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THE CROSSED PRODUCT BY A POINTWISE UNITARY ACTION ON A C^*-ALGEBRA WITH CONTINUOUS TRACE

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Abstract Let G be a locally compact group, A a continuous trace C^*-algebra, and α a pointwise unitary action of G on A. It is a result of Olesen and Raeburn that if A is separable and G is second countable, then the crossed product $A \times_\alpha G$ has continuous trace. We present a new and much more elementary proof of this fact. Moreover, we do not even need the separability assumptions made on A and G.

Keywords: continuous trace C^*-algebra; action; crossed product; pointwise unitary

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1. Introduction

Let A be a C^*-algebra. Let $T(A^+)$ be the set of all $a \in A^+$ such that the trace function $\pi \to \operatorname{tr}\pi(a)$ is finite and continuous on $\hat{A}$. We know by [**3**, 4.5.2] that the linear hull $T(A)$ of $T(A^+)$ is a two-sided self-adjoint ideal in A such that $T(A)^+ = T(A^+)$. We denote the closure of $T(A)$ by $J(A)$. Following [**3**] we say that A is a C^*-algebra with *continuous trace* if $J(A) = A$.

Let (A, G, α) be a *covariant system*. That is, G is a locally compact group, A is a C^*-algebra and $\alpha : G \to \operatorname{Aut}(A)$ is a strongly continuous homomorphism of G into the automorphism group $\operatorname{Aut}(A)$ of A. Here, 'strongly continuous' means that, for each $a \in A$, the map $G \to A$, $s \mapsto \alpha_s(a)$ is continuous. We shall also say that α is an *action of G on A*. Let π be a non-degenerate representation of A on a Hilbert space $\mathcal{H}$, and let u be a unitary representation of G on the same Hilbert space $\mathcal{H}$. The pair (π, u) is called a *covariant representation* of (A, G, α) if

$$\pi(\alpha_s(a)) = U_s\pi(a)U_s^* \quad \text{for all } a \in A, \quad s \in G.$$

The *crossed product* $A \times_\alpha G$ is the enveloping C^*-algebra of a certain Banach-$*$-algebra $L^1(G, A)$ (see, for example, [**8**, Chapter 7.6]). For abbreviation we shall sometimes also write $A \times G$ instead of $A \times_\alpha G$. There is a canonical $*$-homomorphism i_A of A into $M(A \times_\alpha G)$ and there is a canonical group homomorphism i_G of G into $M(A \times_\alpha G)$

with the following property. For each covariant representation (π, u) of (A, G, α) there exists a non-degenerate representation $\pi \times u$ of $A \times_\alpha G$ such that $(\pi \times u) \circ i_A = \pi$ and $(\pi \times u) \circ i_G = u$. Conversely, for each non-degenerate representation ρ of $A \times_\alpha G$, the pair $(\pi, u) := (\rho \circ i_A, \rho \circ i_G)$ is a covariant representation of (A, G, α) such that $\rho = \pi \times u$ (see [**8**, Theorem 7.6.4]).

As in [**8**, Chapter 7.1] we denote by $C^*(G)$ the (full) *group-C^*-algebra* of G, which can also be regarded as the crossed product of the covariant system $(\mathbb{C}, G, \mathrm{id})$. Thus, in particular, each unitary representation $u : G \to \mathcal{U}(\mathcal{H})$ defines a non-degenerate representation of $C^*(G)$ on $\mathcal{H}$. This representation shall also be denoted by u.

Now a covariant system (A, G, α) is called *pointwise unitary* if, for each $\pi \in \hat{A}$, there exists a unitary representation $u : G \to \mathcal{U}(\mathcal{H})$ such that (π, u) is a covariant representation of (A, G, α) (see [**9**]). Pointwise unitary actions of abelian second countable groups on separable C^*-algebras with continuous trace were studied in great detail by Olesen and Raeburn in [**6**]. For example, they showed that the crossed product $A \times_\alpha G$ of such a system has Hausdorff spectrum and that the dual action $\hat{\alpha}$ of $\hat{G}$ on $A \times_\alpha G$ induces an action on $(A \times_\alpha G)^\wedge$, which is free and proper in the sense of [**7**]. Furthermore, they were able to identify $A \times_\alpha G$ with the pull-back $\mathrm{Res}^* A$, where $\mathrm{Res} : (A \times_\alpha G)^\wedge \to \hat{A}$ is the map which sends $\pi \times u \in (A \times_\alpha G)^\wedge$ to $\pi \in \hat{A}$. In particular, it follows that $A \times_\alpha G$ has continuous trace.

