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THE SET OF FINITE OPERATORS IS NOWHERE DENSE

BY

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ABSTRACT. A bounded linear operator A on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$ is called finite if $\|AX - XA - 1\| \geqq 1$ for each $X \in \mathcal{L}(\mathcal{H})$. It is shown that the class of all finite operators is a closed nowhere dense subset of $\mathcal{L}(\mathcal{H})$.

Introduction. In [15], J. P. Williams introduced the notion of finite operator. In a finite dimensional Hilbert space, the commutator of two linear operators has trace 0, and therefore 0 belongs to the numerical range of every commutator. Let $\mathcal{L}(\mathcal{H})$ denoted the algebra of all (bounded linear) operators acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. We say that $A \in \mathcal{L}(\mathcal{H})$ is *finite* if $0 \in W(AX - XA)^-$ for all X in $\mathcal{L}(\mathcal{H})$, where $W(T)$ denotes the numerical range of the operator T. In that article, Williams proves that the class $\mathcal{F}$ of all finite operators is closed in $\mathcal{L}(\mathcal{H})$, and that the following three conditions are equivalent for A in $\mathcal{L}(\mathcal{H})$:

(1) $A \in \mathcal{F}$.

(2) $\|AX - XA - 1\| \geqq 1$ for all $X \in \mathcal{L}(\mathcal{H})$ (that is, the identity operator is "orthogonal" to the range of the inner derivation induced by A).

(3) There exists a state f such that $f(AX) = f(XA)$ for all $X \in \mathcal{L}(\mathcal{H})$.

Furthermore, if $A \in \mathcal{F}$, then the C^*-algebra $C^*(A)$ (generated by A and 1) is included in $\mathcal{F}$.

As J. P. Williams explains in his article, the adjective "finite" used to describe the operators in $\mathcal{F}$ is admittedly ad hoc. It comes from the fact that $\mathcal{F} \supset \mathcal{R}^-$, where $\mathcal{R}_n = \{T \in \mathcal{L}(\mathcal{H}) : T$ has a reducing subspace of dimension $n\}$ and $\mathcal{R} = \cup_{n=1}^{\infty} \mathcal{R}_n$. (The most difficult open problem in this area is the question of whether $\mathcal{F} = \mathcal{R}^-$; see [10],[15].)

The existence of non-finite operators follows immediately from, for instance, the Brown-Pearcy characterization of commutators [4]. (For more information about the class $\mathcal{F}$, the reader is referred to [2],[5],[7],[9],[10].)

In the Introduction of [11] (joint work with S. J. Szarek), the author claims without proof that $\mathcal{R}^-$ is nowhere dense in $\mathcal{L}(\mathcal{H})$. The purpose of this note is to provide such a proof. Indeed, it will be shown that $\mathcal{F}$ is nowhere dense in $\mathcal{L}(\mathcal{H})$.

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By William's results, it is sufficient to show that for each T in a dense subset of $\mathcal{L}(\mathcal{H})$ and for each $\epsilon > 0$, there exists $T_\epsilon \in \mathcal{L}(\mathcal{H})$, with $\|T - T_\epsilon\| < \epsilon$, such that $C^*(T_\epsilon)$ contains some non-finite operator. The proof will be given in Section 3. Section 2 contains all the necessary auxiliary results, including a very general result on approximation of operators that has some interest in itself (see Proposition 3 below).

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Preliminaries on Approximation of Operators. An analytic Cauchy domain Ω is a (not necessarily connected) bounded open subset of the complex plane $\mathbf{C}$ whose boundary consists of finitely many pairwise disjoint Jordan curves. Let $M(\partial\Omega) =$ "multiplication by λ" on $L^2(\partial\Omega)$ (linear Lebesgue measure on $\partial\Omega$), and let $H^2(\partial\Omega)$ denote the closure in $L^2(\partial\Omega)$ of the rational functions with poles outside Ω^-; $H^2(\partial\Omega)$ is invariant under $M(\partial\Omega)$, and we have the decomposition

$$M(\partial\Omega) = \begin{pmatrix} M_+(\partial\Omega) & Z(\partial\Omega) \\ 0 & M_-(\partial\Omega) \end{pmatrix} \begin{matrix} H^2(\partial\Omega) \\ L^2(\partial\Omega)\ominus H^2(\partial\Omega) \end{matrix},$$

