# AN OBSERVATION ON THE DIRICHLET PROBLEM AT INFINITY IN RIEMANNIAN CONES

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**Abstract.** In this short paper, we show a sufficient condition for the solvability of the Dirichlet problem at infinity in Riemannian cones (as defined below). This condition is related to a celebrated result of Milnor that classifies parabolic surfaces. When applied to smooth Riemannian manifolds with a special type of metrics, which generalize the class of metrics with rotational symmetry, we obtain generalizations of classical criteria for the solvability of the Dirichlet problem at infinity. Our proof is short and elementary: it uses separation of variables and comparison arguments for ODEs.

## §1. Introduction

We define a Riemannian cone as follows. Let $(N, g_N)$ be an $(n-1)$-dimensional smooth closed Riemannian manifold, and consider the quotient space

$$M = (N \times [0, \infty)) / (N \times \{0\}),$$

endowed with a complete metric that can be written as

$$g = dr^2 + \phi(r)^2 g_N, \tag{1}$$

where $g_N$ is any smooth metric on $N$. We shall assume that $\phi : [0, \infty) \longrightarrow [0, \infty)$ is smooth, with $\phi(r) > 0$ if $r > 0$, and that $\phi(0) = 0$ and $\phi'(0) = 1$. When $N = \mathbb{S}^{n-1}$ and thus $M$ is a manifold, the conditions imposed on $\phi$ guarantee that the metric $g$ can be extended, as a metric, smoothly up to the pole or vertex of $M$, as we shall refer to the equivalence class of $N \times \{0\}$ (see [10]). It is usual to define the cone metric using $g = r^2$, so our definition of a Riemannian cone is a bit more general. Furthermore, if $N = \mathbb{S}^{n-1}$ and $g$ is smooth up to the pole, with $g_N$ any smooth metric on the unit $(n-1)$-dimensional sphere, this definition of a cone contains a family of metrics in $\mathbb{R}^n$ that includes those with rotational symmetry: recall that $g$ has rotational symmetry when $N = \mathbb{S}^{n-1}$ is endowed with the round metric. Notice that the metrics defined by (1) are also referred to in the literature as warped products.

The Laplacian in a cone can be written for $r > 0$ as

$$\Delta_g = \frac{\partial^2}{\partial r^2} + (n-1) \frac{\phi'}{\phi} \frac{\partial}{\partial r} + \frac{1}{\phi^2} \Delta_N,$$

where $\Delta_N$ is the Laplacian in $N$. A function $u : M \longrightarrow \mathbb{R}$ is called *harmonic* if it satisfies $\Delta_g u = 0$ in $N \times (0, \infty)$, and it is continuous in $M$. Notice that when $M$ is a smooth manifold and $g$ is—or extends to—a smooth Riemannian metric, then the continuity requirement implies that $u$ is actually smooth and that it is harmonic if it satisfies the usual definition,

that is, if $\Delta_g u = 0$ in the whole manifold. Observe also that given a function

$$u : N \times [0, \infty) \longrightarrow \mathbb{R},$$

which is continuous and which satisfies $\Delta_g u = 0$ in $N \times (0, \infty)$, it can be passed to the quotient space $M$ as a harmonic function, as defined above, as long as it is constant on $N \times \{0\}$, and vice versa: a function $u : M \longrightarrow \mathbb{R}$ can be lifted to a function $\tilde{u} : N \times [0, \infty) \longrightarrow \mathbb{R}$ which satisfies $\Delta_g \tilde{u} = 0$ in $N \times (0, \infty)$, is continuous in $N \times [0, \infty)$, and is constant in $N \times \{0\}$.

As is customary, given a manifold $V$, we denote by $C^\infty(V)$ the space of smooth real-valued functions in $V$ and by $C(V)$ the set of real-valued continuous functions.

We prove the following result (see below for applications).

THEOREM 1. *Assume that*

$$\int_1^\infty \frac{1}{\phi(s)} \, ds < \infty, \tag{2}$$

*and, if $dim(N) \geq 2$, that there is an $R_0 > 0$ such that $\phi'(r) \geq 0$ if $r \geq R_0$. Then, for any $f : N \longrightarrow \mathbb{R}$ regular enough (to avoid technicalities, we can just set that $f \in C^\infty(N)$), the Dirichlet problem is uniquely uniformly solvable at infinity. By the Dirichlet problem being uniformly solvable we mean that there is a harmonic function $u : M \longrightarrow \mathbb{R}$ such that*

$$\lim_{r \to \infty} u(\omega, r) = f(\omega) \quad \text{uniformly.}$$

*We call $u$ the harmonic extension of f.*

The main ingredient in our proof of Theorem 1 is separation of variables (for a precursor to our proof, see Dodziuk's work on harmonic $L^2$-forms in [7]). In fact, we shall show that a whole family of harmonic functions can be written as

$$u(\omega, r) = \sum_m \varphi_m(r) \left( \sum_k c_{m,k} f_{m,k}(\omega) \right),$$

