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CUBIC GRAPHS ADMITTING TRANSITIVE NON-ABELIAN CHARACTERISTICALLY SIMPLE GROUPS

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Abstract Let Γ be a graph and let G be a vertex-transitive subgroup of the full automorphism group $\mathrm{Aut}(\Gamma)$ of Γ. The graph Γ is called *G-normal* if G is normal in $\mathrm{Aut}(\Gamma)$. In particular, a Cayley graph $\mathrm{Cay}(G,S)$ on a group G with respect to S is *normal* if the Cayley graph is $R(G)$-normal, where $R(G)$ is the right regular representation of G. Let T be a non-abelian simple group and let $G = T^{\ell}$ with $\ell \geqslant 1$. We prove that if every connected T-vertex-transitive cubic symmetric graph is T-normal, then every connected G-vertex-transitive cubic symmetric graph is G-normal. This result, among others, implies that a connected cubic symmetric Cayley graph on G is normal except for $T \cong A_{47}$ and a connected cubic G-symmetric graph is G-normal except for $T \cong A_7$, A_{15} or $\mathrm{PSL}(4,2)$.

Keywords: Cayley graph; symmetric graph; quasiprimitive graph

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1. Introduction

Throughout this paper graphs are assumed to be finite, simple and undirected. For a graph Γ, let $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ be the vertex set, the edge set and the full automorphism group of Γ, respectively. For a non-negative integer s, an *s-arc* in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \dots, v_s)$ of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leqslant i \leqslant s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i < s$; in other words, a directed walk of length s that never includes any backtracking. For a group G of automorphisms of a graph Γ, the graph Γ is said to be *G–s-arc-transitive* or *G–s-arc-regular* if G is transitive or regular on the set of s-arcs in Γ, respectively. In particular, G–0-arc-transitive (G–0-arc-regular) means *G-vertex-transitive* (*G-vertex-regular*), and G–1-arc-transitive means *G-arc-transitive* or *G-symmetric*. A graph Γ is said to be *vertex-transitive*, *arc-transitive*, *symmetric*, *s-arc-transitive* or *s-arc-regular* if Γ is $\mathrm{Aut}(\Gamma)$*-vertex-transitive*, $\mathrm{Aut}(\Gamma)$*-arc-transitive*, $\mathrm{Aut}(\Gamma)$*-symmetric*, $\mathrm{Aut}(\Gamma)$*–s-arc-transitive* or $\mathrm{Aut}(\Gamma)$*–s-arc-regular*, respectively.

For a finite group G and a subset S of G such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G,\ s \in S\}$. A graph X is isomorphic to a Cayley graph on G if and only if

$\mathrm{Aut}(X)$ has a subgroup isomorphic to G that acts regularly on vertices (see [**24**]). Given $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg$, $x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$ is a permutation group isomorphic to G, called the *right regular representation* of G. The Cayley graph $\mathrm{Cay}(G, S)$ is vertex-transitive because $R(G)$ is a vertex-regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Furthermore, the group $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is also a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$. Actually, $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))_1$, the stabilizer of the vertex 1 in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $\mathrm{Aut}(\mathrm{Cay}(G, S))$ contains $R(G)$ as a normal subgroup.

It was conjectured in [**27**] that 'most' Cayley graphs are normal. In the literature, studying normality itself, or, equivalently, the determination of automorphism groups of Cayley graphs, has become a very active topic in algebraic graph theory, which also plays an important role in the investigation of various symmetry properties of graphs (see, for example, [**2**, **5**–**11**, **15**, **21**]). By [**12**], if $X = \mathrm{Cay}(G, S)$ is normal, then $\mathrm{Aut}(X) = R(G) \rtimes \mathrm{Aut}(G, S)$, which implies that the automorphism group of a normal Cayley graph is known and normal Cayley graphs are just those that have the smallest possible full automorphism groups.

