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A CLASSIFICATION SCHEME FOR NONOSCILLATORY SOLUTIONS OF A HIGHER ORDER NEUTRAL NONLINEAR DIFFERENCE EQUATION

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Abstract

Nonoscillatory solutions of a nonlinear neutral type higher order difference equations are classified by means of their asymptotic behaviors. Existence criteria are then provided for justification of such classifications.

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1. Introduction

Classifications schemes for nonoscillatory solutions of nonlinear difference equations are important since further investigations of some of the qualitative behaviors of nonoscillatory solutions can then be reduced to only a number of cases. In this paper, we are concerned with a class of nonlinear neutral difference equations of the form

$$\Delta^m(x_n + px_{n-\tau}) + f(n, x_{n-\delta}) = 0, \quad n = 0, 1, 2, \dots, \tag{1}$$

where m, τ and δ are integers such that $m \geq 2$, $\tau > 0$, $\delta \geq 0$, and p is a nonnegative real number different from 1. The function $f : \{0, 1, \dots\} \times \mathbb{R} \to \mathbb{R}$ is continuous in the second variable and $xf(n, x) > 0$ for $x \neq 0$ and all $n = 0, 1, 2, \dots$. In some cases, we will also assume that f is superlinear or sublinear. More precisely, f is said

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to be *superlinear* if

$$\frac{f(n,x)}{x} \geq \frac{f(n,y)}{y}, \quad \text{for } x \geq y > 0, \text{ or } x \leq y < 0,$$

and *sublinear* if

$$\frac{f(n,x)}{x} \leq \frac{f(n,y)}{y}, \quad \text{for } x \geq y > 0, \text{ or } x \leq y < 0.$$

Note that if f is superlinear, then when $0 < a \leq x \leq b$, we have

$$f(n,a) \leq f(n,x) \leq f(n,b), \quad n = 0, 1, 2, \dots,$$

and if f is sublinear, then when $0 < a \leq x \leq b$, we have

$$\frac{a}{b} f(n,a) \leq f(n,x) \leq \frac{b}{a} f(n,b), \quad n = 0, 1, \dots .$$

Similar statements can be made when $a \leq x \leq b < 0$.

Since (1) can be written as a recurrence relation

$$x_{n+m} = F(x_{n-\delta}, x_{n-\tau}, x_n, x_{n+1}, \dots, x_{n+m-1}),$$

it is clear that given x_i for $-\max\{\tau, \delta\} \leq i \leq 1$, we can successively calculate $x_2, x_3, \dots$ in a unique manner. Such a sequence $\{x_n\}$ will be called a solution of (1). We will be concerned with the nonoscillatory, that is eventually positive or eventually negative solutions of (1). In particular, we will provide a classification scheme for these solutions. This scheme is then justified by existence criteria.

Nonlinear difference equations have been studied by a number of authors. In particular, Grace and Lalli in [4] studied the equation

$$\Delta^2(x_n + p x_{n-\tau}) + q_n f(x_{n-\delta}) = 0$$

and obtained a number of oscillation criteria for its solutions. Li and Cheng in [10] studied the equation

$$\Delta(r_n \Delta(x_n - p_n x_{n-\tau})) + f(n, x_{n-\delta}) = 0$$

and obtained classification schemes for its nonoscillatory solutions. Fourth order equations of the form

$$\Delta^2(r_{n+1}\Delta^2 x_n) + f(n+2, y_{\tau(n)}) = 0$$

have been studied by a number of people (see [12–14]) and several results are related to classifications of nonoscillatory solutions. Other related results can be found in [1–5, 7–16].

The convention that an empty sum is zero will be adopted in the sequel.

2. Preparatory lemmas

We first establish or quote several preparatory results which will be useful later. Let $\{x_n\}$ be a real sequence defined for $n \geq a - \tau$, the sequence $\{z_n\}$ defined by

$$z_n = x_n + px_{n-\tau}, \quad n = a, a+1, \dots,$$

will be called its associated sequence (relative to the constant p and the integer τ). Two relations between a nonoscillatory solution and its associated sequence will be needed.

LEMMA 1. *Assume that $p \geq 0$ and $p \neq 1$. Assume further that the sequence $\{x_n/n^\alpha\}$ is bounded and eventually positive (or eventually negative), where α is a nonnegative integer. If $\{z_n\}$ is the associated sequence of $\{x_n\}$ and*

$$\lim_{n\to\infty} \frac{z_n}{n^\alpha} = b,$$

then

$$\lim_{n\to\infty} \frac{x_n}{n^\alpha} = \frac{b}{1+p}.$$

PROOF. Without loss of generality, we may assume that $x_n > 0$ for $n \geq 0$. Let

$$Q = \limsup_{n\to\infty} \frac{x_n}{n^\alpha} \quad \text{and} \quad q = \liminf_{n\to\infty} \frac{x_n}{n^\alpha}.$$

Then there exist subsequences $\{n(k)\}$ and $\{j(k)\}$ of the nonnegative integers such that

$$\lim_{k\to\infty} \frac{x_{n(k)}}{n^\alpha(k)} = Q \quad \text{and} \quad \lim_{k\to\infty} \frac{x_{j(k)}}{j^\alpha(k)} = q.$$

If $p \in [0, 1)$, then

$$b = \lim_{k\to\infty} \frac{z_{n(k)}}{n^\alpha(k)} = \lim_{k\to\infty} \frac{x_{n(k)} + px_{n(k)-\tau}}{n^\alpha(k)} \geq Q + pq$$

and

$$b = \lim_{k\to\infty} \frac{z_{j(k)}}{j^\alpha(k)} = \lim_{k\to\infty} \frac{x_{j(k)} + px_{j(k)-\tau}}{j^\alpha(k)} \leq q + pQ.$$

Hence $q + pQ \geq Q + pq$, or $q \geq Q$. Since $q \leq Q$ by definition, we see that $q = Q$.

Similarly, if $p > 1$, we have

$$b = \lim_{k\to\infty} \frac{z_{n(k)+\tau}}{(n(k)+\tau)^\alpha} \geq q + pQ$$

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and

$$b = \lim_{k\to\infty} \frac{z_{j(k)+\tau}}{(j(k)+\tau)^\alpha} \le Q + pq.$$

Thus $Q + pq \ge q + pQ$, or $q \ge Q$. Again, we have $q = Q$.