where $f_{m,k}$ are eigenfunctions of the Laplacian $\Delta_{g_N}$ of $N$ with eigenvalue $\lambda_m^2$. Then we show that the $\varphi_m$ can be chosen to be nonnegative, nondecreasing, and bounded. In the case when the sectional curvature satisfies $K \leq -1$ everywhere, we even have a more explicit estimate, namely,

$$0 \leq \varphi_m(r) \leq A_m \tanh^{\lambda_m} r, \tag{3}$$

for a convenient constant $A_m$. Since a basis for $L^2(N)$ (we give a brief discussion on the definition of this space below, right before Theorem 2) can be constructed from a family of eigenfunctions of the Laplacian of $N$, then we can solve the Dirichlet problem for any given $f \in C^\infty(N)$ as boundary data at infinity. By smooth enough in the statement of Theorem 1, we mean that $f$ *must be regular enough so that its expansion in eigenfunctions of the Laplacian $\Delta_N$ converges absolutely*. The fact that enough regularity of $f$ implies the absolute convergence of its expansion in eigenfunctions of the Laplacian was shown by Peetre in [17].

Proving uniqueness of the harmonic extension of $f$ in the case of $M$ not being a manifold is a bit subtle and, in our proof, it requires showing that the only possible value that $u$ can

take at the vertex of the cone is actually the average of $f$, that is,

$$u\left(0\right) = \overline{f} := \frac{1}{\mathrm{Vol}\left(N\right)} \int_N f\left(\omega\right) dV_{g_N}\left(\omega\right).$$

In a manifold, the fact that $u\left(0\right) = \overline{f}$ follows from Green's identity. In the case of $M$ not being a manifold, our proof also uses Green's identity plus a little extra argument.

The definition of uniform solvability at infinity we used in the statement of Theorem 1 is stronger than the one commonly used when studying the Dirichlet problem at infinity in Cartan–Hadamard manifolds; that is, when both definitions apply, uniform solvability implies solvability in the sense of Choi [5]. Theorem 1 implies that cones with metrics satisfying (2) have a wealth of nontrivial bounded harmonic functions.

For applications of our main result, we specialize to the case when $N = \mathbb{S}^{n-1}$ endowed with an arbitrary smooth metric so that the cone is a smooth Riemannian manifold diffeomorphic to $\mathbb{R}^n$. From Theorem 1, given $f \in C\left(\mathbb{S}^{n-1}\right)$, taking a sequence $\{f_n\}_n$ of smooth functions such that $f_n \to f$ uniformly, then solving the Dirichlet problem for each $f_n$, we obtain a uniformly convergent sequence of harmonic functions (by Choi's asymptotic maximum principle [see Proposition 2.5 in [5]], but see also our discussion in §2.1), and thus we have the following corollary.

COROLLARY 1. *Assume that (2) holds and, if $dim\left(\mathbb{S}^{n-1}\right) \geq 2$, that there is an $R_0 \geq 0$ such that if $r > R_0$, then $\phi'\left(r\right) \geq 0$. Then, for any continuous $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$, the Dirichlet problem is uniquely solvable at infinity. By this, we mean that there is a harmonic function $u : M \longrightarrow \mathbb{R}$ such that*

$$\lim_{r \to \infty} u\left(\omega, r\right) = f\left(\omega\right) \quad \textit{in the cone topology(see §2).}$$

Corollary 1 seems to be new: notice that we do not require the manifold to be of nonpositive curvature, that is, to be Cartan–Hadamard; however, if we assume the manifold to be Cartan–Hadamard, the hypothesis $\phi' > 0$ can be dropped, as it is automatically satisfied.

From the previous corollary, we obtain the following.

COROLLARY 2. *Let $g$ be a metric of the form (1). If there is an $\epsilon > 0$ such that $-\phi''/\phi \leq -\left(1+\epsilon\right)/r^2 \log r$, for large enough $r$, and $\phi$ is unbounded, then for any $f \in C\left(N\right)$, the Dirichlet problem is uniquely solvable at infinity.*

The proof of this corollary is as follows. By Milnor's argument in [14], we have that, under the hypotheses of the corollary, it can be shown that $\phi$ satisfies

$$\int_1^\infty \frac{1}{\phi\left(s\right)} ds < \infty.$$

The condition $\phi' \geq 0$ is automatically satisfied for large enough $r$: if $\phi' < 0$ for all $r$ large enough, then $1/\phi$ would be increasing for $r$ large enough and then it would not be integrable, and hence at some point, say $r_1$, we must have $\phi'\left(r_1\right) \geq 0$, and since the radial curvature $-\phi''/\phi$ is negative, $\phi'$ will be nonnegative from then on; thus, the hypotheses of Corollary 1 hold and Corollary 2 follows. Again, notice that Corollary 2 includes metrics that are *not necessarily rotationally symmetric*, since in the case of $N = \mathbb{S}^{n-1}$ it is not required for the metric carried by the sphere to be the round metric nor to be conformal to it; this is in contrast to the results proved in [5], [13], which require either one (especially [5] where

rotational symmetry is strongly required in the calculations [see §3]; on the other hand, in [13], rotational symmetry does not seem to be essential, in the sense that the metric that $\mathbb{S}^{n-1}$ carries does not need to be the round metric, but the author only proves the transience of the manifold). In this sense, Corollary 2 is new for dimensions greater than or equal to 4. As an aside comment: when Milnor's criterion is used, the "$\phi$ unbounded" part in its statement is usually replaced by saying that the manifold has everywhere nonpositive curvature, and so, again, the assumption of $\phi$ being unbounded might be replaced by assuming the stronger condition that the manifold is a Cartan–Hadamard manifold, which is the assumption made by Choi in [5].