To prove that $A \times_\alpha G$ has Hausdorff spectrum, Olesen and Raeburn used an important result of Rosenberg which says that a pointwise unitary action of a compactly generated abelian group H on a continuous trace C^*-algebra A is automatically locally unitary (see [**10**]). In our paper we give a much more elementary proof of the fact that $A \times_\alpha G$ has continuous trace. We mainly use the fact that the map $\mathrm{Res} : (A \times_\alpha G)^\wedge \to \hat{A}$ as described above is well defined and continuous (see [**9**, Proposition 2.1]). Of course the arguments used in our Theorem 2.1 below do not apply to the other two important results described above, namely that $\hat{G}$ acts properly on $(A \times_\alpha G)^\wedge$ and that $A \times_\alpha G$ is isomorphic to $\mathrm{Res}^* A$. For a more general treatment of this topic see, for example, [**4**], where pointwise unitary *subgroup bundles* are considered, or [**1**], where pointwise unitary *coactions* are considered.

2. Proof of Theorem 2.1

Theorem 2.1. *Let (A, G, α) be a pointwise unitary covariant system such that G is abelian and A has continuous trace. Then the crossed product $A \times_\alpha G$ also has continuous trace.*

Proof. Let $A \times_\alpha G$ be represented faithfully and non-degenerately on a Hilbert space $\mathcal{H}$. Then the pair (i_A, i_G) becomes a covariant representation of (A, G, α). In particular, i_G defines a non-degenerate representation of $C^*(G)$ on $\mathcal{H}$. Furthermore, $A \times_\alpha G$ is generated by elements of the form $i_A(a) i_G(f)$ with $a \in A$ and $f \in C^*(G)$ (cf. the proof of Theorem 7.6.6 in [**8**]).

Define

$$T^M(A \times_\alpha G^+) := \{z \in M(A \times_\alpha G)^+ : \rho \mapsto \operatorname{tr} \rho(z) \text{ is finite and continuous on } (A \times_\alpha G)^\wedge\}.$$

It follows from the properties of the trace function tr that $T^M(A \times_\alpha G^+)$ is closed under addition and that $zz^* \in T^M(A \times_\alpha G^+)$ implies that $z^*z \in T^M(A \times_\alpha G^+)$. Suppose that $w \in M(A \times_\alpha G)^+$ is dominated by an element $z \in T^M(A \times_\alpha G^+)$ (that is, there exists a $\lambda \in \mathbb{R}^+$ such that $w \leqslant \lambda z$). Since $(A \times_\alpha G)^\wedge$ is clearly an open subset of $(M(A \times_\alpha G))^\wedge$, the arguments used in the proof of Lemma 4.4.2 (i) in [**3**] show that $w \in T^M(A \times_\alpha G^+)$. Now it follows from Lemma 4.5.1 in [**3**] that the linear span $T^M(A \times_\alpha G)$ of $T^M(A \times_\alpha G^+)$ is a self-adjoint two-sided ideal in $M(A \times_\alpha G)$ such that $T^M(A \times_\alpha G)^+ = T^M(A \times_\alpha G^+)$.

We claim that $i_A(T(A^+)) \subset T^M(A \times_\alpha G^+)$. Let $a \in T(A^+)$, and suppose that $\pi_\lambda \times \mu_\lambda \to \pi \times \mu \in (A \times_\alpha G)^\wedge$. Since A is of type I and G is abelian, we know by Proposition 2.1 in [**9**] that $\pi_\lambda \to \pi$. Since $a \in T(A^+)$, this implies that

$$\operatorname{tr}(\pi_\lambda \times \mu_\lambda)(i_A(a)) = \operatorname{tr} \pi_\lambda(a) \to \operatorname{tr} \pi(a) = \operatorname{tr}(\pi \times \mu)(i_A(a)).$$

Thus $i_A(a) \in T^M(A \times_\alpha G^+)$, and $i_A(T(A^+)) \subset T^M(A \times_\alpha G^+)$, as claimed.