Thus,

$$b = \lim_{n\to\infty} \frac{z_n}{n^\alpha} = \lim_{n\to\infty} \frac{x_n + px_{n-\tau}}{n^\alpha} = q + pq.$$

This shows that

$$\lim_{n\to\infty} \frac{x_n}{n^\alpha} = q = \frac{b}{1+p}.$$

The proof is complete. □

Given a real function $u(t)$ whose derived function $u^{(r)}(t)$ is sign regular, the intermediate derived functions will also satisfy certain sign conditions. Such results are well known in the theory of ordinary differential equations (see Kiguradze and Chanturia [6]) and their discrete analogs have been reported by a number of authors [13, 16]. Several of these sign regularity conditions are listed here without proofs.

LEMMA 2 (Zhou and Yan [16]). *Let $\{x_k\}$ be a real bounded sequence of fixed sign. Suppose $x_k\Delta^t x_k \le 0$ $(x_k\Delta^t x_k \ge 0)$ for some odd integer $t > 1$ and all large k. Then $(-1)^j x_k\Delta^j x_k \ge 0$ (respectively $(-1)^j x_k\Delta^j x_k \le 0$) for $1 \le j \le t$ and all large k.*

LEMMA 3 (Zafer and Dahiya [13]). *Let N be a positive integer. Let $\{y_n\}_{n=0}^\infty$ be a real sequence such that $\{y_n\}$ and $\{\Delta^N y_n\}$ are of constant sign. Suppose further that $\{\Delta^N y_n\}$ does not vanish identically for all large n and that $y_n\Delta^N y_n \le 0$ for $n \ge 0$. Then*

(i) *for each $j \in \{1, 2, \dots, N-1\}$, the sequence $\{\Delta^j y_n\}$ is of constant sign for all large n; and*

(ii) *there is an integer $k \in \{0, 1, 2, \dots, N-1\}$ such that $(-1)^{N-k-1} = 1$ and for each $j \in \{0, 1, \dots, k\}$, $y_n\Delta^j y_n > 0$ for all large n, and for each $j \in \{k+1, \dots, N-1\}$, $(-1)^{j-k} y_n\Delta^j y_n > 0$ for all large n.*

LEMMA 4 (Zafer and Dahiya [13]). *Let N be a positive integer. Let $\{y_n\}_{n=0}^\infty$ be a real sequence such that for each $j \in \{0, 1, \dots, N-1\}$, $\{\Delta^j y_n\}$ is of constant sign for all large n. Suppose further that $y_n\Delta^N y_n \ge 0$ for all large n. Then either*

(i) *for each $j \in \{1, 2, \dots, N\}$, $y_n\Delta^j y_n \ge 0$ for all large n; or*

(ii) *there is an integer* $k \in \{0, 1, \dots, N-2\}$ *such that* $(-1)^{N-k} = 1$, *and for each* $j \in \{1, \dots, k\}$, $y_n \Delta_n^j y > 0$ *for all large n, and for each* $j \in \{k+1, \dots, N-2\}$, $(-1)^{j-k} y_n \Delta_n^j y_n > 0$ *for all large n.*

We need the following result in our subsequent development. Let B be the linear space of all real sequences $x = \{x_n\}_{n=N}^{\infty}$ endowed with the usual operations and the supremum norm

$$\|x\| = \sup_{k \geq N} \frac{|x_k|}{h_k},$$

where $\{h_k\}_{k=N}^{\infty}$ is a positive sequence with a uniform positive lower bound. Then B is a Banach space. A set Ω of sequences in B under the above norm is said to be *uniformly Cauchy* if for every $\epsilon > 0$, there exists an integer M such that whenever $i, j > M$, we have $\left|(x_i/h_i) - (x_j/h_j)\right| < \epsilon$ for all $x = \{x_k\} \in \Omega$. The following discrete Schauder type fixed point theorem was obtained by Cheng and Patula [2].

LEMMA 5. *Let* Ω *be a closed, bounded and convex subset of the Banach space* B. *Suppose* T *is a continuous mapping such that* $T(\Omega)$ *is contained in* Ω, *and suppose the* $T(\Omega)$ *is uniformly Cauchy. Then* T *has a fixed point in* Ω.

The following well known theorem of Stolz is a discrete analog of l'Hopital's rule (see [1, Theorem 1.7.7 and Corollary 1.7.8]).

LEMMA 6 (Stolz's theorem). *Let* $\{u_k\}$ *and* $\{v_k\}$ *be two real sequences such that* $v_k > 0$ *and* $\Delta v_k > 0$ *for all large k. If* $\lim_{k\to\infty} v_k = \infty$ *and* $\lim_{k\to\infty} \Delta u_k / \Delta v_k = c$, *where c may be infinite, then* $\lim_{k\to\infty} u_k/v_k = c$.

3. Classification scheme

We will propose a classification scheme for the nonoscillatory solutions of (1). For this purpose it is convenient to denote the set of all eventually positive solutions of (1) by S^+ and also make use of the following notations:

$$E_j(\infty, *) = \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty, \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} \in (0, \infty)\right\}$$
$$= \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} \in (0, \infty)\right\},$$
$$E_j(\infty, 0) = \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty, \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0\right\},$$
$$E_j(*, 0) = \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} \in (0, \infty), \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0\right\}$$
$$= \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} \in (0, \infty)\right\},$$

$$O_j(\infty,*) = \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = \infty,\ \lim_{n\to\infty} \frac{x_n}{n^{2j}} \in (0,\infty)\right\}$$
$$= \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j}} \in (0,\infty)\right\},$$
$$O_j(\infty,0) = \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = \infty,\ \lim_{n\to\infty} \frac{x_n}{n^{2j}} = 0\right\},$$
$$O_j(*,0) = \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} \in (0,\infty),\ \lim_{n\to\infty} \frac{x_n}{n^{2j}} = 0\right\}$$
$$= \left\{\{x_n\} \in S^+ \mid \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} \in (0,\infty)\right\},$$

where j is some integer to be specified. Similar notations can also be introduced for eventually negative solutions of (1).

THEOREM 1. *Under the condition that m is even, for every eventually positive solution $\{x_n\}$ of (1), there is some integer $j \in \{1,2,\ldots,m/2\}$ such that $\{x_n\}$ belongs to one of the classes $E_j(\infty,*)$, $E_j(\infty,0)$, or $E_j(*,0)$. Under the condition that m is odd, for every eventually positive solution $\{x_n\}$ of (1), either there is $j \in \{1,2,\ldots,(m-1)/2\}$ such that it belongs to one of the classes $O_j(\infty,*)$, $O_j(\infty,0)$, $O_j(*,0)$, or else it converges.*

PROOF. Assume that $\{x_n\}$ is an eventually positive solution of (1). Then the associated sequence $\{z_n\}$ of $\{x_n\}$ is eventually positive, furthermore, in view of (1), $\{\Delta^m z_n\}$ is eventually negative.