Regarding (2), in a beautiful work [15] (which in turn has been generalized in [16]), Neel shows that in Cartan–Hadamard surfaces, if we write the metric as

$$dr^2 + J\left(r,\theta\right)^2 d\theta^2, \tag{4}$$

and

$$\int_1^\infty \frac{1}{J\left(r,\theta\right)} \, dr < \infty, \tag{5}$$

then the Dirichlet problem at infinity is uniquely solvable. Neel's approach is probabilistic, and his result strengthens the following result of Doyle, at least in the case of surfaces, which is proved in [11]: for a Riemannian manifold with a metric written as (4), condition (5) implies transience.

Furthermore, our proof is quite elementary, perhaps strikingly simple when compared with the published proofs of the results mentioned above, and it reveals that there is a notion of solvability for the Dirichlet problem at infinity on manifolds for boundary data in $L^2$ that has not been treated before in the literature (to the best of our knowledge), and which perhaps deserves more consideration. Recall that given a Riemannian manifold $(N, g_N)$, the space of square integrable functions on $N$ is defined as the set of $f : N \longrightarrow \mathbb{R}$ such that

$$\|f\|_{L^2(N)}^2 := \int_N |f\left(\omega\right)|^2 \, dV_{g_N}\left(\omega\right) < \infty. \tag{6}$$

We shall denote this space by $L^2\left(N\right)$. It is not difficult to show that when $N$ is a compact manifold, if $f$ is square integrable with respect to a given smooth volume form, it is also square-integrable with respect to any smooth volume form, and also, the norms induced by any pair of Riemannian metrics, via its volume form, on the space of square integrable functions are all equivalent: this is why we are not including the metric explicitly in our notation for the space of square-integrable functions on $N$.

We thus have the following result on solvability for boundary data at infinity, when the data belong to $L^2\left(\mathbb{S}^{n-1}\right)$.

THEOREM 2. *Assume that $(M, g)$, with $g$ as in (4), is a smooth Riemannian manifold (thus, $N = \mathbb{S}^{n-1}$). Furthermore, assume that there is an $R_0$ such that $\phi'\left(r\right) \geq 0$ if $r > R_0$, and that (2) holds. Then, for any $f \in L^2\left(\mathbb{S}^{n-1}\right)$, the Dirichlet problem is $L^2$-solvable at infinity. By the Dirichlet problem being $L^2$-solvable we mean that there is a function $u : M \longrightarrow \mathbb{R}$ which is harmonic in $M$, and such that*

$$\lim_{r \to \infty} u\left(\omega, r\right) = f\left(\omega\right) \quad in \quad L^2\left(\mathbb{S}^{n-1}\right).$$

The Dirichlet problem at infinity has a rich history full of deep and interesting results (see, e.g., [1], [2], [5], [12], [18]). Theorem 1 and its consequences represent an improvement in the study of the Dirichlet problem at infinity in the case of rotationally symmetric metrics as given in [5], and it is a natural extension of the classical result of Milnor in [14]. Furthermore, our main result bears some resemblance with that of March in [13], where under the hypothesis

$$\int_1^\infty \phi(r)^{n-3} \int_r^\infty \phi(\rho)^{1-n} \, d\rho \, dr < \infty, \tag{7}$$

and rotational symmetry, the author proves the existence of nonconstant bounded harmonic functions in $M$. The reader will find the proof of Theorem 1 (our main result) in §3 and the proof of Theorem 2 in §4.

## §2. Preliminaries

Here, we define what is to be understood as to solve the Dirichlet problem at infinity. First, we define a compactification for $M$. To this end, define the set $\overline{M}$:

$$\overline{M} = N \times [0, \infty] \, / \, (N \times \{0\}),$$

where $[0, \infty]$ is a compactification of $[0, \infty)$ which is homeomorphic to $[0, 1]$. The subspace $\partial_\infty M := N \times \{\infty\}$ plays the role of a boundary, and in fact, when $N$ is homeomorphic to $\mathbb{S}^{n-1}$, $\overline{M}$ has the structure of a topological manifold with boundary. Given the previous definition, a way of defining that *the Dirichlet problem is solvable at infinity* is as follows: given $f \in C(N)$, there exists a function $u : \overline{M} \to \mathbb{R}$ which is in $C(\overline{M})$, is harmonic in $M$, and such that its restriction to $\partial_\infty M$ is $f$. In the case of Cartan–Hadamard manifolds, this definition of solvability coincides with the definition of solvability given by Choi, who uses the *cone topology* as defined by Eberlein and O'Neill [8], and which is equivalent, in the sense of homeomorphism, to the one defined for the compactification above: the resulting spaces in both cases are homeomorphic to the closed $n$-ball.