where $M_+(\partial\Omega) = M(\partial\Omega)|H^2(\partial\Omega)$, $\sigma(M_+(\partial\Omega)) = \sigma(M_-(\partial\Omega)) = \Omega^-$, $\sigma_e(M_+(\partial\Omega)) = \sigma_e(M_-(\partial\Omega)) = \sigma(M(\partial\Omega)) = \sigma_e(M(\partial\Omega)) = \partial\Omega$, and $-\operatorname{ind}(\lambda - M_+(\partial\Omega)) = \operatorname{ind}(\lambda - M_-(\partial\Omega)) = \operatorname{nul}(\lambda - M_+(\partial\Omega))^* = \operatorname{nul}(\lambda - M_-(\partial\Omega)) = 1$ and $\operatorname{nul}(\lambda - M_+(\partial\Omega)) = \operatorname{nul}(\lambda - M_-(\partial\Omega))^* = 0$ for all $\lambda \in \Omega$ (see, e.g., [8, Chapter 3] for details). Here $\sigma(\cdot)$ and $\sigma_e(\cdot)$ denote the spectrum and, respectively, the essential spectrum of an operator. The reader is referred to [12] for definition and properties of the Fredholm and semi-Fredholm operators, index, stability, etc.

Given $A_1 \in \mathcal{L}(\mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_2)$, $A_1 \oplus A_2$ will denote the direct sum of A_1 and A_2 acting in the usual fashion on the orthogonal direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of the underlying spaces. By $A_1^{(\alpha)}$ we indicate the direct sum of $\alpha (0 \leqq \alpha \leqq \infty)$ copies of A_1 acting on the orthogonal direct sum $\mathcal{H}_1^{(\alpha)}$ of α copies of $\mathcal{H}_1$.

LEMMA 1. *Suppose $A \in \mathcal{L}(\mathcal{H}_0)$, $\sigma(A) \subset \Omega$ (an analytic Cauchy domain) and*

$$S = \begin{pmatrix} A & Z \\ 0 & M_+(\partial\Omega)^{(\alpha)} \end{pmatrix} \begin{matrix} \mathcal{H}_0 \\ H^2(\partial\Omega)^{(\alpha)} \end{matrix} (1 \leqq \alpha \leqq \infty);$$

then the C^*-algebra $C^*(S)$ generated by S and 1 *contains the orthogonal projection onto $\mathcal{H}_0$ and $H^2(\partial\Omega)^{(\alpha)}$.*

PROOF. According to [1], there exists a function ϕ, analytic on a neighborhood of Ω^- such that $\phi(\Omega^-) = \mathbf{D}^-$ and $\phi(\partial\Omega) = \partial\mathbf{D}$ ($\mathbf{D}$:= open unit disk). Clearly, $\phi(M_+(\partial\Omega)^{(\alpha)})$, $\phi(A)$ and $\phi(S)$ are well-defined via functional calculus; moreover,

$$\phi(M_+(\partial\Omega)^{(\alpha)}) = \text{“multiplication by } \phi(\lambda)\text{” on } H^2(\partial\Omega)^{(\alpha)}$$

is an isometry, and $\sigma(\phi(A)) = \phi(\sigma(A)) \subset \phi(\Omega) = \mathbf{D}$.

Therefore, $\phi(M_+(\partial\Omega)^{(\alpha)})^m$ is an isometry for all $m = 1, 2, \ldots$, and

$$\|\phi(A)^m\| \longrightarrow 0$$

exponentially, as $m \longrightarrow \infty$, because the spectral radius of $\phi(A)$ is less than 1.