In most situations, it is difficult to determine the normality of Cayley graphs. A more general problem is to determine the so-called G-normality of a G-vertex-transitive graph. Let G be a vertex-transitive group of automorphisms of a graph Γ. The graph Γ is said to be *G-normal* if G is normal in $\mathrm{Aut}(\Gamma)$. In particular, a Cayley graph $\mathrm{Cay}(G, S)$ is normal if and only if the Cayley graph is $R(G)$-normal. A non-abelian characteristically simple group is the direct product $T \times T \times \cdots \times T$ (ℓ times), denoted by T^ℓ, of a non-abelian simple group T. The T-normality was investigated for T-vertex-regular cubic symmetric graphs in [**28**] and for T-symmetric cubic graphs in [**14**]. Note that a T-vertex-regular graph is a Cayley graph on T. In this paper we consider T^ℓ-normality for T^ℓ-vertex-transitive cubic symmetric graphs. The following theorem is the main result.

Theorem 1.1. *Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 1$. Assume that every connected T-vertex-transitive cubic symmetric graph is T-normal. Then every connected G-vertex-transitive cubic symmetric graph is G-normal.*

Let Γ be a connected cubic symmetric Cayley graph on T. By [**28**, Theorem 1.1], if Γ is not normal, then $T \cong A_{47}$, and by Theorem 1.1 we have the following corollary.

Corollary 1.2. *Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 1$. Then every connected cubic symmetric Cayley graph on G is normal except for $T \cong A_{47}$.*

Let Γ be a connected T-symmetric cubic graph. By [**14**, Proposition 7.1.3], if Γ is not T-normal, then $T \cong A_7$, A_{15} or $\mathrm{PSL}(4, 2)$.

Corollary 1.3. *Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 1$. Every connected cubic G-symmetric graph is then G-normal except for $T \cong A_7$, A_{15} or* $\mathrm{PSL}(4, 2)$.

2. Preliminary results

This section collects some notation and preliminary results that will be used later. We start by stating a well-known result on permutation group theory.

Proposition 2.1 (Wielandt [26, Proposition 4.3]). *Let G be a transitive permutation group on a set Ω. The centralizer of G in the symmetric group S_Ω on Ω is then semiregular: that is, the centralizer has trivial stabilizer for each point in Ω.*

The following proposition is due to Burnside.

Proposition 2.2 (Robinson [23, Theorem 8.5.3]). *Let p and q be primes and let m and n be non-negative integers. Any group of order $p^m q^n$ is then solvable.*

Let G be a group. The *inner automorphism group* $\mathrm{Inn}(G)$ of G is the group of automorphisms of G induced by conjugate action of elements in G, which is a normal subgroup in the full automorphism group $\mathrm{Aut}(G)$ of G. The quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G. By the classification of finite simple groups, we have the following proposition, which is the famous Schreier conjecture.

Proposition 2.3 (Gorenstein [13, Theorem 1.46]). *Every finite simple group has a solvable outer automorphism group.*

A typical method for studying vertex-transitive graphs is to take certain quotients. Let Γ be a graph and let N be a subgroup of $\mathrm{Aut}(X)$. Denote by Γ_N the quotient graph of Γ corresponding to the orbits of N: that is, the graph having the orbits of N as vertices with two orbits adjacent in Γ_N whenever there is an edge in Γ between vertices lying in these two orbits. In view of Theorem 9 of [**17**] (see also [**18**]), we have the following proposition.

Proposition 2.4. *Let $N \trianglelefteq X$ and let Γ be a connected X–s-arc-transitive graph of prime valency p for some $s \geqslant 1$. If N has more than two orbits, then N is semiregular on $V(X)$ and the quotient graph X_N of X corresponding to the orbits of N is a connected symmetric graph of valency p. Furthermore, N is the kernel of X acting on $V(\Gamma_N)$ and Γ_N is X/N–s-arc-transitive.*

An *imprimitive block* of a transitive permutation group G on a set Ω is a non-empty subset Δ of Ω such that for all $g \in G$ we have either $\Delta^g \cap \Delta = \emptyset$ or $\Delta^g = \Delta$. A permutation group G on Ω is *primitive* if it is transitive and its only imprimitive blocks are Ω and the singleton sets $\{\omega\}$ for $\omega \in \Omega$. A permutation group G is *quasiprimitive* if all its non-trivial normal subgroups are transitive. It is elementary and well known that any primitive group is quasiprimitive. For a group G, let $\mathrm{soc}(G)$ denote the *socle* of G: that is, the product of all minimal normal subgroups of G. The *holomorph* $\mathrm{Hol}(G)$ of a group G is the semiproduct $G \rtimes \mathrm{Aut}(G)$.

