i-1)\kappa_{d-2}}{2(d-1)\kappa_{d-1}}V_2(C_n) \geq n. \tag{19}$$

Denote $\frac{(i-1)\kappa_{d-2}}{2(d-1)\kappa_{d-1}}$ by $A(d,i) \geq \frac{1}{2\sqrt{3}}$ then (19) holds by (18). So let $A(d,i) < \frac{1}{2\sqrt{3}}$ and $n \geq 3$. Assume that C_n is the trapezoid such that the sides with lengths $2[(n-2)/2]$ and $2[(n-1)/2]$ are parallel with distance $\sqrt{3}$ apart, and the other sides have length 2. Then $C_n \in \mathcal{F}_n^d$ and

$$1 + \frac{1}{2}V_1(C_n) + \frac{1}{2\sqrt{3}}V_2(C_n) = n,$$

which in turn yields that (19) does not hold.

Since $\frac{\kappa_{d-2}}{\kappa_{d-1}}$ is either rational or transcendental, $A(d,i) \neq \frac{1}{2\sqrt{3}}$. Finally, observe that $A(d,i) > \frac{1}{2\sqrt{3}}$ is equivalent to $i > \frac{(d-1)\kappa_{d-1}}{\sqrt{3}\kappa_{d-2}} + 1$. ∎

Let $C_n \in \mathcal{F}_n^3$. According to [3] (see ii) in Section 3), we have $V(S_n+B^4) \leq V(C_n+B^4)$ in $\mathbf{E}^4$. Expanding it with the help of (17) and using similar considerations as in the proof of Theorem 4.1 show that $\mathrm{SP}(d,i,3)$ holds if $d \geq 4$ and $i \geq \frac{3(d-1)\kappa_{d-1}}{4\kappa_{d-2}} + 1 \sim \frac{3}{4}\sqrt{2\pi d}$.

THEOREM 4.2. *$\mathrm{SP}(d,i,i)$ does not hold if $d \geq 4$ and $2 \leq i \leq \frac{1}{2}d$.*

PROOF. Let $2 \leq i < m \leq d$. Note that as n tends to infinity,

$$V_i(S_n + B^d) = \binom{d}{d-i}\frac{\kappa_d}{\kappa_{d-i}} + \binom{d}{d-i}\frac{\kappa_{d-1}}{\kappa_{d-i}}2(n-1) \sim \binom{d-1}{d-i}\frac{2\kappa_{d-1}}{\kappa_{d-i}} \cdot n. \tag{20}$$

By the definition of the packing density, there exists a sequence $\{C_n\}$, $C_n \in \mathcal{F}_n^d$ with $\dim C_n = i$ such that as n tends to infinity, $V_i(C_n + B^d) \sim \frac{n\kappa_i}{\delta_i}$. Combining this with (11) and (20) yields that if

$$\delta_i^{-1} < \binom{d-1}{d-i}\frac{2\kappa_{d-1}}{\kappa_i\kappa_{d-i}} \quad \text{then } \mathrm{SP}(d,i,i) \text{ does not hold.} \tag{21}$$

Set $B(d,i) = \binom{d-1}{d-i}\frac{2\kappa_{d-1}}{\kappa_i\kappa_{d-i}}$. It can be written in the form

$$B(d,i) = \frac{\kappa_{d-1}}{\kappa_d} \cdot \frac{i}{d} \cdot \frac{\Gamma(d)\Gamma(\frac{1}{2}i)\Gamma\big(\frac{1}{2}(d-i)\big)}{\Gamma(d-i)\Gamma(i)\Gamma(\frac{1}{2}d)}.$$

For $t \geq 1$, we have Stirling's formula (see [1])

$$\Gamma(t) = \sqrt{2\pi} \cdot t^{t-\frac{1}{2}} \cdot e^{-t+\frac{\theta}{12t}},$$

where $0 < \theta < 1$. This and (7) yield that

$$B(d,i) > \frac{i}{\sqrt{\pi d}} \frac{d^{d/2}}{(d-i)^{(d-i)/2} i^{i/2}} e^{-\frac{1}{12}\left(\frac{1}{i}+\frac{1}{d-i}+\frac{2}{d}\right)}.$$

Recall that $i \leq \frac{1}{2}d$. By Theorem 4.1, we may assume $i > \sqrt{\frac{2\pi}{3}}\sqrt{d}$, and hence also $d \geq 10$. Observe that

$$\frac{d^{d/2}}{(d-i)^{(d-i)/2} i^{i/2}} = \left(\left(1+\frac{i}{d-i}\right)^{\frac{d-i}{i}} \cdot \frac{d}{i}\right)^{i/2} \geq 2^i.$$

On the other hand, $\delta_i^{-1} < 2^{i-1}$ according to Minkowski's theorem (see [12]). Now $\frac{1}{2}d \geq i > \sqrt{\frac{2\pi}{3}}\sqrt{d}$ and $d \geq 10$ yield $\delta_1^{-1} < B(d,i)$, and hence $\mathrm{SP}(d,i,i)$ does not hold by (21). ■

Finally, we investigate the property $\mathrm{SP}(d, d-1, k)$, $k = 2, \ldots, d-1$.