Observe that $\sigma(A) \subset \{\lambda \in \mathbf{C} : \lambda - M_+(\partial\Omega)^{(\alpha)}$ is *left invertible*$\}$. Thus, according to [6] (or [8, Chapter 3]), there exists W invertible, $W = \left(\begin{smallmatrix} 1 & X \\ 0 & 1 \end{smallmatrix}\right)$, such that

$$S = W[A \oplus M_+(\partial\Omega)^{(\alpha)}]W^{-1} = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & M_+(\partial\Omega)^{(\alpha)} \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned} \phi(S)^m &= \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(A)^m & 0 \\ 0 & \phi(M_+(\partial\Omega)^{(\alpha)}) \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \phi(A)^m & X\phi(M_+(\partial\Omega)^{(\alpha)})^m - \phi(A)^m X \\ 0 & \phi(M_+(\partial\Omega)^{(\alpha)})^m \end{pmatrix} \end{aligned}$$

and

$$[\phi(S)^m]^*\phi(S)^m = \begin{pmatrix} 0 & 0 \\ 0 & 1 + [\phi(M_+(\partial\Omega)^{(\alpha)})^m]^* X^* X [\phi(M_+(\partial\Omega)^{(\alpha)}]^m \end{pmatrix} + O(\|\phi(A)^m\|).$$

Since $1 \leqq 1 + [\phi(M_+(\partial\Omega)^{(\alpha)})^m]^* X^* X [\phi(M_+(\partial\Omega)^{(\alpha)}]^m \leqq 1 + \|X\|^2$, and $\|\phi(A)^m\|$ converges exponentially to 0, it is not difficult to conclude that the sequence

$$\left\{([\phi(S)^m]^*\phi(S)^m)^{1/\sqrt{m}}\right\}_{m=1}^{\infty}$$

converges in the norm to the orthogonal projection onto $H^2(\partial\Omega)^{(\alpha)}$ (and therefore this projection belongs to $C^*(S)$.

Since $C^*(S)$ contains the identity, the orthogonal projection onto $\mathcal{H}_0$ also belongs to $C^*(S)$. □

REMARK 2. The conclusion is the same if S is replaced by

$$\begin{pmatrix} A & 0 \\ Z & M_-(\partial\Omega)^{(\infty)} \end{pmatrix} \begin{matrix} H_o \\ [L^2(\partial\Omega)\Theta H^2(\partial\Omega)]^{(\alpha)} \end{matrix}$$

$(A \in \mathcal{L}(\mathcal{H}_0)$, $\sigma(A) \subset \Omega)$.

PROPOSITION 3. *Let $T \in \mathcal{L}(\mathcal{H})$; T can be uniformly approximated by operators of the form $S = R_1 \oplus R_2 \oplus R_3$, where the R_j's ($R_j \in \mathcal{L}(\mathcal{R}_j)$) satisfy the following condition: given any three operators R_{12}, R_{13} and R_{23} ($R_{ij} : \mathcal{R}_j \longrightarrow \mathcal{R}_i$), the C^*-algebra $C^*(S')$ generated by*

$$S' = \begin{pmatrix} R_1 & R_{12} & R_{13} \\ 0 & R_2 & R_{23} \\ 0 & 0 & R_3 \end{pmatrix} \begin{matrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \end{matrix}$$

and 1 *contains the orthogonal projections $\{P_j\}_{j=1}^3$ onto the subspaces $\{\mathcal{R}_j\}_{j=1}^3$.*

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PROOF. Let $\mathcal{R}ho : C^*(\tilde{T}) \longrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{R}}ho)$ be a faithful unital *-representation of the C^*-algebra $C^*(\tilde{T})$, generated by $\tilde{T}$ and $\tilde{1}$ onto a separable Hilbert space $\mathcal{H}_{\mathcal{R}}ho$, where $\tilde{T} = T + \mathcal{K}(\mathcal{H}) \in \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ ($\mathcal{K}(\mathcal{H})$ denotes the ideal of all compact operators). By Voiculescu's theorem [14], given $\epsilon > 0$ there exists $T_1 \in \mathcal{L}(\mathcal{H})$ such that $T - T_1 \in \mathcal{K}(\mathcal{H})$, $\|T - T_1\| < \epsilon$, and $T_1 \simeq T \oplus A \oplus A$, where $A = \mathcal{R}ho(\tilde{T})^{(\infty)}$.