Suppose m is even. Then by Lemma 3, there exist integers n_1 and $N = 2j-1$, where $j \in \{1,\ldots,m/2\}$, such that

$$\Delta^k z_n > 0, \quad n = n_1, n_1+1, \ldots;\ k = 0,1,\ldots,N-1,$$

and

$$(-1)^{k+1}\Delta^k z_n > 0, \quad n = n_1, n_1+1, \ldots;\ k = N, N+1, \ldots, m-1.$$

In particular, $\Delta^{2j-2} z_n > 0$, $\Delta^{2j-1} z_n > 0$ and $\Delta^{2j} z_n < 0$ for $n \ge n_1$. Thus

$$0 \le \lambda_{2j-1} \equiv \lim_{n\to\infty} \Delta^{2j-1} z_n < \infty \quad \text{and} \quad 0 < \lambda_{2j-2} \equiv \lim_{n\to\infty} \Delta^{2j-2} z_n \le \infty.$$

If $\lambda_{2j-1} > 0$, then by Stolz's theorem, we have

$$\lim_{n\to\infty} \frac{z_n}{n^{2j-1}} = \lim_{n\to\infty} \frac{\Delta z_n}{(2j-1)n^{2j-2}} = \cdots = \lim_{n\to\infty} \frac{\Delta^{2j-1} z_n}{(2j-1)!} = \frac{\lambda_{2j-1}}{(2j-1)!} < \infty.$$

Since $0 \le x_n/n^{2j-1} \le z_n/n^{2j-1}$, we see that x_n/n^{2j-1} is bounded for $n \ge n_1$. In view of Lemma 1, we have

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = \frac{\lambda_{2j-1}}{(2j-1)!(1+p)} \ne 0.$$

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It follows that

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty,$$

and hence $\{x_n\} \in E_j(\infty, *)$.

If $\lambda_{2j-1} = 0$ and $\lambda_{2j-2} = \infty$, then by Stolz's theorem, it is easy to see that

$$\lim_{n\to\infty} \frac{z_n}{n^{2j-1}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{z_n}{n^{2j-2}} = \infty.$$

In view of Lemma 1, we see further that

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty.$$

Hence $\{x_n\}$ belongs to $E_j(\infty, 0)$.

If $\lambda_{2j-1} = 0$ and $0 < \lambda_{2j-2} < \infty$, then by Stolz's theorem, we have

$$\lim_{n\to\infty} \frac{z_n}{n^{2j-2}} = \frac{\lambda_{2j-2}}{(2j-1)!} \neq 0.$$

In view of Lemma 1, we see that

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \frac{\lambda_{2j-2}}{(2j-1)!(1+p)} \neq 0.$$

It follows that $\lim_{n\to\infty} x_n/n^{2j-1} = 0$. Hence $\{x_n\}$ belongs to $E_j(*, 0)$.

When m is odd, in view of Lemma 3, there is an even integer $t \in \{0, 1, \ldots, m-1\}$ such that for each $s \in \{0, 1, \ldots, t\}$, $\Delta^s x_n > 0$ for all large n, and for each $s \in \{t+1, \ldots, m-1\}$, $(-1)^{s-t}\Delta^s x_n > 0$ for all large k. In case $t \in \{1, 2, \ldots, m-1\}$, the proof is similar to that given above. In case $t = 0$, then $x_n > 0$, $\Delta x_n < 0$ and $\Delta^2 x_n > 0$ for all large n. It follows that $\{x_n\}$ converges to some nonnegative constant. The proof is complete. □

4. Existence criteria when *m* is even

Eventually positive (and by analogy eventually negative) solutions of (1) have been classified according to Theorem 1. We remark, however, that there is an uncertainty involved, namely, the integer j which is needed in the definitions of E_j and O_j. We now justify our classification schemes by finding existence criteria for solutions in E_j and O_j.

THEOREM 2. *Suppose that m is even. Suppose further that f is superlinear or sublinear. If equation* (1) *has an eventually positive solution in $E_j(\infty, *)$ for some $j \in \{1, 2, \dots, m/2\}$, then there is a constant $C > 0$ such that*

$$\sum_{n=0}^{\infty} n^{m-2j} f\left(n, C(n-\delta)^{2j-1}\right) < \infty. \tag{2}$$

The converse is also true.

PROOF. Let $\{x_n\}$ be an eventually positive solution in $E_j(\infty, *)$ so that $\lim_{n\to\infty} x_n/n^{2j-2} = \infty$ and $\lim_{n\to\infty} x_n/n^{2j-1} = a > 0$. Then its associated sequence $\{z_n\}$ is eventually positive. Since $\{\Delta^m z_n\}$ is eventually negative in view of (1), we may apply Lemma 3 and conclude that $\{\Delta^i z_n\}$ is eventually monotonic for each $i \in \{1, 2, \dots, m-1\}$. Let n_1 be an integer such that

$$\frac{an^{2j-1}}{2} \le x_n \le \frac{3an^{2j-1}}{2}$$

for $n \ge n_1$. Note first that

$$\lim_{n\to\infty} \Delta^{2j-1} z_n = (2j-1)!(1+p)a,$$

which holds since by the theorem of Stolz

$$\lim_{n\to\infty} \frac{z_n}{n^{2j-1}} = \cdots = \lim_{n\to\infty} \frac{\Delta^{2j-1} z_n}{(2j-1)!} = (1+p)a.$$

Next, since $\{\Delta^i z_n\}$ is eventually monotonic for $i \in \{2j, 2j+1, \dots, m-1\}$, we see further that

$$\lim_{n\to\infty} \Delta^{2j} z_n = \lim_{n\to\infty} \Delta^{2j+1} z_n = \cdots = \lim_{n\to\infty} \Delta^{m-1} z_n = 0.$$