Now let $L := \{i_G(f)\ i_A(a)\ i_G(f) : f \in C^*(G)^+,\ a \in T(A^+)\}$. Then

$$L \subset T^M(A \times_\alpha G^+) \cap A \times_\alpha G = T(A \times_\alpha G^+) \subset J(A \times_\alpha G).$$

Since $J(A \times_\alpha G)$ is an ideal in $A \times_\alpha G$ and hence in $M(A \times_\alpha G)$, and since $i_G(C^*(G)) \subset M(A \times_\alpha G)$, it follows that

$$Li_G(C^*(G)) := \{ab : a \in L,\ b \in i_G(C^*(G))\} \subset J(A \times_\alpha G).$$

But $A = \overline{T(A)}$, and therefore $A \times_\alpha G$ is generated by elements of the form $i_A(a)i_G(f)$ with $a \in T(A^+)$ and $f \in C^*(G)$. Furthermore, $i_G : C^*(G) \to M(A \times_\alpha G) \subset \mathcal{L}(\mathcal{H})$ is a non-degenerate representation. Thus, using an approximate identity for $C^*(G)$ and the fact that $C^*(G)$ is abelian, we see that $A \times_\alpha G$ is generated by $Li_G(C^*(G))$. Thus $A \times_\alpha G = J(A \times_\alpha G)$. It follows that $A \times_\alpha G$ has continuous trace. □

Remark 2.2. Let A be an arbitrary C^*-algebra. By transfinite induction, there exists a minimal ordinal number γ and an increasing family $(J_\beta)_{0 \leqslant \beta \leqslant \gamma}$ of closed ideals in A such that

$$\left.\begin{aligned} J_0 = 0, \quad J(A/J_\gamma) = 0, \quad J_{\beta+1}/J_\beta = J(A/J_\beta) \quad \text{for } 0 \leqslant \beta < \gamma, \\ \text{and } J_{\beta'} = \overline{\bigcup_{\beta < \beta'} J_\beta} \text{ if } \beta' \text{ is a limit ordinal.} \end{aligned}\right\} \tag{2.1}$$

Following [**2**] we say that A is a C^*-algebra with *generalized continuous trace* if $J_\gamma = A$. For example, if A is a type I C^*-algebra such that the points in $\hat{A}$ are separated, then A has generalized continuous trace (see [**2**]). In particular, each type I C^*-algebra with Hausdorff spectrum has generalized continuous trace.

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Now suppose that (A, G, α) is a pointwise unitary covariant system such that G is abelian and A has generalized continuous trace. Let $(J_\beta)_{0 \leqslant \beta \leqslant \gamma}$ be an increasing family of closed ideals in A such that $J_\gamma = A$ and such that condition (2.1) holds. Since α is pointwise unitary, it follows that each closed ideal I is α-invariant in the sense that $\alpha_s(I) = I$ for all $s \in G$. In particular, for each β, α induces an action α_{J_β} on I_β and an action α^{J_β} on A/I_β. By the results of Green [**5**] we obtain an increasing family $(J_\beta \times G)_{0 \leqslant \beta \leqslant \gamma}$ of closed ideals in $A \times_\alpha G$ such that $J_0 \times G = 0$, $J_\gamma \times G = A \times_\alpha G$ and

$$J_{\beta'} \times G = \overline{\bigcup_{\beta < \beta'} (J_\beta \times G)} \text{ if } \beta' \text{ is a limit ordinal.}$$

Moreover, it also follows from [**5**] that $(A \times_\alpha G)/(J_\beta \times G) = (A/J_\beta) \times G$ for each β and it is clear that the action α^{J_β} of G on A/J_β is still pointwise unitary.

Since A is of type I by Proposition 4.3.4 in [**3**], the proof of the theorem above (applied to the action of G on A/J_β) shows that

$$\begin{aligned}(J_{\beta+1} \times G)/(J_\beta \times G) &= (J_{\beta+1}/J_\beta) \times G \\ &\subset J((A/J_\beta) \times G) = J((A \times_\alpha G)/(J_\beta \times G))\end{aligned}$$

for each $0 \leqslant \beta < \gamma$. But, by Proposition 11 in [**2**], this suffices to conclude that $A \times_\alpha G$ has generalized continuous trace.

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