A stronger definition of solvability was used in the statement of Theorem 1; let us recall it. *We shall say that the Dirichlet problem is uniformly solvable at infinity* if there is a harmonic function $u : M \longrightarrow \mathbb{R}$ such that

$$\lim_{r \to \infty} u(\omega, r) = f(\omega)$$

uniformly on $\omega \in N$. Notice that using this definition, if $f \in C(N)$, then we can extend $u$ to $\overline{M}$ continuously and hence this definition of solvability implies the one given above. In the case of $N = \mathbb{S}^{n-1}$ (not necessarily with the round metric), uniform solvability implies solvability in the sense of Choi in [5].

Observe that with the definitions of solvability given in the previous paragraph, if the Dirichlet problem is solvable at infinity for given continuous data, then the corresponding harmonic extension is bounded; thus, solving the Dirichlet problem at infinity gives a method for proving the existence of bounded nonconstant harmonic functions.

Furthermore, in the case of $M$ being a smooth Riemannian manifold, we can define the concept of solvability at infinity for the Dirichlet problem with boundary data in $L^2$. In this case, given $f \in L^2(\mathbb{S}^{n-1})$, we say that the Dirichlet problem is $L^2$-solvable at infinity with

boundary data $f \in L^2\left(\mathbb{S}^{n-1}\right)$ if there is a function $u : M \longrightarrow \mathbb{R}$ which is harmonic, and such that

$$\lim_{r \to \infty} u\left(\omega, r\right) = f\left(\omega\right) \quad \text{in} \quad L^2\left(\mathbb{S}^{n-1}\right).$$

### 2.1 The Maximum Principle

To prove uniqueness, we shall make use of the maximum principle. What follows is not new, but we have written it down for the convenience of the reader and for easy reference (see also Choi's asymptotic maximum principle [5]).

THEOREM 3. *Let $\overline{M} = N \times [0, \infty]$ such that $N \times (0, \infty)$ is endowed with the metric*

$$g = dr^2 + \phi\left(r\right)^2 g_N,$$

*with $\phi$ smooth and $\phi > 0$ on $(0, \infty)$. Let $u : \overline{M} \longrightarrow \mathbb{R}$ be continuous function which is harmonic in $N \times (0, \infty)$ with respect to the metric $g$. Then, $u$ reaches its maximum and its minimum at $N \times \{0\} \cup N \times \{\infty\}$. If $M$ is a smooth Riemannian manifold (this, of course, requires that $g$ extends smoothly to $M$ at $r = 0$), and $u : \overline{M} \longrightarrow \mathbb{R}$ is harmonic, then it reaches its maximum and minimum at $\partial_\infty M$.*

*Proof.* Assume that $u$ reaches its maximum at a point in $N \times (0, \infty)$, and let the value of this maximum (minimum) be $a$. Consider the set

$$A = \{(\omega, r) \in N \times (0, \infty) : u\left(\omega, r\right) = a\}.$$

By the strong maximum principle for elliptic operators (Theorem 3.5 in [9]), there is a small ball around any point in $A$ where $u$ is constant and equal to $a$. Thus, $A$ is open in $N \times (0, \infty)$. However, $A$ is also closed by the continuity of $u$, and as $N \times (0, \infty)$ is connected, we must have that $A = N \times (0, \infty)$, and thus $u$ is constant in $N \times (0, \infty)$ and, as a consequence, by continuity, also in $N \times [0, \infty]$: this implies the first part of the statement of the theorem. In the case of $M$ being a smooth Riemannian manifold, if $u$ is harmonic, it is smooth in $M$, and thus if it reaches its maximum in $M$, similar arguments as above show that $u$ must be constant; the same holds if $u$ reaches its minimum in $M$. Thus, we conclude that the maximum and the minimum are reached at $\partial_\infty M$. $\square$

As is well known, an important consequence of the maximum principle is a uniqueness result for the Dirichlet problem, and which we shall use in what follows.

COROLLARY 3. *Let $g$ be as in the previous theorem. Let $u_1, u_2 : N \times [0, \infty] \longrightarrow \mathbb{R}$ be continuous functions which are harmonic in $N \times (0, \infty)$. If $u_1 = u_2$ on $N \times \{0\} \cup N \times \{\infty\}$, then $u_1 \equiv u_2$ in $N \times [0, \infty]$. If $M$ is a smooth Riemannian manifold, and $u_1, u_2 : \overline{M} \longrightarrow \mathbb{R}$ are harmonic in $M$, then if $u_1 = u_2$ in $\partial_\infty M$, then $u_1 \equiv u_2$.*

### §3. A proof of Theorem 1

We first solve the following problem in $N \times [0, \infty]$. Given $f \in C^\infty\left(N\right)$, we shall construct a harmonic function $u$ in $N \times (0, \infty)$ such that as $r \to \infty$, $u\left(\omega, r\right) \to f\left(\omega\right)$ uniformly, and such that

$$u\left(\omega, r\right) \to \overline{f} := \frac{1}{\text{Vol}\left(N\right)} \int_N f \, dV_{g_N}$$

also uniformly as $r \to 0$, so that $u$ passes to the quotient $\overline{M}$ as a continuous function which is harmonic, as it has been defined in the introduction, and such that it also solves the Dirichlet problem at infinity. To prove this, we use separation of variables.