According to [3] (or [8, Chapter 6]), we can find

$$R_1 \simeq \begin{pmatrix} S'_{+,1} \oplus S_{+,1} & * & * \\ 0 & N_1 & * \\ 0 & 0 & S'_{-,1} \oplus S_{-,1} \end{pmatrix} \begin{matrix} \mathcal{R}_{+,1} \\ \mathcal{R}_{0,1} \\ \mathcal{R}_{-+,1} \end{matrix}, \|T - R_1\| < \epsilon,$$

where $\sigma(N_1)$, $\sigma(S'_{+,1})$, $\sigma(S_{+,1})$, $\sigma(S_{-,1})$ and $\sigma(S'_{-,1})$ are pairwise disjoint, N_1 is algebraic (and therefore $\sigma(N_1)$ is a finite set),

$$S'_{+,1} \simeq \oplus_{i=1}^{m'} M_+(\partial\Omega'_{1,i})^{(p_{1,i})}, \quad S'_{-,1} \simeq \oplus_{k=1}^{n'} M_-(\partial\Phi'_{1,k})^{(q_{1,k})},$$
$$S_{+,1} \simeq \oplus_{i=1}^{m} M_+(\partial\Omega_{1,i})^{(\infty)}, \quad S_{-,1} \simeq \oplus_{k=1}^{n} M_-(\partial\Phi_{1,k})^{(\infty)},$$

$1 \leqq p_{1,i}, q_{1,k} < \infty$ (for all i and all k) and the analytic Cauchy domains $\{\Omega'_{1,i}\}_{i=1'}^{m'}$, $\{\Phi'_{1,k}\}_{k=1'}^{n'}$, $\{\Omega_{1,i}\}_{i=1}^{m}$ and $\{\Phi_{1,k}\}_{k=1}^{n}$ have pairwise disjoint closures.

Clearly, $\sigma(A) = \sigma_e(A) = \sigma_e(T)$, $\mathcal{R}ho_{s-F}(A) = \mathcal{R}ho_{s-F}(T)$ and for each $\lambda \in \mathcal{R}ho_{s-F}(T)$,

$$\operatorname{ind}(A - \lambda) = \begin{cases} 0, & \text{if } -\infty < \operatorname{ind}(T - \lambda) < \infty, \\ \infty, & \text{if } \operatorname{ind}(T - \lambda) = \infty, \\ -\infty, & \text{if } \operatorname{ind}(T - \lambda) = -\infty. \end{cases}$$

Thus, by proceeding as above, we can find

$$R'_j = \begin{pmatrix} S_{+,1} & * & * \\ 0 & N'_j & * \\ 0 & 0 & S_{-,1} \end{pmatrix} \begin{matrix} \mathcal{R}_{+,j} \\ \mathcal{R}_{0,j} \\ \mathcal{R}_{-,j} \end{matrix}, \|A - R'_j\| < \epsilon,$$

where $\sigma(S_{+,1})$, $\sigma(N'_j)$ and $\sigma(S_{-,1})$ are pairwise disjoint, and N_j is algebraic ($j = 2, 3$).

Furthermore, the results of [3] (see, especially, the comments in the first part of [8, Chapter 6] on this subject) indicate that we have some flexibility on our choice of the Cauchy domains $\Omega_{1,j}$ and $\Phi_{1,k}$. By using this flexibility, R'_j can be replaced by

$$R_j = \begin{pmatrix} S_{+,j} & * & * \\ 0 & N_j & * \\ 0 & 0 & S_{-,j} \end{pmatrix} \begin{matrix} \mathcal{R}_{+,j} \\ \mathcal{R}_{0,j} \\ \mathcal{R}_{-,j} \end{matrix}, \|A - R_j\| < \epsilon,$$

where $S_{+,j} \simeq \oplus_{i=1}^{m} M_+(\partial\Omega_{j,i})^{(\infty)}$, $S_{-,j} \simeq \oplus_{k=1}^{n} M_-(\partial\Phi_{j,k})^{(\infty)}$, $(j = 2, 3)$, $\Omega_{1,i} \subset (\Omega_{1,i})^- \subset \Omega_{2,i} \subset (\Omega_{2,i})^- \subset \Omega_{3,i} \subset (\Omega_{3,i})^-$, $(\Omega_{3,i})^-$ is disjoint from $\sigma(S'_{+,1}) \cup \sigma(S'_{-,1}) \cup \sigma(S_{-,1}) \cup \{\bigcup_{3,i}^{3} \sigma(N_j)\}$, and $(\Phi_{1,k})^- \supset \Phi_{1,k} \supset (\Phi_{2,k})^- \supset \Phi_{2,k} \supset (\Phi_{3,k})^- \supset \Phi_{3,k}$.