Thus by summing (1) successively, we obtain

$$\Delta^{2j} z_n = (-1)^{m-2j+1} \sum_{i=n}^{\infty} \frac{(i-n+1)\cdots(i-n+m-2j-1)}{(m-2j-1)!} f(i, x_{i-\delta})$$

for $n \ge n_1$. Summing again, we obtain

$$\begin{aligned} &\Delta^{2j-1} z_{n_1} - (2j-1)!(1+p)a \\ &= (-1)^{m-2j+1} \sum_{i=n_1}^{\infty} \frac{(i-n_1+1)\cdots(i-n_1+m-2j)}{(m-2j)!} f(i, x_{i-\delta}), \end{aligned}$$

which implies

$$\sum_{i=0}^{\infty} i^{m-2j} f(i, x_{i-\delta}) < \infty.$$

If f is superlinear, then $f(n, an^{2j-1}/2) \leq f(n, x_n)$ for $n \geq n_1$. Hence

$$\sum_{i=0}^{\infty} i^{m-2j} f\left(i, \frac{a}{2}(i-\delta)^{2j-1}\right) < \infty.$$

Similarly, if f is sublinear, then

$$\sum_{i=0}^{\infty} i^{m-2j} f\left(i, \frac{3a}{2}(i-\delta)^{2j-1}\right) < \infty.$$

Next, we assume that (2) holds for some $j \in \{1, 2, \ldots, m/2\}$ and $C > 0$. Let $D = C/2$ if f is superlinear and let $D = C$ if f is sublinear. Let

$$\Gamma(n) = n^{2j-1}, \quad n = 0, 1, \ldots .$$

We need to consider the cases $p = 0$, $0 < p < 1$ and $p > 1$. If $p \in (0, 1)$, we have

$$\lim_{n\to\infty} p \frac{\Gamma(n)}{\Gamma(n-\tau-\delta)} = p$$

and

$$\lim_{n\to\infty} \frac{\Gamma(n-\tau)}{\Gamma(n)} = 1 > 1 - \frac{1-p}{4p}.$$

Choose $p_1 \in (p, 1)$ and $M \geq \tau + \delta$ such that

$$\frac{\Gamma(M)}{\Gamma(M-\tau-\delta)} < 2,$$

$$p\frac{\Gamma(n)}{\Gamma(n-\tau-\delta)} < p_1, \quad \text{and} \quad \frac{\Gamma(n-\tau)}{\Gamma(n)} > 1 - \frac{1-p}{4p} \tag{3}$$

for $n \geq M$, and

$$\sum_{i=M}^{\infty} \frac{(i-M+1)\cdots(i-M+m-2j)}{(m-2j)!} f\left(i, C(i-\delta)^{2j-1}\right) < \frac{(1-p)D}{8}. \tag{4}$$

Let $N = M - \tau - \delta$. Then $p\Gamma(M)/\Gamma(N) < p_1$ by (3). We introduce the Banach space B of all real sequences $x = \{x_n\}_{n=N}^{\infty}$ endowed with the usual operations and the norm

$$\|x\| = \sup_{n\geq N} \frac{|x_n|}{\Gamma^2(n)},$$

and let Ω be the subset of B defined by

$$\Omega = \{\{x_n\} \in B \mid D\Gamma(n) \le x_n \le 2D\Gamma(n), n \ge N\}.$$

Let $T : \Omega \to \Omega$ be defined as follows: for $x = \{x_n\} \in \Omega$,

$$(Tx)_n = D\Gamma(M), \quad N \le n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2}\Gamma(n) - px_{n-\tau} + \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1} H_{i_{m-2j+2}}^{m-2j}(i) f(i, x_{i-\delta}), \quad n \ge M, \tag{5}$$

where we have employed the convenient notation

$$H_t^s(i)u_i \equiv \sum_{i=t}^{\infty} \frac{(i-t+1)\cdots(i-t+s)}{s!} u_i.$$

The subset Ω is bounded, convex and closed. We assert further that T maps Ω into Ω, T is continuous, and that $T\Omega$ is uniformly Cauchy. Indeed, if $x = \{x_n\} \in \Omega$, then from the definitions of M, N and from (3) as well as (4), we see that

$$\begin{aligned} D\Gamma(n) &< D\Gamma(M) = (Tx)_n < 2D\Gamma(n), \quad N \le n < M, \\ (Tx)_n &\ge \frac{3D(1+p)}{2}\Gamma(n) - px_{n-\tau} \ge \frac{3D(1+p)}{2}\Gamma(n) - 2pD\Gamma(n-\tau) \\ &\ge \frac{3D(1+p)}{2}\Gamma(n) - 2pD\Gamma(n) \ge D\Gamma(n), \quad n \ge M, \end{aligned}$$

and

$$\begin{aligned} (Tx)_n &\le \frac{3D(1+p)}{2}\Gamma(n) - px_{n-\tau} + \frac{(1-p)D}{8}\Gamma(n) \\ &\le \frac{3D(1+p)}{2}\Gamma(n) - Dp\left(1 - \frac{1-p}{4p}\right)\Gamma(n) + \frac{(1-p)D}{8}\Gamma(n) \\ &\le \frac{15D + Dp}{8}\Gamma(n) \le 2D\Gamma(n), \quad n \ge M. \end{aligned}$$

This shows that $Tx \in \Omega$. Next, let $\{x^{(k)}\}_{k=1}^{\infty}$ be a sequence in Ω such that $\lim_{k\to\infty} x^{(k)} = x$. Since Ω is closed, $x \in \Omega$. Furthermore, for $n \ge M$, we have

$$\frac{\left|(Tx^{(k)})_n - (Tx)_n\right|}{\Gamma^2(n)} \le p\frac{\left|x_{n-\tau}^{(k)} - x_{n-\tau}\right|}{\Gamma^2(n)}$$

$$+\frac{1}{\Gamma^2(n)}\sum_{i_{m-1}=M}^{n-1}\sum_{i_{m-2}=M}^{i_{m-1}-1}\cdots\sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1}H^{m-2j}_{i_{m-2j+1}}(i)\left|f\left(i,x^{(k)}_{i-\delta}\right)-f\left(i,x_{i-\delta}\right)\right|$$

$$\le p\left\|x^{(k)}-x\right\|+\frac{1}{\Gamma^2(n)}\frac{(n-M)^{2j-1}}{(2j-1)!}H^{m-2j}_M(i)\left|f\left(i,x^{(k)}_{i-\delta}\right)-f\left(i,x_{i-\delta}\right)\right|$$

$$\le p\left\|x^{(k)}-x\right\|+\frac{1}{\Gamma(n)}\frac{1}{(2j-1)!}H^{m-2j}_M(i)\left|f\left(i,x^{(k)}_{i-\delta}\right)-f\left(i,x_{i-\delta}\right)\right|.$$

Since

$$\left|f\left(i,x^{(k)}_{i-\delta}\right)-f\left(i,x_{i-\delta}\right)\right|\le f\left(i,x^{(k)}_{i-\delta}\right)+f\left(i,x_{i-\delta}\right)\le 4f\left(i,C(i-\delta)^{2j-1}\right)$$

and since f is continuous in the second variable, by the Lebesgue's dominated convergence theorem, we see that

$$\lim_{k\to\infty}\left\|Tx^{(k)}-Tx\right\|=\lim_{k\to\infty}\sup_{n\ge N}\frac{\left|(Tx^{(k)})_n-(Tx)_n\right|}{\Gamma^2(n)}=0.$$

This shows that T is continuous.

