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ON SOME PEXIDER-TYPE FUNCTIONAL EQUATIONS CONNECTED WITH THE ABSOLUTE VALUE OF ADDITIVE FUNCTIONS. PART II

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Abstract

We investigate the Pexider-type functional equation

$$\max\{f(x+y), f(x-y)\} = f(x)g(y) + h(y), \quad x, y \in G,$$

where f, g, h are real functions defined on an abelian group G. We solve this equation under the assumptions $G = \mathbb{R}$ and f is continuous.

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1. Introduction

In [3] we introduced two Pexider functional equations:

$$\max\{f(x+y), f(x-y)\} = f(x)g(y) + h(y) \tag{1.1}$$

and

$$\max\{f(x+y), f(x-y)\} = f(y)g(x) + h(x), \tag{1.2}$$

with $f, g, h : G \to \mathbb{R}$, where G is an abelian group. These are common generalizations of two functional equations:

$$\max\{f(x+y), f(x-y)\} = f(x) + f(y) \tag{1.3}$$

and

$$\max\{f(x+y), f(x-y)\} = f(x)f(y). \tag{1.4}$$

For the solution of (1.2) see [3, Theorem 1.1]. Here, in Theorem 3.1, we are going to describe the solutions of (1.1) under the additional assumptions that $G = \mathbb{R}$ and f is continuous. However, some results concerning this equation in more general settings, that is for an arbitrary abelian group G and $f, g, h : G \to \mathbb{R}$, will be given in Section 2.

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2. Equation (1.1) on groups

As we will see, there is a relationship between solutions of (1.1) and (1.2). First, for a given $f : G \to \mathbb{R}$, we introduce the following functions:

$$\begin{aligned} F(x) &:= \max\{f(x), f(-x)\}, \quad x \in G; \\ \tilde{f}(x) &:= f(x) - f(0), \quad x \in G; \\ \tilde{F}(x) &:= \max\{\tilde{f}(x), \tilde{f}(-x)\} = F(x) - f(0), \quad x \in G. \end{aligned}$$

Of course, F and $\tilde{F}$ are even; moreover, $\tilde{f}(0) = 0$ and $\tilde{F}(0) = 0$.

Lemma 2.1. *Let G be an abelian group, and let $f, g, h : G \to \mathbb{R}$ satisfy (1.1). Then*

$$\max\{\tilde{f}(x+y), \tilde{f}(x-y)\} = \tilde{f}(x)g(y) + \tilde{F}(y), \quad x, y \in G. \tag{2.1}$$

Moreover, if $g \geq 0$ then F, g, h and $\tilde{F}, g, \tilde{F}$ satisfy (1.1).

Proof. With $x = 0$ in (1.1) we obtain

$$\max\{f(y), f(-y)\} = f(0)g(y) + h(y),$$

whence

$$h(y) = F(y) - f(0)g(y), \quad y \in G. \tag{2.2}$$

Therefore,

$$\max\{f(x+y), f(x-y)\} = f(x)g(y) + F(y) - f(0)g(y),$$

and by subtracting $f(0)$ from each side we get

$$\max\{\tilde{f}(x+y), \tilde{f}(x-y)\} = \tilde{f}(x)g(y) + \tilde{F}(y).$$

Further, assume that $g \geq 0$. We have

$$\begin{aligned} \max\{F(x+y), F(x-y)\} &= \max\{\max\{f(x+y), f(-x-y)\}, \max\{f(x-y), f(-x+y)\}\} \\ &= \max\{\max\{f(x+y), f(x-y)\}, \max\{f(-x-y), f(-x+y)\}\} \\ &= \max\{f(x)g(y) + h(y), f(-x)g(y) + h(y)\} \\ &= \max\{f(x), f(-x)\}g(y) + h(y) = F(x)g(y) + h(y). \end{aligned}$$

This implies

$$\begin{aligned} \max\{\tilde{F}(x+y), \tilde{F}(x-y)\} &= \max\{F(x+y), F(x-y)\} - f(0) \\ &= F(x)g(y) + h(y) - f(0) = F(x)g(y) + F(y) - f(0)g(y) - f(0) \\ &= \tilde{F}(x)g(y) + \tilde{F}(y). \end{aligned}$$

This concludes the proof. □

We proceed with the following simple observations.

Remark 2.2. Let G be an abelian group and consider $f, g, h : G \to \mathbb{R}$. Then, f, g, h satisfy (1.2) if and only if f, g, h satisfy (1.1) and f is even.

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Proof. First assume that f, g, h satisfy (1.2). If f is constant, g is an arbitrary function and $h(x) = f(x)(1 - g(x))$, then f is obviously even and (1.1) is fulfilled by f, g, h. However, if f is not constant, then f, g, h are even (see [3, Remark 3.1]), and we have

$$\begin{aligned}\max\{f(x+y), f(x-y)\} &= \max\{f(-x-y), f(x-y)\}\\ &= f(x)g(-y) + h(-y) = f(x)g(y) + h(y).\end{aligned}$$

Now, assume that f, g, h satisfy (1.1) and f is even. Then,

$$\begin{aligned}\max\{f(x+y), f(x-y)\} &= \max\{f(-x-y), f(x-y)\}\\ &= f(-y)g(x) + h(x) = f(y)g(x) + h(x).\end{aligned}$$

This concludes the proof. □

Remark 2.3. Let G be an abelian group, and consider $x_0 \in G$ and $f, g, h : G \to \mathbb{R}$. Then f, g, h satisfy (1.1) if and only if $f(\cdot - x_0), g, h$ satisfy (1.1).