We let $f_{m,k}$, $k = 0, 1, 2, \ldots, k_m$, be eigenfunctions of the $m$th eigenvalue, $\lambda_m^2$, $m = 0, 1, 2, \ldots$, such that the set $\{f_{m,k}\}_{m,k}$ is an orthogonal basis for $L^2(N)$ with respect to the inner product induced by (6). We use the convention that $\lambda_m > 0$, if $m > 0$, and that $\lambda_0 = 0$ is the trivial eigenvalue of the Laplacian of $N$ whose eigenfunctions are the constant functions; we shall also fix the convention $f_{0,0} \equiv 1$. Next, we must find functions $\varphi_m$ so that the product $\varphi_m(r) f_{m,k}(\Omega)$ satisfies $\Delta_g(\varphi_m f_{m,k}) = 0$; this time we use the convention $\varphi_0 \equiv 1$. Then, it is elementary to prove that the equation to be satisfied by $\varphi_m$, $m > 0$, is

$$\mathcal{L}_m \varphi_m := \varphi_m'' + (n-1) \frac{\phi'}{\phi} \varphi_m' - \frac{\lambda_m^2}{\phi^2} \varphi_m = 0. \tag{8}$$

First, we show that there is a solution to (8) such that $\varphi_m(0) = 0$, if $m \neq 0$, and that we can choose $\varphi_m > 0$ near 0 (for $m = 0$, as said before, we just choose the constant function 1). Indeed, since $\phi(r) \sim r$, $r = 0$ is a regular singular point of the equation, and thus, near $r = 0$, a solution can be written as

$$q(r) = r^s p(r),$$

where $s = \dfrac{-(n-2) + \sqrt{(n-2)^2 + 4\lambda_m^2}}{2} > 0$ satisfies the indicial equation

$$s(s-1) + (n-1)s - \lambda_m^2 = 0,$$

$p$ is smooth function, and $p(0) = 1$ (see page 45 in [3], a classical paper by Bôcher, and Chapters 4 and 5 in [6]). From this, our assertion follows.

Next, we show that $\varphi_m$ is nondecreasing. From its general form, it is clear that $\varphi_m'(r) > 0$ near 0. Assume then that at some point $r_0 > 0$, $\varphi_m'(r_0) = 0$ occurs for the first time. Then, since for $0 < r < r_0$ $\varphi_m(r) > 0$, using equation (8) shows that $\varphi_m''(r_0) > 0$, which in turn implies that $\varphi'(r) > 0$ a little beyond $r_0$. This shows that

$$\sup \{r : \varphi_m'(\rho) \geq 0 \quad \text{for all} \quad \rho \in [0,r]\} = \infty,$$

and thus our claim follows.

In order to show that $\varphi_m$ is bounded, define the function

$$\eta_m(r) = \exp\left[-\int_r^\infty \frac{\lambda_m}{\phi(s)} \, ds\right].$$

Of course, here is where we need the fundamental assumption that

$$\int_1^\infty \frac{1}{\phi(s)} \, ds < \infty,$$

which implies that $\eta_m$ is bounded on $[R_0, \infty)$ for any $R_0 > 0$ (the bound depends on $R_0$).

A straightforward computation shows that if $r \geq R_0$, $R_0 > 0$ as in the statement of Theorem 1, then

$$\mathcal{L}_m \eta_m = (n-2) \frac{\phi'}{\phi} \lambda_m \eta_m \geq 0. \tag{9}$$

This is why we need the hypothesis $\phi'(r) \geq 0$ for $r$ large enough. We note in passing that when $\dim(N) = n - 1 = 1$, the $\eta_m$'s thus defined give explicit solutions to (8) (this was pointed out to me by J. E. Bravo): this is the reason why in dimension 2 the assumption $\phi' \geq 0$ is not required.

Next, we are going to use inequality (9) to show that for $r \geq R_0 > 0$, it holds that $0 \leq \varphi_m(r) \leq A_m \eta_m(r)$ for an appropriate constant $A_m$. So, choose $A_m > 0$ large enough so that

$$\varphi(R_0) < A_m \eta_m(R_0), \quad \text{and} \quad \varphi'_m(R_0) < A_m \eta'_m(R_0).$$

We now claim that $(A_m \eta_m - \varphi_m)'(r) \geq 0$ must also hold for all $r \geq R_0$. In order to prove our claim, let

$$r_0 = \sup \left\{ r : (A_m \eta_m - \varphi_m)'(r) \geq 0 \quad \text{in} \quad [R_0, r] \right\},$$

and let us show that $r_0 = \infty$. Indeed, assume that $r_0 < \infty$. By continuity, it is clear that $(A_m \eta_m - \varphi_m)'(r) \geq 0$ up to $r_0$ and equality holds at $r = r_0$. Hence, by our choice of $A_m$, it also holds that $h_m := A_m \eta_m - \varphi_m > 0$ up to $r_0$. Then, using (9), we have that

$$h''_m(r_0) \geq \frac{\lambda_m^2}{\phi^2} h_m(r_0) > 0,$$

and hence a bit beyond $r_0$ we would still have $h'_m > 0$, which leads to a contradiction, and thus we must have $r_0 = \infty$. This shows that for all $r \in [R_0, \infty)$, $(A_m \eta_m - \varphi_m)'(r) \geq 0$, which together with the fact that $\varphi(R_0) < A_m \eta_m(R_0)$ implies that $A_m \eta_m \geq \varphi_m$ for all $r \in [R_0, \infty)$, and this shows that $\varphi_m$ is bounded above.