Finally, we assert that $T\Omega$ is uniformly Cauchy. To see this, we have to show that given any $\varepsilon>0$ there exists an integer P such that for $t,n\ge P$

$$\left|\frac{(Tx)_t}{\Gamma^2(t)}-\frac{(Tx)_n}{\Gamma^2(n)}\right|<\varepsilon,$$

for any $\{x_n\}\in\Omega$. Indeed, assuming without loss of generality that $t>n$, we have

$$\left|\frac{(Tx)_t}{\Gamma^2(t)}-\frac{(Tx)_n}{\Gamma^2(n)}\right|\le\frac{3D(1+p)}{2\Gamma(t)}+\frac{3D(1+p)}{2\Gamma(n)}+p\left(\frac{x_{t-\tau}}{\Gamma^2(t)}+\frac{x_{n-\tau}}{\Gamma^2(n)}\right)$$

$$+\frac{1}{\Gamma^2(n)}\sum_{i_{m-1}=M}^{t-1}\sum_{i_{m-2}=M}^{i_{m-1}-1}\cdots\sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1}H^{m-2j}_{i_{m-2j+2}}(i)\left(f\left(i,x^{(k)}_{i-\delta}\right)+f\left(i,x_{i-\delta}\right)\right)$$

$$\le\frac{3D(1+p)}{\Gamma(n)}+\frac{4Dp}{\Gamma(n)}+\frac{(1-p)D}{8}\left(\frac{1}{\Gamma(t)}+\frac{1}{\Gamma(n)}\right)$$

$$\le\frac{3D(1+p)}{\Gamma(n)}+\frac{4Dp}{\Gamma(n)}+\frac{(1-p)D}{4\Gamma(n)}$$

$$\le\frac{6D(1+p)+8Dp+(1-p)D}{4\Gamma(n)}\le\frac{5D}{\Gamma(n)}.$$

Since $\Gamma(n)$ tends to infinity as n tends to infinity, we see that our assertion holds.

As a consequence, we may now apply Lemma 5 to conclude that T has a fixed point $x^*=\{x_n^*\}$ in Ω. In view of (5), we now see that

$$x_n^*+px^*_{n-\tau}=\frac{3D(1+p)}{2}\Gamma(n)+\sum_{i_{m-1}=M}^{n-1}\sum_{i_{m-2}=M}^{i_{m-1}-1}\cdots\sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1}H^{m-2j}_{i_{m-2j+2}}(i)f\left(i,x^*_{i-\delta}\right).$$

By taking differences on both sides of the above equality, we see that x^* is a solution of (1). Furthermore, by applying the theorem of Stolz, we also see that

$$\begin{aligned}\lim_{n\to\infty} \frac{x_n^* + px_{n-\tau}^*}{n^{2j-1}} &= \frac{3(1+p)D}{2} + \lim_{n\to\infty} \frac{1}{n^{2j-1}} \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1} H_{i_{m-2j+2}}^{m-2j}(i) f(i, x_{i-\delta}^*) \\ &= \frac{3(1+p)D}{2} + \lim_{n\to\infty} \frac{1}{(2j-1)!} H_n^{m-2j}(i) f(i, x_{i-\delta}^*) = \frac{3(1+p)D}{2}.\end{aligned}$$

Finally, by Lemma 1, we have

$$\lim_{n\to\infty} \frac{x_n^*}{n^{2j-1}} = \frac{3D}{2} \neq 0,$$

which also implies that

$$\lim_{n\to\infty} \frac{x_n^*}{n^{2j-2}} = \infty.$$

In other words, we have found a solution in $E_j(\infty, *)$ for the case $p \in (0, 1)$.

For the case $p = 0$, we modify the definition of M in the above arguments so that

$$H_M^{m-2j}(i) f(i, C(i-\delta)^{2j-1}) < \frac{D}{8},$$

and modify the definition of T so that

$$(Tx)_n = \frac{3D}{2}\Gamma(n) + \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1} H_{i_{m-2j+2}}^{m-2j}(i) f(i, x_{i-\delta})$$

for $n \geq M$. Then under the same N, B, and Ω, we may show that (1) has a solution in $E_j(\infty, *)$.

For the case $p > 1$, we first choose p_1 such that $0 < 1/p < p_1 < 1$. We also change the definition of M so that $M \geq \tau + \delta$ and

$$\frac{1}{p} \frac{\Gamma^2(n+\tau)}{\Gamma^2(n-\tau-\delta)} < p_1 < 1, \quad n \geq M,$$

$$\frac{\Gamma(n+\tau)}{\Gamma(n)} < 1 + \frac{p-1}{4}, \quad n \geq M,$$

and

$$H_M^{m-2j}(i) f(i, C(i-\delta)^{2j-1}) < \frac{(p-1)D}{8}.$$

Finally, we modify the definition of T so that

$$(Tx)_n = \frac{3D(1+p)}{2p}\Gamma(n) - \frac{1}{p}\frac{\Gamma(n)}{\Gamma(M)}x_{M+\tau}, \quad N \le n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2p}\Gamma(n) - \frac{1}{p}x_{n+\tau} + \frac{1}{p}\sum_{i_{m-1}=M+\tau}^{n+\tau-1}\sum_{i_{m-2}=M}^{i_{m-1}-1}\cdots\sum_{i_{m-2j+1}=M}^{i_{m-2j+2}-1} H_{i_{m-2j+2}}^{m-2j}(i)f(i, x_{i-\delta}), \quad n \ge M.$$

The same reasoning described above will lead to the conclusion that (1) has a solution in $E_j(\infty, *)$. The proof is complete. □

Next we turn our attention to solutions in $E_j(*, 0)$.

THEOREM 3. *Suppose that m is even. Suppose further that f is superlinear or sublinear. If equation* (1) *has an eventually positive solution in $E_j(*, 0)$ for some $j \in \{1, 2, \dots, m/2\}$, then there is a constant $C > 0$ such that*

$$\sum_{n=0}^{\infty} n^{m-2j+1} f\left(n, C(n-\delta)^{2j-2}\right) < \infty. \tag{6}$$

The converse is also true.