We can formulate the following corollary.

Corollary 2.4. *Let G be an abelian group, and consider $x_0 \in G$ and $f, g, h : G \to \mathbb{R}$. If f, g, h satisfy (1.2) then $f(\cdot - x_0), g, h$ satisfy (1.1).*

In view of [3, Theorem 1.1] we have the following solutions of (1.1):

(1) $\begin{cases} f(x) = b, \\ g \text{ is an arbitrary function,} \\ h(x) = b(1 - g(x)), \end{cases}$

where $b \in \mathbb{R}$;

(2) $\begin{cases} f(x) = c\phi(x - x_0) + b, \\ g(x) = \phi(x), \\ h(x) = b(1 - \phi(x)), \end{cases}$

where $x_0, c, b \in \mathbb{R}$, $c > 0$, and $\phi : G \to \mathbb{R}$ is a solution of (1.4);

(3) $\begin{cases} f(x) = c\phi(x - x_0) + b, \\ g(x) = \phi(x), \\ h(x) = b(1 - \phi(x)), \end{cases}$

where $x_0, c, b \in \mathbb{R}$, $c < 0$, and $\phi : G \to \mathbb{R}$ is a solution of

$$\min\{\phi(x+y), \phi(x-y)\} = \phi(x)\phi(y); \tag{2.3}$$

(4) $\begin{cases} f(x) = \phi(x - x_0) + b, \\ g(x) = 1, \\ h(x) = \phi(x), \end{cases}$

where $x_0, b \in \mathbb{R}$, and $\phi : G \to \mathbb{R}$ is a solution of (1.3).

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For solutions of (1.3), (1.4) and (2.3) see [3, Section 2]. Nevertheless, as we will see, these are not all solutions to (1.1).

Other partial results can be derived from [3, Theorem 1.1], Lemma 2.1 and Remark 2.2, as in the following corollary.

Corollary 2.5. *Let G be an abelian group and suppose that $f, g, h : G \to \mathbb{R}$ satisfy (1.1). Suppose also that $g \geq 0$. Then:*

(1) $$\begin{cases} F(x) = b, \\ g \text{ is an arbitrary function}, \\ h(x) = b(1 - g(x)), \end{cases}$$
where $b \in \mathbb{R}$; or

(2) $$\begin{cases} F(x) = c\phi(x) + b, \\ g(x) = \phi(x), \\ h(x) = b(1 - \phi(x)), \end{cases}$$
where $c, b \in \mathbb{R}$, $c > 0$, and $\phi : G \to \mathbb{R}$ is a solution of (1.4); or

(3) $$\begin{cases} F(x) = c\phi(x) + b, \\ g(x) = \phi(x), \\ h(x) = b(1 - \phi(x)), \end{cases}$$
where $c, b \in \mathbb{R}$, $c < 0$, and $\phi : G \to \mathbb{R}$ is a solution of (2.3); or

(4) $$\begin{cases} F(x) = \phi(x) + b, \\ g(x) = 1, \\ h(x) = \phi(x), \end{cases}$$
where $b \in \mathbb{R}$, and $\phi : G \to \mathbb{R}$ is a solution of (1.3).

We finish this section with one more simple observation, in view of, for example, (2.1) and (2.2).

Corollary 2.6. *Suppose that $f, g, h : G \to \mathbb{R}$ satisfy (1.1) and f is not constant. Then $g(0) = 1$ and g and h are even.*

3. Solution of (1.1) on $\mathbb{R}$

In this section we are going to show the main result of this paper, that is, the following theorem.

Theorem 3.1. *Let $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy equation (1.1) and assume that f is continuous. Then the functions f, g, h are of one of the following forms:*

(1) $$\begin{cases} f(x) = b, \\ g \text{ is an arbitrary function}, \\ h(x) = b(1 - g(x)), \end{cases}$$
where $b \in \mathbb{R}$;

$$(2)\quad \begin{cases} f(x) = ce^{a|x-x_0|} + b, \\ g(x) = e^{a|x|}, \\ h(x) = b(1 - e^{a|x|}), \end{cases}$$

where $x_0, b \in \mathbb{R}$, $ac > 0$*;*

$$(3)\quad \begin{cases} f(x) = a|x - x_0| + b, \\ g(x) = 1, \\ h(x) = a|x|, \end{cases}$$

where $b, x_0 \in \mathbb{R}$, $a > 0$*;*

$$(4)\quad \begin{cases} f(x) = ce^{ax} + b, \\ g(x) = e^{\operatorname{sgn}(c)|ax|}, \\ h(x) = b(1 - e^{\operatorname{sgn}(c)|ax|}), \end{cases}$$

where $a, b, c \in \mathbb{R}$*;*

$$(5)\quad \begin{cases} f(x) = ax + b, \\ g(x) = 1, \\ h(x) = |ax|, \end{cases}$$

where $a, b \in \mathbb{R}$*.*

Conversely, if f, g, h *are of one of the forms (1)–(5), then they satisfy (1.1).*

The 'if' part of this theorem (the converse) can be checked directly. The remainder of the paper is concerned with proving that if $f, g, h : \mathbb{R} \to \mathbb{R}$ satisfy (1.1) and f is continuous then f, g, h are of the forms (1)–(5). First notice that if $f \equiv b$ is constant and g is an arbitrary function then $h(y) := b(1 - g(y))$. So, from now on, we will work under the assumptions

$$f, g, h : \mathbb{R} \to \mathbb{R}, \text{ satisfy (1.1), and } f \text{ is continuous and not constant.} \tag{H}$$

Remark 3.2. Suppose (H). Then the functions g and h are continuous.