Notice that in the case that the sectional curvature satisfies $K \leq -1$ everywhere, by the Bishop–Gromov theorem (which in the case of rotational symmetry reduces to a simple ODE comparison argument), $\phi(r) \geq \sinh r$, and hence we have that $0 \leq \varphi_m \leq A_m \tanh^{\lambda_m} r$, and the claimed estimate (3) holds. In any case, the previous estimates show that, by multiplying by appropriate constants, we may assume that $\lim_{r \to \infty} \varphi_m(r) = 1$.

Given $f \in C^\infty(N)$, we can represent it as

$$f(\omega) = \sum_{m \geq 0} \sum_k c_{m,k} f_{m,k}(\omega),$$

where the $f_{m,k}$'s are eigenfunctions of the Laplacian as defined above. Notice that the coefficient $c_{0,0}$ of $\varphi_0 f_{0,0}$ is the average value of $f$:

$$c_{0,0} = \frac{1}{\mathrm{Vol}(N)} \int_N f \, dV_{g_N}.$$

From the Fourier representation of $f$, we get a harmonic extension

$$u(\omega, r) = \sum_{m \geq 0} \varphi_m(r) \sum_k c_{m,k} f_{m,k}(\omega). \tag{10}$$

Since $f \in C^\infty(N)$, it is not difficult to show that $u$ is twice differentiable and that it satisfies $\Delta_g u = 0$ in $N \times (0, \infty)$. All we need to prove next is that $u$ satisfies the boundary conditions at infinity and that it is continuous at $r = 0$. We start by proving that the boundary condition at infinity is satisfied. Let $\epsilon > 0$, we estimate as follows: By the triangular

inequality,

$$|f(\omega) - u(\omega, r)| \leq \sum_{m \geq 0} (1 - \varphi_m(r)) \left| \sum_k c_{m,k} f_{m,k}(\omega) \right|.$$

Pick $H$ such that

$$\sum_{m \geq H} \left| \sum_k c_{m,k} f_{m,k}(\omega) \right| \leq \frac{\epsilon}{2}.$$

This can be done by Peetre's result on the absolute convergence of a Fourier series as soon as $f$ is smooth enough [17]. Let $R > 0$ be such that if $r \geq R$, for $m = 0, 1, \ldots, H$,

$$1 - \varphi_m(r) \leq \frac{\epsilon}{2L},$$

where $L$ bounds

$$\sum_{m \leq H} \left| \sum_k c_{m,k} f_{m,k}(\omega) \right|$$

from above. Under this considerations, we obtain that, for $r \geq R$,

$$|f(\omega) - u(\omega, r)| < \epsilon,$$

and our claim is now proved: the boundary condition at infinity is satisfied.

Notice that $u = \overline{f}$ at $r = 0$, so, to show continuity at $r = 0$, we must show that $u(\omega, r) \to \overline{f}$ as $r \to 0$. To do so, just observe that

$$\left| u(\omega, r) - \overline{f} \right| \leq \sum_{m > 0} \varphi_m(r) \left| \sum_k c_{m,k} f_{m,k}(\omega) \right|,$$

and as for $m > 0$, $\varphi_m(r) \to 0$ when $r \to 0$, by the dominated convergence theorem our assertion follows.

Hence, the function $u$ defined by (10) passes to the quotient and its value at $r = 0$ is given by $\overline{f}$. The uniqueness statement of Theorem 1 when $M$ is a smooth Riemannian manifold follows from Corollary 3. So all is left to show to finish the proof of Theorem 1 is to prove that there is uniqueness of the solution constructed above in the case of $M$ not being necessarily a manifold.

To prove uniqueness of the harmonic extension of $f$ when $M$ is not a manifold, we first show that the only possible value that a solution $u$ to the Dirichlet problem at infinity with boundary condition (at infinity) equals to $f$ can take at the vertex of the cone is $\overline{f}$, the average of $f$ over $N$. To proceed, we let

$$\overline{u}(r) = \frac{1}{\text{Vol}(N)} \int_N u(\omega, r) \, dV_{g_N}(\omega).$$

We claim that $\overline{u}$ is constant. Indeed, let $0 < r < r_1 < \infty$; then, by Green's theorem, we have that

$$\phi(r_1)^{n-1} \frac{d\overline{u}}{dr}(r_1) - \phi(r)^{n-1} \frac{d\overline{u}}{dr}(r) = \frac{1}{\text{Vol}(N)} \int_{N \times [r, r_1]} \Delta u(\omega, r) \, dV_g(\omega, r) = 0,$$

that is,

$$\frac{d\overline{u}}{dr}(r) = \frac{\phi(r_1)^{n-1}}{\phi(r)^{n-1}}\frac{d\overline{u}}{dr}(r_1) = \frac{A}{\phi(r)^{n-1}}.$$