PROOF. If equation (1) has an eventually positive solution in $E_j(*, 0)$, then by modifying the arguments in the proof of Theorem 2, it is not difficult to see that (6) holds for some positive constant C. The proof of the converse is also similar to that of Theorem 2. We first retain the definition of the Banach space B and its subspace Ω in the proof of Theorem 2. We then change the definition of $\Gamma(n)$ to $\Gamma(n) = n^{2j-2}$ for $n \ge 0$. Furthermore, for the case $p \in (0, 1)$, we choose $p_1 \in (p, 1)$ and $M \ge \tau + \delta$ such that

$$p\frac{\Gamma(n)}{\Gamma(n-\tau-\delta)} < p_1, \qquad \frac{\Gamma(n-\tau)}{\Gamma(n)} > 1 - \frac{1-p}{4p}$$

for $n \ge M$, and

$$\sum_{i=M}^{\infty} \frac{(i-M+1)\cdots(i-M+m-2j+1)}{(m-2j+1)!} f\left(i, C(i-\delta)^{2j-2}\right) < \frac{(1-p)D}{8}.$$

We modify the definition of the operator T by

$$(Tx)_n = D\Gamma(M), \quad N \le n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2}\Gamma(n) - px_{n-\tau} + \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+2}=M}^{i_{m-2j+3}-1} H_{i_{m-2j+3}}^{m-2j+1}(i) f(i, x_{i-\delta}), \quad n \geq M.$$

For the case where $p = 0$, we modify the definition of M in Theorem 2 so that

$$H_M^{m-2j-1}(i) f\left(i, C(i-\delta)^{2j-2}\right) < \frac{D}{8}.$$

Let $N = M - \tau - \delta$. We modify the definition of T by

$$(Tx)_n = D\Gamma(M), \quad N \leq n < M,$$

and

$$(Tx)_n = \frac{3D}{2}\Gamma(n) + \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+2}=M}^{i_{m-2j+3}-1} H_{i_{m-2j+3}}^{m-2j+1}(i) f(i, x_{i-\delta})$$

for $n \geq M$.

Finally for the case $p > 1$, we choose $p_1 \in (1/p, 1)$ and $M \geq \tau + \delta$ such that

$$\frac{1}{p}\frac{\Gamma^2(n+\tau)}{\Gamma^2(n-\tau-\delta)} < p_1 < 1, \quad n \geq M,$$

$$\frac{\Gamma(n+\tau)}{\Gamma(n)} < 1 + \frac{p-1}{4}, \quad n \geq M,$$

and

$$H_M^{m-2j-1}(i) f\left(i, C(i-\delta)^{2j-2}\right) < \frac{(p-1)D}{8}.$$

We also modify the definition of T so that

$$(Tx)_n = \frac{3D(1+p)}{2p}\Gamma(n) - \frac{1}{p}\frac{\Gamma(n)}{\Gamma(M)}x_{M+\tau}, \quad N \leq n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2p}\Gamma(n) - \frac{1}{p}x_{n+\tau} + \frac{1}{p}\sum_{i_{m-1}=M+\tau}^{n+\tau-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+2}=M}^{i_{m-2j+3}-1} H_{i_{m-2j+3}}^{m-2j+1}(i) f(i, x_{i-\delta}), \quad n \geq M.$$

Then the same reasoning described in the proof of Theorem 2 will lead to the conclusion that (1) has a solution in $E_j(*, 0)$. □

Next we turn our attention to solutions in $E_j(\infty, 0)$.

THEOREM 4. *Suppose that m is even. Suppose further that for each fixed* $n \geq 0$, $f(n, x)$ *is nonincreasing in* x *over the interval* $(0, \infty)$. *If equation* (1) *has an eventually positive solution in* $E_j(\infty, 0)$ *for some* $j \in \{1, 2, \dots, m/2\}$, *then*

$$\sum_{n=0}^{\infty} n^{m-2j} f(n, a(n-\delta)^{2j-1}) < \infty \tag{7}$$

for every $a > 0$, *and*

$$\sum_{n=0}^{\infty} n^{m-2j+1} f(n, b(n-\delta)^{2j-2}) = \infty \tag{8}$$

for every $b > 0$. *Conversely, if (7) holds for every* $a > 0$ *and*

$$\sum_{n=0}^{\infty} n^{m-2j} f(n, C(n-\delta)^{2j-2}) < \infty \tag{9}$$

for some $C > 0$, *then equation* (1) *has a positive solution in* $E_j(\infty, 0)$.

PROOF. Let $\{x_n\}$ be an eventually positive solution in $E_j(\infty, 0)$ so that

$$\limsup_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty,$$

and

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0.$$

For any $a > 0$ and $b > 0$, there exists $n_1 \geq 0$ such that

$$bn^{2j-2} \leq x_n \leq an^{2j-1}, \quad n \geq n_1,$$

which, in view of the decreasing property of f, implies that

$$f(n, an^{2j-1}) \leq f(n, x_n) \leq f(n, bn^{2j-2}), \quad n \geq n_1.$$

Now we may follow the arguments in the proof of Theorem 2 and conclude that (7) holds for any $a > 0$. Next, note that the associated sequence $\{z_n\}$ of $\{x_n\}$ satisfies

$$\lim_{n\to\infty} \Delta^{2j-2} z_n = \infty,$$

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which holds since, by the theorem of Stolz,

$$\limsup_{n\to\infty} \frac{x_n}{n^{2j-2}} \le \limsup_{n\to\infty} \frac{z_n}{n^{2j-2}} = \lim_{n\to\infty} \frac{z_n}{n^{2j-2}} = \cdots = \lim_{n\to\infty} \frac{\Delta^{2j-2} z_n}{(2j-2)!} = \infty.$$

Since $\{\Delta^i z_n\}$ is eventually monotonic for $i \in \{2j-1, 2j, \dots, m-1\}$, we see further that

$$\lim_{n\to\infty} \Delta^{2j-1} z_n = \cdots = \lim_{n\to\infty} \Delta^{n-1} z_n = 0.$$