Proof. By (2.1) with $x_0 \in \mathbb{R}$ such that $\tilde{f}(x_0) \neq 0$,

$$g(y) = \frac{\max\{\tilde{f}(x_0 + y), \tilde{f}(x_0 - y)\} - \tilde{F}(0)}{\tilde{f}(x_0)}.$$

Therefore g is continuous. Moreover, from (2.2), h is continuous, too. □

Lemma 3.3. *Suppose (H). Then* $g \geq 0$*.*

Proof. We consider three cases.

(I) $\tilde{F} \equiv 0$. Since f is not constant, we can find an $x_0 \neq 0$ such that $\tilde{f}(x_0) < 0$. The inequality $g(y) < 0$ for some $y \in \mathbb{R}$ implies, by (2.1),

$$0 \geq \max\{\tilde{f}(x_0 + y), \tilde{f}(x_0 - y)\} = \tilde{f}(x_0)g(y) + \tilde{F}(y) = \tilde{f}(x_0)g(y) > 0,$$

which is impossible. Hence, $g \geq 0$.

(II) $\tilde{F} \le 0$ and $\tilde{F} \not\equiv 0$. Suppose that $g(y) = 0$ for some $y \in \mathbb{R}$. Since $g(0) = 1$ and g is continuous and even, we can define $y_0 := \min\{y > 0 : g(y) = 0\}$. We have, using (2.1) again,

$$\max\{\tilde{f}(x + y_0), \tilde{f}(x - y_0)\} = \tilde{f}(x)g(y_0) + \tilde{F}(y_0) = \tilde{F}(y_0), \quad x \in \mathbb{R}. \tag{3.1}$$

Putting $x = -y_0$ in the above, we obtain

$$0 = \tilde{f}(0) \le \max\{\tilde{f}(0), \tilde{f}(x - y_0)\} = \tilde{F}(y_0) \le 0.$$

Hence we have proved that

$$\tilde{F}(y_0) = 0. \tag{3.2}$$

Now we will prove the implication

$$\tilde{F}(x) < 0 \Rightarrow \forall_{z \in (x - y_0, x + y_0)} \tilde{f}(z) < 0. \tag{3.3}$$

Fix an x with $\tilde{F}(x) < 0$ and $y \in (-y_0, y_0)$. By (2.1) we have

$$\max\{\tilde{f}(x + y), \tilde{f}(x - y)\} = \tilde{f}(x)g(y) + \tilde{F}(y) < 0,$$

as $\tilde{f}(x) < 0$, $g(y) > 0$ and $\tilde{F}(y) \le 0$. Therefore, we have proved (3.3).

Now fix an x_0 such that $\tilde{F}(x_0) < 0$. By (3.1) and (3.2) we have

$$\max\{\tilde{f}(x_0 + y_0), \tilde{f}(x_0 - y_0)\} = 0.$$

Suppose, without loss of generality, that

$$\tilde{f}(x_0 - y_0) = 0. \tag{3.4}$$

For an arbitrary $x_1 \in (x_0 - y_0, x_0)$ we have $\tilde{f}(x_1) < 0$ (see (3.3)). Hence, using (3.3) for x_1 we infer that $\tilde{f}(z) < 0$ for $z \in (x_1 - y_0, x_1 + y_0)$. But $x_0 - y_0 \in (x_1 - y_0, x_1 + y_0)$, so $\tilde{f}(x_0 - y_0) < 0$, which is a contradiction with (3.4). Therefore $g > 0$.

(III) $\tilde{F}(z_0) > 0$ for some $z_0 \in \mathbb{R}$. If $g > 0$ then the proof is finished. So, assume that it is not the case. Then, by continuity of g, the fact that g is even and $g(0) = 1$, we can define

$$y_0 := \min\{y > 0 : g(y) = 0\}.$$

Put $M := \sup\{\tilde{f}(x) : x \in \mathbb{R}\}$. We have, of course, $M > 0$. Moreover,

$$M = \tilde{F}(y), \quad y \in g^{-1}(\{0\}). \tag{3.5}$$

Indeed, by (2.1),

$$\max\{\tilde{f}(x + y), \tilde{f}(x - y)\} = \tilde{f}(x)g(y) + \tilde{F}(y) = \tilde{F}(y), \quad x \in \mathbb{R}, y \in g^{-1}(\{0\}). \tag{3.6}$$

This implies

$$\tilde{f}(x) \le \tilde{F}(y), \quad x \in \mathbb{R}, y \in g^{-1}(\{0\}).$$

Hence $M \le \tilde{F}(y)$, for every $y \in g^{-1}(\{0\})$. Therefore, by the definition of M and $\tilde{F}$ we infer that $M = \tilde{F}(y)$, for every $y \in g^{-1}(\{0\})$. Now, (3.6) implies

$$[\tilde{f}(x) < M \text{ and } y \in g^{-1}(\{0\})] \Rightarrow \tilde{f}(x + 2y) = \tilde{f}(x - 2y) = M. \tag{3.7}$$