Therefore, if $d\overline{u}(r_1)/dr \neq 0$, we would have that $\lim_{r\to 0^+}\overline{u}(r) = \pm\infty$. Indeed, by the Fundamental Theorem of Calculus,

$$\overline{u}(r) = \overline{u}(r_1) + \int_{r_1}^{r}\frac{A}{\phi(\rho)^{n-1}}\,d\rho.$$

However, near 0, $\phi(r)^{n-1} \sim r^{n-1}$, $n \geq 2$, and thus the previous identity shows our claim because, depending on the sign of $A$, which in turn depends on the sign of $\overline{u}'(r_1)$, we then would have that

$$\lim_{r\to 0^+}\int_{r_1}^{r}\frac{A}{\phi(\rho)^{n-1}}\,d\rho = \pm\infty.$$

However, by the continuity of $u$, we must have $\lim_{r\to 0^+}\overline{u}(r) = u(0)$. Therefore,

$$d\overline{u}(r_1)/dr = 0$$

must hold. Since $r_1$ is arbitrary, this shows that $\overline{u}$ is constant, and in fact, since we must have that $\lim_{r\to\infty}\overline{u} = \overline{f}$, that $u(0) = \overline{f}$. The maximum principle (Corollary 3) implies the uniqueness statement even in the case when the cone is not a smooth manifold.

## §4. $L^2$-solvability

If we only require $f \in L^2\left(\mathbb{S}^{n-1}\right)$, the harmonic extension $u$ constructed above is locally integrable and harmonic in the sense of distributions, and hence almost everywhere equal to a smooth harmonic function, and thus harmonic, and we can also show that $u(\omega, r) \to f(\omega)$ as $r \to \infty$ in $L^2\left(\mathbb{S}^{n-1}\right)$, and in consequence that the Dirichlet problem is $L^2$-solvable. Let us give a proof of these claims.

We let

$$u_l(\omega, r) = \sum_{m\leq l}\varphi_m(r)h_m(\omega),$$

where

$$h_m(\omega) = \sum_k c_{m,k}f_{m,k}(\omega).$$

First of all, the sequence $\{u_l\}_{l=1,2,3,\dots}$ converges in $L^2_{loc}(M)$. Indeed, for $R > 0$, we can compute as follows (we employ the notation $B_R(0)$ for the geodesic ball of radius $R > 0$ centered at the pole of the manifold):

$$\int_{B_R(0)}\left|\sum_{m_1 < m \leq m_2}\varphi_m(r)h_m(\omega)\right|^2 dV_g$$

$$=$$

$$\int_0^R \int_{\mathbb{S}^{n-1}} \sum_{m_1 < m, m' \le m_2} \varphi_m(r) \varphi_{m'}(r) h_{m'}(\omega) h_m(\omega) \, dV_{g_{\mathbb{S}^{n-1}}}(\omega) \, \phi(r) \, dr$$

$$=$$

$$\int_0^R \sum_{m_1 < m \le m_2} \varphi_m(r)^2 \|h_m\|_{L^2(\mathbb{S}^{n-1})}^2 \phi(r) \, dr$$

$$\le$$

$$C_R \sum_{m_1 < m \le m_2} \|h_m\|_{L^2(\mathbb{S}^{n-1})}^2 = C_R \left\| \sum_{m_1 < m \le m_2} h_m \right\|_{L^2(\mathbb{S}^{n-1})}^2,$$

where we have used the orthogonality in $L^2(\mathbb{S}^{n-1})$ of eigenfunctions of the Laplacian with different eigenvalues. Since $\sum_{m \le l} h_m \to f$ in $L^2(\mathbb{S}^{n-1})$, given $\epsilon > 0$, there is an $L$ such that whenever $m_2, m_1 \ge L$, then

$$C_R \left\| \sum_{m_1 < m \le m_2} h_m \right\|_{L^2(\mathbb{S}^{n-1})} < \epsilon,$$

which shows that $\{u_l\}_l$ is a Cauchy sequence in $L^2(B_R(0))$, which in turn implies that the sequence converges in $L^2(B_R(0))$. This proves our claim. We call $u$ the limit of the sequence $u_l$ in $L_{loc}^2(M)$. Notice that $u \in L_{loc}^1(M)$.

Next, we show that $u$ is weakly harmonic. This is standard: given $\varphi \in C_0^\infty(\mathbb{S}^{n-1})$ as each $u_l$ is harmonic, it then holds that

$$\int_M u_l(x) \Delta_g \varphi(x) \, dV_g = 0.$$

However, $u_l \to u$ in $L_{loc}^2(M)$, and so, since $\Delta_g \varphi \in C_0^\infty(M)$, we must have that

$$\int_M u(x) \Delta_g \varphi(x) \, dV_g = 0,$$

which is what we wanted to show. Since $u$ is weakly harmonic and locally integrable, Weyl's lemma applies and we can conclude that $u$ is almost everywhere equal to a proper harmonic function on $M$.