Thus by summing (1) successively, we obtain

$$\Delta^{2j-1} z_n = (-1)^{m-2j} \sum_{i=n}^{\infty} \frac{(i-n+1)\cdots(i-n+m-2j)}{(m-2j)!} f(i, x_{i-\delta})$$

for $n \ge n_1$. Summing again, we obtain

$$\begin{aligned} &\Delta^{2j-2} z_n - \Delta^{2j-2} z_{n_1} \\ &\quad = (-1)^{m-2j} \sum_{i=n_1}^{n-1} \sum_{k=i}^{\infty} \frac{(k-i+1)\cdots(k-i+m-2j)}{(m-2j)!} f(k, x_{k-\delta}) \\ &\quad \le \sum_{n=n_1}^{\infty} \frac{(n-n_1+1)\cdots(n-n_1+m-2j+1)}{(m-2j+1)!} f(n, x_{n-\delta}), \end{aligned}$$

which, since $\lim_{n\to\infty} \Delta^{2j-2} z_n = \infty$ and $f(n, x_n) \le f(n, bn^{2j-2})$, implies

$$\sum_{n=n_1+\delta}^{\infty} n^{m-2j+1} f(n, b(n-\delta)^{2j-2}) = \infty$$

for every $b > 0$. This shows that

$$\sum_{n=0}^{\infty} n^{m-2j+1} f(n, b(n-\delta)^{2j-2}) = \infty$$

for every $b > 0$.

Conversely, assuming that (8) holds for every $b > 0$ and (9) holds for some $C > 0$, we may proceed as in the proof of Theorem 2. We first retain the definitions of the Banach space B and its subspace Ω. Then we change the definition of $\Gamma(n)$ to $\Gamma(n) = n^{2j-2}$ for $n \ge 0$. We have three cases to consider: $p = 0$, $0 < p < 1$ and $p > 1$. We will only consider the case $p \in (0, 1)$, for the other two cases can be proved in similar manners. Choose $p_1 \in (p, 1)$ and $M \ge \tau + \delta$ such that

$$\frac{\Gamma(n)}{\Gamma(n-\tau-\delta)} < p_1, \qquad \frac{\Gamma(n-\tau)}{\Gamma(n)} > 1 - \frac{1-p}{4p}$$

for $n \geq M$, and

$$\sum_{i=M}^{\infty} \frac{(i-M+1)\cdots(i-M+m-2j)}{(m-2j)!} f\left(i, C(i-\delta)^{2j-2}\right) < \frac{(1-p)D}{8}.$$

We modify the definition of the operator T by

$$(Tx)_n = D\Gamma(M), \quad N \leq n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2}\Gamma(n) - px_{n-\tau} + \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_m-1} \cdots \sum_{i_{m-2j+3}=M}^{i_{m-2j+4}-1} H_{i_{m-2j+4}}^{m-2j}(i) f(i, x_{i-\delta}), \quad n \geq M.$$

Using similar argument as in the proof of Theorem 2 we conclude that (1) has a solution x^* in Ω. Furthermore, applying the theorem of Stolz, we see that

$$\lim_{n\to\infty} \frac{x^* + px^*_{n-\tau}}{n^{2j-2}} = \frac{3(1+p)D}{2} + \lim_{n\to\infty} \frac{1}{(2j-2)!} \sum_{k=M}^{n-1} H_k^{m-2j}(i) f(i, x^*_{i-\delta})$$

and

$$\lim_{n\to\infty} \frac{x^* + px^*_{n-\tau}}{n^{2j-1}} = \lim_{n\to\infty} \frac{1}{(2j-1)!} H_n^{m-2j}(i) f(i, x^*_{i-\delta}) = 0.$$

Hence, by Lemma 1, we have

$$\lim_{n\to\infty} \frac{x^*_n}{n^{2j-1}} = 0. \tag{10}$$

Since the associated sequence $\{z^*_n\}$ of $\{x^*_n\}$ satisfies $\Delta^{2j-2}z^* > 0$ and $\Delta^{2j-1}z^* < 0$ for all large n, the sequence $\{\Delta^{2j-2}z^*_n\}$ is eventually positive and increasing. Thus $\{\Delta^{2j-2}z^*_n\}$ either converges to some positive limit or diverges to ∞. If the first case holds, then

$$\lim_{n\to\infty} \frac{z^*_n}{n^{2j-2}} = \theta,$$

for some constant θ, and hence by Lemma 1,

$$\lim_{n\to\infty} \frac{x^*_n}{n^{2j-2}} = \frac{\theta}{1+p}.$$

But then x^* will belong to $E_j(*, 0)$ so that (6) holds for some constant $C > 0$. This is contrary to the assumption that (8) holds for every $b > 0$. In other words, x^* either belongs to $E_j(*, 0)$ or $E_j(\infty, 0)$. The latter case is, however, excluded by (10). The proof is complete. □

5. Existence criteria when m is odd

For the case where m is odd, we have four results. The first three correspond to Theorem 2, Theorem 3 and Theorem 4 in the last section. The proofs of these results are similar in nature to those above and hence are sketched or omitted.

THEOREM 5. *Suppose that m is odd. Suppose further that f is superlinear or sublinear. If equation* (1) *has an eventually positive solution in* $O_j(\infty, *)$ *for some* $j \in \{1, 2, \ldots, (m-1)/2\}$, *then there is a constant* $C > 0$ *such that*

$$\sum_{n=0}^{\infty} n^{m-2j-1} f\left(n, C(n-\delta)^{2j}\right) < \infty. \tag{11}$$

The converse is also true.

The proof is again similar to that of Theorem 2. In particular, to show the converse, we first retain the definitions of the Banach space B and its subspace Ω in the proof of Theorem 2. We then change the definition of $\Gamma(n)$ to $\Gamma(n) = n^{2j}$ for $n \geq 0$. Furthermore, for the case $p \in [0, 1)$, we modify the definition of T so that

$$\begin{aligned}
(Tx)_n &= D\Gamma(M), \quad N \leq n < M, \\
(Tx)_n &= \frac{3D(1+p)}{2} T(n) - p x_{n-\tau} \\
&\quad + \sum_{i_{m-1}=M}^{n-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j}=M}^{i_{m-2j+1}-1} H_{i_{m-2j+1}}^{m-2j-1}(i) f(i, x_{i-\delta}), \quad n \geq M;
\end{aligned}$$

and for the case $p > 1$, we modify the definition of T so that

$$\begin{aligned}
(Tx)_n &= \frac{3D(1+p)}{2p} \Gamma(n) - \frac{1}{p} \frac{\Gamma(n)}{\Gamma(M)} x_{M+\tau}, \quad N \leq n < M, \\
(Tx)_n &= \frac{3D(1+p)}{2p} \Gamma(n) - \frac{1}{p} x_{n+\tau} \\
&\quad + \frac{1}{p} \sum_{i_{m-1}=M+\tau}^{n+\tau-1} \sum_{i_{m-2}=M}^{i_{m-1}-1} \cdots \sum_{i_{m-2j}=M}^{i_{m-2j+1}-1} H_{i_{m-2j+1}}^{m-2j-1}(i) f(i, x_{i-\delta}), \quad n \geq M.
\end{aligned}$$

Then the same reasoning described in the proof of Theorem 2 will lead to the conclusion that (1) has a solution in $O_j(\infty, *)$.