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In particular, since $\tilde{f}(0) = 0 < M$,

$$\tilde{f}(2y) = \tilde{f}(-2y) = M, \quad y \in g^{-1}(\{0\}). \tag{3.8}$$

Put

$$\delta := \min\{\varepsilon > 0 : \tilde{f}(-\varepsilon) = M\},$$
$$\eta := \min\{\varepsilon > 0 : \tilde{f}(\varepsilon) = M\}.$$

This means that $\tilde{f}(-\delta) = \tilde{f}(\eta) = M$ and $\tilde{f}(x) < M$ for $x \in (-\delta, \eta)$. We also have

$$\tilde{F}(-\delta) = \tilde{F}(\eta) = M. \tag{3.9}$$

By (3.7), $\tilde{f}(x + 2y_0) = M$ for every $x \in (-\delta, \eta)$. Hence $\tilde{F}(x + 2y_0) = M$, for $x \in (-\delta, \eta)$. Continuity of $\tilde{F}$ assures us that we also have

$$\tilde{F}(2y_0 - \delta) = \tilde{F}(2y_0 + \eta) = M. \tag{3.10}$$

Using (2.1) we get

$$\max\{\tilde{f}(2y_0 + \eta), \tilde{f}(2y_0 - \eta)\} = \tilde{f}(2y_0)g(\eta) + \tilde{F}(\eta).$$

This, together with (3.10), (3.8) and (3.9), gives $g(\eta) = 0$. However, from the definition of y_0, this means that $y_0 \le \eta$, but, from the definition of η,

$$\tilde{f}(x) < M, \quad x \in [0, y_0). \tag{3.11}$$

Similarly,

$$\max\{\tilde{f}(2y_0 + \delta), \tilde{f}(2y_0 - \delta)\} = \tilde{f}(2y_0)g(-\delta) + \tilde{F}(-\delta).$$

This, together with (3.10), (3.8) and (3.9), gives $g(-\delta) = 0$. However, from the definition of y_0, this means that $y_0 < \delta$, but, from the definition of δ,

$$\tilde{f}(x) < M, \quad x \in (-y_0, 0).$$

This, together with (3.11), means that $\tilde{f}(x) < M$ for every $x \in (-y_0, y_0)$. Now, in view of (3.7) (and the continuity of $\tilde{f}$), we infer

$$\tilde{f}(x) = M, \quad x \in [y_0, 3y_0]. \tag{3.12}$$

On account of (2.1),

$$\max\{\tilde{f}(3y_0 - x), \tilde{f}(3y_0 + x)\} = \tilde{f}(3y_0)g(x) + \tilde{F}(x), \quad x \in \mathbb{R},$$

and hence, by (3.12),

$$g(x) = 0, \quad x \in [y_0, 2y_0]. \tag{3.13}$$

Moreover, from the definition of y_0 and $g(0) = 1$, we know that

$$g(x) > 0, \quad x \in [0, y_0). \tag{3.14}$$

Now, we will prove the implication

$$g(y) = 0 \Rightarrow g(2y) = 0, \quad y \in \mathbb{R}. \tag{3.15}$$

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Notice that in view of (3.13), (3.15) together with (3.14) (and the fact that g is even), implies that $g(x) > 0$ for $x \in (-y_0, y_0)$ and $g(x) = 0$ for $x \in (-\infty, -y_0] \cup [y_0, \infty)$. Hence, proving (3.15) will complete the proof.

Suppose that $g(y) = 0$. By (3.5) we have $\tilde{F}(y) = M$. Let $x \in \mathbb{R}$ be such that $0 < \tilde{f}(x) < M$ (such an x exists, since $\tilde{f}(0) = 0$ and $\tilde{F}(y_0) = M$). Therefore, using (2.1),

$$\max\{\tilde{f}(x), \tilde{f}(x + 2y)\} = \tilde{f}(x + y)g(y) + \tilde{F}(y).$$

This implies $\tilde{f}(x + 2y) = M$, since $\tilde{f}(x) < M$, $g(y) = 0$ and $\tilde{F}(y) = M$. Using (2.1) again, we obtain

$$\max\{\tilde{f}(x + 2y), \tilde{f}(x - 2y)\} = \tilde{f}(x)g(2y) + \tilde{F}(2y).$$

Consequently, $g(2y) = 0$, since $\tilde{f}(x + 2y) = M$, $\tilde{f}(x) > 0$ and $\tilde{F}(2y) = M$ (see (3.8)). The proof of (3.15) is complete. □

Lemma 3.4. *Suppose (H). Then $\tilde{F}$ is not constant.*

Proof. Suppose, on the contrary, that $\tilde{F} \equiv 0$. Since f is not constant, there exists an $x_0 \neq 0$ such that $\tilde{f}(x_0) < 0$. Without loss of generality, we can assume that $x_0 > 0$. Let $z_0 \geq 0$ be such that $\tilde{f}(z_0) = 0$ and $\tilde{f}(x) < 0$ for every $x \in (z_0, x_0]$. For an arbitrary $x \in (z_0, x_0]$, by (2.1),

$$0 = \max\{\tilde{f}(2x - z_0), \tilde{f}(z_0)\} = \tilde{f}(x)g(x - z_0) + \tilde{F}(x - z_0) = \tilde{f}(x)g(x - z_0).$$

In view of $\tilde{f}(x) < 0$ we infer $g(x - z_0) = 0$. Passing with x to z_0, on account of the continuity of g, gives $g(0) = 0$. However, we know from Corollary 2.6 that $g(0) = 1$. This is a contradiction. □

We now reformulate Corollary 2.5 under the assumption (H) (see [3, Section 2]).