Let $S = R_1 \oplus R_2 \oplus R_3$; then $\|T - S\| < 2\epsilon$.

Given $R_{ij} \in \mathcal{L}(\mathcal{R}_j, \mathcal{R}_i)$ $1 \leqq i < j \leqq 3$), let

$$S' = \begin{pmatrix} R_1 & R_{12} & R_{13} \\ 0 & R_2 & R_{23} \\ 0 & 0 & R_3 \end{pmatrix} \begin{matrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \end{matrix},$$

and let P_j denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}_j$ $(j = 1, 2, 3)$.

Since a C^*-algebra of operators is always inverse-closed, $C^*(S')$ contains the orthogonal projection onto every Riesz spectral subspace. Thus, in particular, $C^*(S')$ contains the projections $P_{0,j}$ onto the subspace $\mathcal{R}_{0,j}$ $(j = 1, 2, 3)$, as well as the projections $Q'_{+,1'}$, $Q'_{-,1}$ and Q_1 onto the Riesz subspaces corresponding to $(\bigcup_{i=1}^{m'} \Omega'_{1,i})^-$, $(\bigcup_{k=1}^{n'} \Phi'_{1,k})^-$ and, respectively, $(\Omega_{3,1})^-$.

Observe that

$$S'|\mathcal{R}\, an Q_1 \simeq \begin{pmatrix} M_+(\partial\Omega_{1,1})^{(\infty)} & * & * \\ 0 & M_+(\partial\Omega_{2,1})^{(\infty)} & * \\ 0 & 0 & M_+(\partial\Omega_{3,1})^{(\infty)} \end{pmatrix}$$
$$= \begin{pmatrix} M'_+(\partial\Omega_{3,1}) & * \\ 0 & M_+(\partial\Omega_{3,1})^{(\infty)} \end{pmatrix},$$

where $\sigma(M'_+(\partial\Omega_{3,1})) = (\Omega_{2,1})^- \subset \Omega_{3,1} =$ interior $\sigma[M_+(\partial\Omega_{3,1})^{(\infty)}]$.

By Lemma 1, $C^*(S')$ contains the orthogonal projections $P^+_{3,1}$ onto the image of the subspace $\{0\} \oplus \{0\} \oplus H^2(\partial\Omega_{3,1})^{(\infty)}$ under the unitary equivalence.

By a formal repetition of the same argument, we infer that $C^*(S')$ also contains $P^+_{2,1}$ and $P^+_{1,1}$ (defined in the obvious way).

By repeating the operations with $\Omega_{3,2}, \Omega_{3,3}, \ldots, \Omega_{3,m}$, we deduce that $C^*(S')$ contains the orthogonal projections $P_{+,j}$ onto the subspaces $\mathcal{R}_{+,j}$ $(j = 1, 2, 3;\ P_{+,1} = Q'_{+,1} + \sum_{i=1}^m P^+_{1,i}, P_{+,j} = \sum_{i=1}^m P^+_{j,i},\ j = 2, 3)$.

Another repetition of the same argument (with help of Lemma 1 and Remark 2) shows that $C^*(S')$ contains the orthogonal projections $P_{-,j}$ onto the subspaces $\mathcal{R}_{-,j}$ $(j = 1, 2, 3)$, whence we conclude that

$$P_j = P_{+,j} + P_{0,j} + P_{-,j} \in C^*(S') \quad (j = 1, 2, 3).$$

The proof of Proposition 3 is now complete. □

$\mathcal{F}$ is nowhere dense in $\mathcal{L}(\mathcal{H})$. According to our observations in the Introduction, it suffices to show that for each S as in Proposition 3 and each $\epsilon > 0$, there exists $S_\epsilon \in \mathcal{L}(\mathcal{H})$, with $\|S - S_\epsilon\| < \epsilon$, such that $C^*(S_\epsilon)$ contains a non-finite operator.