PROPOSITION 4.3. *$\mathrm{SP}(d, d-1, k)$ holds if either $d \geq 5$ and $k \leq \min\{d-3, 9\}$ or $d \geq 21$ and $k \leq \frac{7}{12}(d-3)$. On the other hand, $\mathrm{SP}(d, d-1, d-1)$ does not hold if $d \leq 10$.*

PROOF. Let $\dim C_n \leq d-2$ and B^{d-2} be a unit $(d-2)$-ball such that $C_n, S_n \subset \operatorname{aff} B^{d-2}$. Then we have $V_{d-1}(S_n + B^d) \leq V_{d-1}(C_n + B^d)$ if and only if $V_{d-2}(S_n + B^{d-2}) < V_{d-2}(C_n + B^{d-2})$ (*cf.* [16]). Hence the first statement follows by [3] and [4] (see i) in Section 3).

Using the table in [6] p. 15, (21) yields that $\mathrm{SP}(d, d-1, d-1)$ does not hold if $d \leq 10$. ■

THEOREM 4.4. *Let $d \geq 14$ and $\min\{d-1, 6\ln d\} \leq m \leq d-1$. If*

$$r_m(C_n) \geq 2(2+\sqrt{2})\left(1 + O\left(\frac{\ln d}{m}\right)\right)\frac{\ln d}{m}$$

for a $C_n \in \mathcal{F}_n^{d-1}$ then $V_{d-1}(S_n + B^d) < V_{d-1}(C_n + B^d)$.

PROOF. Let $d = 14$ and $C_n \in \mathcal{F}_n^{13}$. Note that by (11) and (17), if $r_{13}(C_n)$ is large then $V_{13}(C_n + B^d)$ is close to $n\kappa_{13}/\delta_{13}$, and hence greater than $V_{13}(S_n + B^d)$ by 20. Here we used the table in [6] p. 15 to estimate δ_{13}.

Now assume $d \geq 15$. Let B^{d-1} be a unit $(d-1)$-ball such that $C_n, S_n \subset \operatorname{aff} B^{d-1}$ and $r = r_m(C_n)$. Note that by (9),

$$V_{d-1}(C_n + B^d) > V_{d-1}(C_n + B^{d-1}) \geq \left(\frac{\sqrt{2}r + \sqrt{2}}{r + \sqrt{2}}\right)^m \frac{2\kappa_{d-1}}{d+1} \cdot n.$$

On the other hand, (20) and (7) yield that $V_{d-1}(S_n + B^d) < (d-1)\kappa_{d-1} \cdot n$. Thus it is sufficient to consider the equation

d	$\phi = \phi(d)$	lower bound for	
		r_ϕ	R_ϕ
6	6	1690.3813	4183.4804
7	7	19.6265	51.9268
8	8	9.2130	26.2208
9	9	6.0494	18.1483
10	10	4.5203	14.3541
11	10	5.7206	62.9273
12	10	7.3818	81.2007
13	10	9.8417	108.2589
14	10	13.8717	152.5895
15	10	21.7058	238.7646
16	10	43.6246	479.8715
17	10	462.5658	5088.2239
18	11	15.2755	183.3061
19	11	21.4220	257.0645
20	12	9.2274	119.9569
21	12	10.9718	142.6342
22	13	6.6048	92.4679
23	13	7.3840	103.3760
24	14	5.1437	77.1558
25	15	3.9635	63.4168
26	15	4.2139	67.4236
27	16	3.3995	57.7916
28	16	3.5710	60.7080
29	17	2.9767	53.5810
30	17	3.1003	55.8059

TABLE 1

$$\left(\frac{\sqrt{2}r + \sqrt{2}}{r + \sqrt{2}}\right)^m \frac{2\kappa_{d-1}}{d+1} \cdot n = (d-1)\kappa_{d-1} \cdot n,$$

which is equivalent to

$$\left(\frac{\sqrt{2}r + \sqrt{2}}{r + \sqrt{2}}\right)^m = \frac{d^2 - 1}{2}.$$

Since $\sqrt{2}^m > (d^2 - 1)/2$ because of the conditions $m \geq \min\{d - 1, 6 \ln d\}$ and $d \geq 15$, this equation has a non-negative solution. The asymptotic behavior of the solution can be derived as it was done in Theorem 3.1, which in turn yields the required lower bound for $r_m(C_n)$. ∎

We note that the analogue of Corollary 3.3 also holds. For example, $R_m(C_n) \geq 2(2 + \sqrt{2})\big(1 + O(\frac{\ln d}{m})\big) \ln d$ yields $V_{d-1}(S_n + B^d) < V_{d-1}(C_n + B^d)$. The observations suggest that $\mathrm{SP}(d, d-1, d-1)$ holds if d is large enough.

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