Corollary 3.5. *Suppose (H). Then we have one of the following cases:*

$$\text{(U)} \quad \begin{cases} \tilde{F}(x) = C(e^{A|x|} - 1), \\ F(x) = C(e^{A|x|} - 1) + f(0), \\ g(x) = e^{A|x|}, \\ h(x) = (f(0) - C)(1 - e^{A|x|}), \end{cases}$$

where $A, C > 0$;

$$\text{(V)} \quad \begin{cases} \tilde{F}(x) = A|x|, \\ F(x) = A|x| + f(0), \\ g(x) = 1, \\ h(x) = A|x|, \end{cases}$$

where $A > 0$;

$$\text{(W)} \quad \begin{cases} \tilde{F}(x) = C(1 - e^{-A|x|}), \\ F(x) = C(1 - e^{-A|x|}) + f(0), \\ g(x) = e^{-A|x|}, \\ h(x) = (f(0) + C)(1 - e^{-A|x|}), \end{cases}$$

where $A, C > 0$.

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In particular, $g(x) > 0$, $x \in \mathbb{R}$, and F and $\tilde{F}$ are strictly decreasing on $(-\infty, 0]$ *and strictly increasing on* $[0, \infty)$.

We see that it is enough to determine the formula for f in each of the cases (U), (V) and (W). Now, we are going to prove four technical lemmas, which will be summarized in Corollary 3.10.

LEMMA 3.6. *Suppose (H). Then we have the following results.*

- *If there exists an $x_0 < 0$ such that $\tilde{f}(x_0) < \tilde{F}(x_0)$, then*

$$\begin{aligned} \tilde{f}(x) &< \tilde{F}(x), \qquad x \in (-\infty, 0), \\ \tilde{f}(x) &= \tilde{F}(x), \qquad x \in [0, \infty). \end{aligned}$$

- *If there exists an $x_0 > 0$ such that $\tilde{f}(x_0) < \tilde{F}(x_0)$, then*

$$\begin{aligned} \tilde{f}(x) &< \tilde{F}(x), \qquad x \in (0, \infty), \\ \tilde{f}(x) &= \tilde{F}(x), \qquad x \in (-\infty, 0]. \end{aligned}$$

PROOF. Let us assume that there exists an $x_0 < 0$ such that $\tilde{f}(x_0) < \tilde{F}(x_0)$ (the second implication can be considered analogously). Suppose that there is a $y < x_0$ such that $\tilde{f}(y) = \tilde{F}(y)$. We can choose $x_1 < 0$ and $y_1 > 0$ with $x_1 + y_1 < 0$, $\tilde{f}(x_1) < \tilde{F}(x_1)$ and $\tilde{f}(x_1 - y_1) = \tilde{F}(x_1 - y_1)$. Using (2.1), the positiveness of g, the fact that $\tilde{F}, g, \tilde{F}$ satisfy (1.1) and the monotonicity of $\tilde{F}$, we have

$$\begin{aligned} \tilde{F}(x_1 - y_1) = \tilde{f}(x_1 - y_1) &\leq \max\{\tilde{f}(x_1 + y_1), \tilde{f}(x_1 - y_1)\} \\ &= \tilde{f}(x_1)g(y_1) + \tilde{F}(y_1) < \tilde{F}(x_1)g(y_1) + \tilde{F}(y_1) \\ &= \max\{\tilde{F}(x_1 + y_1), \tilde{F}(x_1 - y_1)\} = \tilde{F}(x_1 - y_1). \end{aligned}$$

This is a contradiction. Hence, we have proved that if $\tilde{f}(x) < \tilde{F}(x)$, for some $x < 0$, then $\tilde{f}(y) < \tilde{F}(y)$, for every $y < x$. Let $z := \sup\{x < 0 : \tilde{f}(x) < \tilde{F}(x)\}$. We have

$$\tilde{f}(z) = \tilde{F}(z) \tag{3.16}$$

and

$$\tilde{f}(x) < \tilde{F}(x), \qquad x < z. \tag{3.17}$$

Let $y > -2z$. Hence, $-z - y < z \leq 0$. Therefore, $\tilde{f}(-z - y) < \tilde{F}(-z - y) = \tilde{F}(z + y)$. Hence,

$$\tilde{f}(z + y) = \tilde{F}(z + y). \tag{3.18}$$

Using the inequalities $0 < z + y \leq y - z$, the fact that $\tilde{F}$ is even and the results concerning its monotonicity from Corollary 3.5, we obtain

$$\tilde{F}(z + y) \leq \tilde{F}(z - y). \tag{3.19}$$

Suppose that

$$\tilde{F}(z + y) < \tilde{F}(z - y). \tag{3.20}$$

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Using (2.1), (3.16), Lemma 2.1 and (3.19), we obtain

$$\begin{aligned}\max\{\tilde{f}(z+y), \tilde{f}(z-y)\} &= \tilde{f}(z)g(y) + \tilde{F}(y) \\ &= \tilde{F}(z)g(y) + \tilde{F}(y) = \max\{\tilde{F}(z+y), \tilde{F}(z-y)\} = \tilde{F}(z-y).\end{aligned}$$