THEOREM 6. *Suppose that m is odd. Suppose further that f is superlinear or sublinear. If equation* (1) *has an eventually positive solution in* $O_j(*, 0)$ *for some*

$j \in \{1, 2, \ldots, (m-1)/2\}$, then there is a constant $C > 0$ such that

$$\sum_{n=0}^{\infty} n^{m-2j} f\left(n, C(n-\delta)^{2j-1}\right) < \infty. \tag{12}$$

The converse is also true.

The proof of this result is similar to those of Theorem 2, Theorem 3 and Theorem 4, and is thus omitted.

THEOREM 7. *Suppose that m is odd. Suppose further that for each fixed $n \geq 0$, $f(n, x)$ is nonincreasing in x over the interval $(0, \infty)$. If equation (1) has an eventually positive solution in $O_j(\infty, 0)$ for some $j \in \{1, 2, \ldots, (m-1)/2\}$, then*

$$\sum_{n=0}^{\infty} n^{m-2j-1} f(n, a(n-\delta)^{2j}) < \infty \tag{13}$$

for every $a > 0$, and

$$\sum_{n=0}^{\infty} n^{m-2j} f(n, b(n-\delta)^{2j-1}) = \infty \tag{14}$$

for every $b > 0$. Conversely, if (14) holds for every $b > 0$ and (12) holds for some $C > 0$, then equation (1) has an eventually positive solution in $O_j(\infty, 0)$.

Our final result is concerned with the existence of convergent positive solutions.

THEOREM 8. *Suppose that m is odd. Suppose further that f is superlinear or sublinear. If equation (1) has an eventually positive solution which converges to a positive constant, then there is a constant $C > 0$ such that*

$$\sum_{n=0}^{\infty} n^{m-1} f(n, C) < \infty.$$

The converse is also true.

Again, the proof is similar to that of Theorem 2. In particular, to show the converse, we first retain the definitions of the Banach space B and its subspace Ω in the proof of Theorem 2. We then change the definition of $\Gamma(n)$ to $\Gamma(n) = 1$ for $n \geq 0$. Furthermore, for the case $p \in [0, 1)$, we modify the definition of T so that

$$(Tx)_n = D\Gamma(M), \quad N \leq n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2} - px_{n-\tau} + H_n^{m-1}(i)f(i, x_{i-\delta}), \quad n \geq M;$$

and for the case $p > 1$, we modify the definition of T so that

$$(Tx)_n = \frac{3D(1+p)}{2p} - \frac{1}{p}x_{M+\tau}, \quad N \leq n < M,$$

and

$$(Tx)_n = \frac{3D(1+p)}{2p} - \frac{1}{p}x_{n-\tau} + \frac{1}{p}H_{n+\tau}^{m-1}(i)f(i, x_{i-\delta}), \quad n \geq M.$$

The same reasoning as described in the proof of Theorem 2 leads to the conclusion that (1) has an eventually positive solution which converges to a positive constant.

As a final remark, when $m = 4$, $p = 0$ and $j = 2$ in Theorem 2, and when $m = 4$, $p = 0$ and $j = 1$ in Theorem 3, the corresponding results have been derived by Zhang and Cheng [14, Theorem 3.1 and Theorem 3.2]. But in [14], only results related to fourth order equations are discussed.

References

[1] R. P. Agarwal, *Difference equations and inequalities* (Academic Press, New York, 1992).

[2] S. S. Cheng and W. T. Patula, 'An existence theorem for a nonlinear difference equation', *Nonlinear Anal.* **30** (1993), 193–203.

[3] S. S. Cheng, G. Zhang and W. T. Li, 'On a higher order neutral difference equation', *Recent Trends in Math. Anal. Appl.*, to appear.

[4] S. R. Grace and B. S. Lalli, 'Oscillation theorems for second order delay and neutral difference equations', *Util. Math.* **45** (1994), 199–211.

[5] X. Z. He, 'Oscillation and asymptotic behavior of second order nonlinear difference equation', *J. Math. Anal. Appl.* **175** (1993), 482–498.

[6] I. T. Kiguradze and T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations* (Kluwer, Amsterdam, 1993).

[7] B. S. Lalli, 'Oscillation theorems for neutral difference equations', *Comput. Math. Appl.* **28** (1994), 191–202.

[8] B. S. Lalli and B. G. Zhang, 'On existence of positive solution and bounded oscillation for neutral difference equations', *J. Math. Anal. Appl.* **166** (1992), 272–287.

[9] B. S. Lalli, B. G. Zhang and J. Z. Li, 'On the oscillation and existence of positive solutions of neutral difference equations', *J. Math. Anal. Appl.* **158** (1991), 213–233.

[10] W. T. Li and S. S. Cheng, 'Classifications and existence of positive solutions of second order nonlinear neutral difference equations', *Funkcial. Ekvac.* **40** (1997), 371–393.

[11] B. Liu and J. R. Yan, 'Oscillation theorems for nonlinear neutral difference equations', *J. Differ. Equations Appl.* **1** (1995), 307–315.

[12] B. Smith and W. E. Taylor, Jr., 'Oscillatory and asymptotic behavior of certain fourth order difference equations', *Rocky Mountain J. Math.* **16** (1986), 403–406.
[13] A. Zafer and R. Dahiya, 'Oscillation of a neutral difference equation', *Appl. Math. Lett.* **6** (1993), 71–74.
[14] B. G. Zhang and S. S. Cheng, 'On a class of nonlinear difference equations', *J. Differ. Equations Appl.* **1** (1995), 391–411.
[15] G. Zhang and S. S. Cheng, 'Oscillation criteria for a neutral difference equation with delay', *Appl. Math. Lett.* **8** (1995), 13–17.
[16] X. L. Zhou and J. R. Yan, 'Oscillatory properties of higher order nonlinear difference equations', *Comput. Math. Appl.* **31** (1996), 61–68.

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