Let X be any non-finite operator, and define

$$S_\epsilon = \begin{pmatrix} R_1 & (\epsilon/2)1 & (\epsilon/2\|X\|)X \\ 0 & R_2 & (\epsilon/2)1 \\ 0 & 0 & R_3 \end{pmatrix} \begin{matrix} \mathcal{R}_1 \\ \mathcal{R}_2 ; \\ \mathcal{R}_3 \end{matrix}$$

then $\|S - S_\epsilon\| < 2.(\epsilon/2) = \epsilon$, and (by Proposition 3) $P_j \in C^*(S_\epsilon)$ $(j = 1, 2, 3)$.

Therefore

$$A(X) := \frac{2}{\epsilon}(P_1S_\epsilon P_2 + P_2S_\epsilon P_3 + \|X\|P_1S_\epsilon P_3) = \begin{pmatrix} 0 & 1 & X \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \end{matrix} \in C^*(S).$$

But, according to [15, Theorem 8], $A(X) \in \mathcal{F}$ if and only if $X \in \mathcal{F}$. Hence, $S_\epsilon \notin \mathcal{F}$.

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We conclude that $\mathcal{F}$ is nowhere dense in $\mathcal{L}(\mathcal{H})$. □

A Concluding Remark. Theorem 8 of [15] admits many variations (see, e.g., [7, p. 605]).

(i) For instance, if $X \notin \mathcal{F}(\mathcal{H})$ and $Q = (Q_{ij})_{i,j=1}^{n} \in \mathcal{L}(\mathcal{H}^{(n)})$ $(n \geqq 3)$ is a nilpotent operator of the form

$$Q_{ij} = \begin{cases} 1, & \text{if } j = i+1, i = 1,2,\ldots,n-1, \\ 0, & \text{if } 1 \leqq i < j \leqq n, \end{cases}$$

and $Q_{ij} = X$ for some (i,j) with $j - i \geqq 2$, then Williams's argument shows that $Q \notin \mathcal{F}$.

(ii) If $F : \mathcal{H} \longrightarrow \mathcal{H}_0$ $(1 \leqq \dim \mathcal{H}_0 \leqq \infty)$ is onto, then

$$Q_- = \begin{pmatrix} 0 & F \\ 0 & X \end{pmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H} \end{matrix} \quad \text{is not finite.}$$

Indeed, if $f = (f_{ij})_{i,j=0}^{1}$ is a state such that $f(Q_-Y) = f(YQ_-)$ for all $Y = (Y_{ij})_{i,j=0}^{1} \in \mathcal{L}(\mathcal{H}_0 \oplus \mathcal{H})$, then

$$Q_-Y - YQ_- = \begin{pmatrix} FY_{10} & FY_{11} - Y_{01}X \\ XY_{10} & XY_{11} - Y_{11}X \end{pmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H} \end{matrix}$$

and $f(Q_-Y - YQ_-) = 0$, whence we obtain $f_{00}(FY_{10}) = 0$ for all $Y_{10} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_0)$ and $f_{11}(XY_{11} - Y_{11}X) = 0$ for all $Y_{11} \in \mathcal{L}(\mathcal{H})$. Since F is onto, it readily follows that $f_{00} = 0$, and therefore $f_{01} = f_{10} = 0$ and $f = 0 \oplus f_{11}$ (where f_{11} is a state on $\mathcal{L}(\mathcal{H})$) because f is a positive map. But $f_{11}(XY_{11} - Y_{11}X) = 0$ for all $Y_{11} \in \mathcal{L}(\mathcal{H})$ is impossible because X is not finite, a contradiction.

Hence $Q_- \notin \mathcal{F}$.

(iii) If $F : \mathcal{H}_0 \longrightarrow \mathcal{H}$ $(1 \leqq \dim \mathcal{H}_0 \leqq \infty)$ is bounded below, then

$$Q_+ = \begin{pmatrix} X & F \\ 0 & 0 \end{pmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H}_0 \end{matrix} \quad \text{is not finite.}$$

Observe that the class $\mathcal{F}$ is self-adjoint. Now the result follows immediately from (ii) by taking adjoints.

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