However, since we have shown (3.18) and assumed (3.20), then it follows that $\tilde{f}(z-y) = \tilde{F}(z-y)$, which is impossible, as $z-y<z$ and (3.17) holds. Therefore, (3.20) leads to a contradiction. Hence, in view of this and (3.19) we find $\tilde{F}(z+y) = \tilde{F}(z-y)$, which, due to the properties of $\tilde{F}$, implies $z=0$. Hence, by (3.17), we have proved that $\tilde{f}(x) < \tilde{F}(x)$, for $x<0$ and, consequently, $\tilde{f}(x) = \tilde{F}(x)$, for $x \geq 0$. □

Lemma 3.7. *Suppose (H). For any $c \in \mathbb{R}$ define function $H_c : \mathbb{R} \to \mathbb{R}$ by the formula $H_c(x) := cg(x) + \tilde{F}(x)$. Then we have the following results.*

- *For every $x_0 < 0$ and $c \in [0, \tilde{F}(x_0)]$,*

$$H_c(x - x_0) > \tilde{F}(x), \qquad x \geq 0.$$

- *For every $x_0 > 0$ and $c \in [0, \tilde{F}(x_0)]$,*

$$H_c(x - x_0) > \tilde{F}(x), \qquad x \leq 0.$$

Proof. Using the formulas for g and $\tilde{F}$ from Corollary 3.5, we can directly check the inequalities in each of the cases (U), (V) and (W). We omit the calculations here. □

Lemma 3.8. *Suppose (H). Then the following implications hold true.*

- *If there exist $x_0, x_1 < 0$ such that $\tilde{f}(x_0) \geq 0$ and $\tilde{f}(x_1) < \tilde{F}(x_1)$, then $\tilde{f}(x_0 - y) = H_{\tilde{f}(x_0)}(y)$ for $y \geq |x_0|$.*
- *If there exist $x_0, x_1 > 0$ such that $\tilde{f}(x_0) \geq 0$ and $\tilde{f}(x_1) < \tilde{F}(x_1)$, then $\tilde{f}(x_0 + y) = H_{\tilde{f}(x_0)}(y)$ for $y \geq |x_0|$.*

Proof. We will prove only the first part. From Lemma 3.6 we infer

$$\tilde{f}(x) = \tilde{F}(x), \qquad x \in [0, \infty). \tag{3.21}$$

Fix $y \geq |x_0|$. Using (2.1) we obtain

$$\max\{\tilde{f}(x_0 - y), \tilde{f}(x_0 + y)\} = \tilde{f}(x_0)g(y) + \tilde{F}(y) = H_{\tilde{f}(x_0)}(y). \tag{3.22}$$

Since $x_0 + y \geq 0$, by (3.21) and Lemma 3.7 we obtain

$$\tilde{f}(x_0 + y) = \tilde{F}(x_0 + y) < H_{\tilde{f}(x_0)}(x_0 + y - x_0) = H_{\tilde{f}(x_0)}(y),$$

which together with (3.22) gives

$$\tilde{f}(x_0 - y) = H_{\tilde{f}(x_0)}(y).$$

This concludes the proof. □

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Lemma 3.9. *Suppose (H). The following implications hold true.*

- *If there exists a* $y < 0$ *such that* $\tilde{f}(y) < \tilde{F}(y)$ *then* $\tilde{f}(x) < 0$ *for* $x \in (-\delta, 0)$ *for some* $\delta > 0$.
- *If there exists a* $y > 0$ *such that* $\tilde{f}(y) < \tilde{F}(y)$ *then* $\tilde{f}(x) < 0$ *for* $x \in (0, \delta)$ *for some* $\delta > 0$.

Proof. Again, we will prove the first part only. From Lemma 3.6 we infer

$$\tilde{f}(x) < \tilde{F}(x), \quad x \in (-\infty, 0). \tag{3.23}$$

Suppose, on the contrary, that there is an increasing sequence $(x_n : n \in \mathbb{N})$ tending to 0 as $n \to \infty$, and such that $\tilde{f}(x_n) \geq 0$ for $n \in \mathbb{N}$. Choose $z \leq 2x_1$. We have $x_n - z \geq |x_n|$, $n \in \mathbb{N}$. Using Lemma 3.8 we obtain

$$\tilde{f}(z) = H_{\tilde{f}(x_n)}(x_n - z) = \tilde{f}(x_n)g(x_n - z) + \tilde{F}(x_n - z).$$

Letting $n \to \infty$ in the above, we get $\tilde{f}(z) = \tilde{F}(z)$, contrary to (3.23). □

Up to now, we have shown the following corollary.

Corollary 3.10. *Suppose (H). Then we have the following possibilities:*

(i) $$\tilde{f} = \tilde{F};$$

(ii) $$\begin{cases} \tilde{f}(x) = \tilde{F}(x), & x \in [0, \infty), \\ \tilde{f}(x) < \tilde{F}(x), & x \in (-\infty, 0), \\ \tilde{f}(x) < 0, & x \in (-\delta, 0), \text{ for some } \delta > 0; \end{cases}$$

(iii) $$\begin{cases} \tilde{f}(x) = \tilde{F}(x), & x \in (-\infty, 0], \\ \tilde{f}(x) < \tilde{F}(x), & x \in (0, \infty), \\ \tilde{f}(x) < 0, & x \in (0, \delta), \text{ for some } \delta > 0. \end{cases}$$

In view of Corollary 3.5 we can consider further only cases (ii) and (iii). Moreover, in each case we distinguish two possibilities:

(1) either $\tilde{f}$ is negative on the whole half line $\mathbb{R}_-$, $\mathbb{R}_+$, respectively; or
(2) it is not.