The following computation, which was originally suggested in [4], and that we reproduce for the convenience of the reader, shows that the boundary data are satisfied in an $L^2$-sense as described above. Again, using the orthogonality of the eigenfunctions of the Laplacian with different eigenvalues, we can estimate

$$\|f(\omega) - u(\omega, r)\|_{L^2(\mathbb{S}^{n-1})}^2 = \left\| \sum_m (1 - \varphi_m(r)) \sum_k c_{m,k} f_{k,m}(\omega) \right\|_{L^2(\mathbb{S}^{n-1})}^2$$

$$= \sum_m |1 - \varphi_m(r)|^2 \left\| \sum_k c_{m,k} f_{m,k}(\omega) \right\|_{L^2(\mathbb{S}^{n-1})}^2.$$

Pick $H$ such that

$$\sum_{m \geq H} \left\| \sum_k c_{m,k} f_{m,k}(\omega) \right\|_{L^2(\mathbb{S}^{n-1})}^2 \leq \frac{\epsilon^2}{8},$$

and let $R > 0$ such that if $r \geq R$, for $m = 0, 1, \ldots, M$, we have that

$$|1 - \varphi_m(r)| \leq \frac{\epsilon}{2K},$$

where $K$ is such that

$$\sum_{m \leq H} \left\| \sum_k c_{m,k} f_{m,k}(\omega) \right\|_{L^2(\mathbb{S}^{n-1})}^2 \leq K.$$

Notice that $|1 - \varphi_m(r)| \leq 1$. Therefore, putting all this together, we find that given any $\epsilon > 0$ for $r$ large enough,

$$\|f(\cdot) - u(\cdot, r)\|_{L^2(\mathbb{S}^{n-1})} < \epsilon,$$

which is what we wanted to prove.

## §5. Some remarks

In the case when $N$ is a Lie group, Taylor in [19] gave sufficient conditions for the eigenfunction expansion of $f$ to converge uniformly to $f$. For instance, if $N = \mathbb{S}^3$ with the round metric, then if $f \in C^{\frac{1}{2}}(\mathbb{S}^3)$, that is, if $f$ is Hölder continuous with exponent $\frac{1}{2}$, the Dirichlet problem is not just solvable, but uniformly solvable at infinity in $\mathbb{R}^4$ endowed with a rotationally symmetric metric such that the factor $\phi$ satisfies (2) and that $\phi' \geq 0$, or if $\mathbb{R}^4$ with the given metric is a Cartan–Hadamard manifold. On the other hand, if $N$ is a Lie group of dimension $4k$, it is only required that $f \in H^{2k}(N)$ for the limit $\lim_{r \to \infty} u(\omega, r) = f(\omega)$ to be uniform.

## References

[1] M. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature*, J. Differ. Geom. **18** (1983), 701–721.

[2] M. Anderson and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. Math. **121** (1985), 429–461.

[3] M. Bôcher, *On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic*, Trans. Amer. Math. Soc. **1** (1900), no. 1, 40–52.

[4] J. E. Bravo, J. C. Cortissoz, and D. Peters-Stein, *Some observations on Liouville's theorem and the Dirichlet problem at infinity on surfaces*, Lobachevskii J. Math. **43** (2022), 71–77.

[5] H. Choi, *In asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds*, Trans. Amer. Math. Soc. **281** (1984), no. 2, 691–716.

[6] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York–Toronto–London, 1955, xii + 429 pp.

[7] J. Dodziuk, $L^2$ *harmonic forms on rotationally symmetric Riemannian manifolds*, Proc. Amer. Math. Soc. **77** (1979), 395–400.

[8] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pac. J. Math. **46** (1973), 45–109.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Grundlehren Math. Wiss. **224**, Springer, Berlin, 1983, xiii + 513 pp.

[10] R. Greene and W. Wu, *Function Theory of Manifolds Which Possess a Pole*, Lecture Notes in Math. **699**, Springer, Berlin, 1979.

[11] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. **36** (1999), 135–249.

[12] R. Ji, *The asymptotic Dirichlet problems on manifolds with unbounded negative curvature*, Math. Proc. Camb. Philos. Soc. **167** (2019), 133–157.

[13] P. March, *Brownian motion and harmonic functions on rotationally symmetric manifolds*, Ann. Probab. **14** (1986), 793–801.

[14] J. Milnor, *On deciding whether a surface is parabolic or hyperbolic*, Amer. Math. Mon. **84** (1977), 43–46.

[15] R. Neel, *Brownian motion and the Dirichlet problem at infinity on two-dimensional Cartan–Hadamard manifolds*, Potential Anal. **41** (2014), 443–462.

[16] R. Neel, *Geometric and Martin boundaries of a Cartan–Hadamard surface*, ALEA Lat. Amer. J. Probab. Math. Stat. **18** (2021), 1669–1687.

[17] J. Peetre, *Absolute convergence of eigenfunction expansions*, Math. Ann. **169** (1967), 307–314.

[18] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Differ. Geom. **18** (1983), 723–732.

[19] M. E. Taylor, *Fourier series on compact lie groups*, Proc. Amer. Math. Soc. **19** (1968), 1103–1105.

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