Finally, we will determine the formula for $\tilde{f}$ (and f) according to cases (U), (V) and (W). Namely, we consider the following possibilities.

(1U(ii)) $\tilde{F}$, F, g and h are given by (U), (ii) holds and $\tilde{f}(x) < 0$ for every $x \in (-\infty, 0)$.
(1V(ii)) $\tilde{F}$, F, g and h are given by (V), (ii) holds and $\tilde{f}(x) < 0$ for every $x \in (-\infty, 0)$.
(1W(ii)) $\tilde{F}$, F, g and h are given by (W), (ii) holds and $\tilde{f}(x) < 0$ for every $x \in (-\infty, 0)$.
(1U(iii)) $\tilde{F}$, F, g and h are given by (U), (iii) holds and $\tilde{f}(x) < 0$ for every $x \in (0, \infty)$.
(1V(iii)) $\tilde{F}$, F, g and h are given by (V), (iii) holds and $\tilde{f}(x) < 0$ for every $x \in (0, \infty)$.
(1W(iii)) $\tilde{F}$, F, g and h are given by (W), (iii) holds and $\tilde{f}(x) < 0$ for every $x \in (0, \infty)$.
(2U(ii)) $\tilde{F}$, F, g and h are given by (U), (ii) holds and there is $x < 0$ such that $\tilde{f}(x) = 0$.

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(2V(ii)) $\tilde{F}$, F, g and h are given by (V), (ii) holds and there is $x < 0$ such that $\tilde{f}(x) = 0$.

(2W(ii)) $\tilde{F}$, F, g and h are given by (W), (ii) holds and there is $x < 0$ such that $\tilde{f}(x) = 0$.

(2U(iii)) $\tilde{F}$, F, g and h are given by (U), (iii) holds and there is $x > 0$ such that $\tilde{f}(x) = 0$.

(2V(iii)) $\tilde{F}$, F, g and h are given by (V), (iii) holds and there is $x > 0$ such that $\tilde{f}(x) = 0$.

(2W(iii)) $\tilde{F}$, F, g and h are given by (W), (iii) holds and there is $x > 0$ such that $\tilde{f}(x) = 0$.

First, we deal with (1U(ii)). Fix $x < 0$. By (2.1) we obtain

$$\max\{\tilde{f}(2x), \tilde{f}(0)\} = \tilde{f}(x)g(x) + \tilde{F}(x),$$

whence, since $\tilde{f}(2x) < 0 = \tilde{f}(0)$ and $g(x) > 0$,

$$\tilde{f}(x) = -\frac{\tilde{F}(x)}{g(x)}, \qquad x < 0. \tag{3.24}$$

Therefore,

$$\tilde{f}(x) = -\frac{Ce^{A|x|} - C}{e^{A|x|}} = -C + Ce^{Ax}, \qquad x < 0;$$

but

$$\tilde{f}(x) = \tilde{F}(x), \qquad x \geq 0, \tag{3.25}$$

so

$$\tilde{f}(x) = -C + Ce^{Ax}, \qquad x \in \mathbb{R},$$
$$f(x) = -C + Ce^{Ax} + f(0), \qquad x \in \mathbb{R}.$$

Similarly, using formulas (3.24) and (3.25), we obtain

(1V(ii)) $$f(x) = Ax + f(0), \qquad x \in \mathbb{R},$$

(1W(ii)) $$f(x) = C - Ce^{-Ax} + f(0), \qquad x \in \mathbb{R}.$$

Analogously, one can show that in cases (1U(iii)), (1V(iii)) and (1W(iii))

$$\tilde{f}(x) = -\frac{\tilde{F}(x)}{g(x)}, \qquad x > 0,$$

and

$$\tilde{f}(x) = \tilde{F}(x), \qquad x \leq 0;$$

hence

(1U(iii)) $$f(x) = -C + Ce^{-Ax} + f(0), \qquad x \in \mathbb{R},$$

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$$\text{(1V(iii))} \qquad f(x) = -Ax + f(0), \quad x \in \mathbb{R},$$

$$\text{(1W(iii))} \qquad f(x) = C - Ce^{Ax} + f(0), \quad x \in \mathbb{R}.$$

Compare points (4) and (5) from Theorem 3.1.

Secondly, we will consider case (2U(ii)). Put

$$x_0 := \tfrac{1}{2}\max\{x < 0 : \tilde{f}(x) = 0\}.$$

Of course, $x_0 < 0$, $\tilde{f}(2x_0) = 0$ and $\tilde{f}(x) < 0$ for $x \in (2x_0, 0)$. We will calculate the formula for $\tilde{f}$ in a few steps.

– We already know that

$$\tilde{f}(x) = \tilde{F}(x), \quad x \geq 0. \tag{3.26}$$

– Suppose $x < 2x_0$. Notice that

$$\tilde{f}(2x_0 + y) < \tilde{F}(y), \quad y > 0. \tag{3.27}$$

Indeed, either $y \in (0, -2x_0)$ and then $2x_0 + y < 0$, whence

$$\tilde{f}(2x_0 + y) < 0 < \tilde{F}(y);$$

or $y \geq -2x_0$ and then $2x_0 + y \geq 0$, so consequently

$$\tilde{f}(2x_0 + y) = \tilde{F}(2x_0 + y) < \tilde{F}(y),$$

as $\tilde{F}$ increases on $[0, \infty)$.

From (2.1) we have

$$\max\{\tilde{f}(2x_0 + y), \tilde{f}(2x_0 - y)\} = \tilde{f}(2x_0)g(y) + \tilde{F}(y) = \tilde{F}(y).$$

On account of (3.27) we infer

$$\tilde{f}(2x_0 - y) = \tilde{F}(y), \quad y > 0,$$

which can be reformulated as

$$\tilde{f}(x) = \tilde{F}(2x_0 - x), \quad x < 2x_0. \tag{3.28}$$

Therefore,

$$\tilde{f}(x) = C(e^{A|2x_0 - x|} - 1), \quad x < 2x_0.$$

– Suppose $x \in (x_0, 0)$. Choose $y \in (-x, x - 2x_0)$. By (2.1) we have

$$\max\{\tilde{f}(x + y), \tilde{f}(x - y)\} = \tilde{f}(x)g(y) + \tilde{F}(y). \tag{3.29}$$

Since $x + y > 0$ we have $\tilde{f}(x + y) = \tilde{F}(x + y) > 0$; moreover, $x - y \in (2x_0, 0)$, and hence $\tilde{f}(x - y) < 0$. Therefore, (3.29) implies

$$\tilde{F}(x + y) = \tilde{f}(x)g(y) + \tilde{F}(y).$$

Consequently,

$$\tilde{f}(x) = \frac{\tilde{F}(x + y) - \tilde{F}(y)}{g(y)}, \quad x \in (x_0, 0), y \in (-x, x - 2x_0). \tag{3.30}$$

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Hence

$$\tilde{f}(x) = C(e^{Ax} - 1), \qquad x \in (x_0, 0).$$

– Suppose $x \in (2x_0, x_0)$ and choose $y \in (x - 2x_0, -x)$. Then

$$\max\{\tilde{f}(x+y), \tilde{f}(x-y)\} = \tilde{f}(x)g(y) + \tilde{F}(y).$$

Hence, because $\tilde{f}(x+y) < 0$ (since $x + y \in (2x_0, 0)$) and $\tilde{f}(x-y) > 0$ (we have $x - y < 2x_0$, and so $\tilde{f}(x-y) = \tilde{F}(2x_0 - (x-y)) > 0$), we infer $\tilde{f}(x-y) = \tilde{f}(x)g(y) + \tilde{F}(y)$. Therefore,

$$\tilde{f}(x) = \frac{\tilde{F}(2x_0 - (x-y)) - \tilde{F}(y)}{g(y)}, \qquad x \in (2x_0, x_0),\ y \in (x - 2x_0, -x). \tag{3.31}$$

We can calculate that

$$\tilde{f}(x) = C(e^{A(x_0 - x)} - 1), \qquad x \in (2x_0, x_0).$$

– Of course, we also have $\tilde{f}(x_0) = C(e^{Ax_0} - 1)$.

Finally, we can summarize the formula for $\tilde{f}$:

$$\tilde{f}(x) = C(e^{A(|x - x_0| + x_0)} - 1), \qquad x \in \mathbb{R}.$$

Therefore,

(2Uii) $$f(x) = C(e^{A(|x - x_0| + x_0)} - 1) + f(0), \qquad x \in \mathbb{R}.$$

Similarly, in cases (2V(ii)) and (2W(iii)), using (3.26), (3.28), (3.30) and (3.31), we obtain

(2V(ii)) $$f(x) = A(|x - x_0| + x_0) + f(0), \qquad x \in \mathbb{R},$$

(2W(ii)) $$f(x) = C(1 - e^{-A(|x - x_0| + x_0)}) + f(0), \qquad x \in \mathbb{R}.$$

Analogously, in cases (2U(iii)), (2V(iii)) and (2W(iii)), with $x_0 := \frac{1}{2}\min\{x > 0 : \tilde{f}(x) = 0\}$, we get the formulas:

$$\tilde{f}(x) = \tilde{F}(x), \qquad x \le 0;$$
$$\tilde{f}(x) = \tilde{F}(x - 2x_0), \qquad x > 2x_0;$$
$$\tilde{f}(x) = \frac{\tilde{F}(x-y) - \tilde{F}(y)}{g(y)}, \qquad x \in (0, x_0), y \in (x, 2x_0 - x);$$
$$\tilde{f}(x) = \frac{\tilde{F}(x + y - 2x_0) - \tilde{F}(y)}{g(y)}, \qquad x \in (x_0, 2x_0), y \in (2x_0 - x, x).$$

Hence,

(2U(iii)) $$f(x) = C(e^{A(|x - x_0| - x_0)} - 1) + f(0), \qquad x \in \mathbb{R},$$

$$f(x) = A(|x - x_0| - x_0) + f(0), \quad x \in \mathbb{R}, \tag{2V(iii)}$$

$$f(x) = C(1 - e^{-A(|x-x_0|-x_0)}) + f(0), \quad x \in \mathbb{R}. \tag{2W(iii)}$$

Compare points (2) and (3) from Theorem 3.1.

This completes the proof. □

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