There are eight types of primitive permutation groups identified in [**16**, **19**]. Analogous to these eight types, quasiprimitive permutation groups were also given in eight types in [**20**, **22**], from which one may deduce the following proposition.

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Proposition 2.5 (Baddeley and Praeger [1, Theorem 3.1]). *A quasiprimitive permutation group G on a set Ω has one of the following eight types.*

HA: *G has a unique minimal normal subgroup M that is abelian and regular. In this case, $\mathrm{soc}(G) = M$, $C_G(M) = M$ and $G \leqslant \mathrm{Hol}(M)$.*

HS: *G has precisely two minimal normal subgroups M and N that are non-abelian and simple, isomorphic and regular. In this case, $\mathrm{soc}(G) = M \times N$, $C_G(M) = N$ and $G \leqslant \mathrm{Hol}(M)$.*

HC: *G has precisely two minimal normal subgroups M and N that are direct products of at least two isomorphic non-abelian simple groups. In this case, M and N are regular, $M \cong N$, $\mathrm{soc}(G) = M \times N$, $C_G(M) = N$ and $G \leqslant \mathrm{Hol}(M)$.*

The above three types correspond to primitive permutation groups and the five remaining types correspond to quasiprimitive permutation groups that may be primitive or imprimitive.

AS: *G has a unique minimal normal subgroup M that is non-abelian and simple. In this case, $\mathrm{soc}(G) = M$, $M \leqslant G \leqslant \mathrm{Aut}(M)$ and M is either regular or not.*

TW: *G has a unique minimal normal subgroup M that is regular and a direct product of at least two isomorphic non-abelian simple groups. In this case, $\mathrm{soc}(G) = M$ and $C_G(M) = 1$.*

Quasiprimitive permutation groups of the three remaining types have a unique minimal normal subgroup M that is not regular and a direct product of at least two isomorphic non-abelian simple groups: say T^k with $k > 1$. In these cases, $\mathrm{soc}(G) = M$ and $C_G(M) = 1$. The types are distinguished by the point-stabilizer M_u of $u \in \Omega$ in M, which is necessarily non-trivial.

SD: *M_u is a full diagonal subgroup of M.*

CD: *M_u is a direct product of at least two disjoint non-trivial full strips of M. (Groups of this type are the blow-ups of groups of type* SD.*)*

PA: *$M_u = R^k$ for a proper subgroup R of T.*

Baddeley and Praeger [**1**] considered almost simple groups containing a direct product of at least two isomorphic non-abelian simple groups.

Proposition 2.6 (Baddeley and Praeger [1, Theorem 1.4]). *Let H be an almost simple group: that is, $S \leqslant H \leqslant \mathrm{Aut}(S)$ for a non-abelian simple group S, and suppose that $H = AB$, where A is a proper subgroup of H not containing S, and $B \cong T^k$ for a non-abelian simple group T and integer $k \geqslant 2$. Then $S = A_n$ and $A \cap S = A_{n-1}$, where $n = |H : A| = |S : A \cap S| \geqslant 10$.*

3. Proof of Theorem 1.1

Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 1$. Assume that every connected T-vertex-transitive cubic symmetric graph is T-normal. Let $\varGamma$ be a connected G-vertex-transitive cubic symmetric graph and let $A = \mathrm{Aut}(\varGamma)$. Denote by $\mathrm{rad}(A)$ the *radical* of A: that is, the largest normal solvable subgroup of A.

The proof of Theorem 1.1 is organized as follows. It is proved that $\mathrm{rad}(A)G = \mathrm{rad}(A) \times G$ in Lemma 3.1. The quotient graph $\varGamma_{\mathrm{rad}(A)}$, by Proposition 2.4, is a connected cubic $A/\mathrm{rad}(A)$-symmetric graph. Furthermore, $\mathrm{rad}(A)G/\mathrm{rad}(A) \cong G$ and $\varGamma_{\mathrm{rad}(A)}$ is $\mathrm{rad}(A)G/\mathrm{rad}(A)$-vertex-transitive. To use induction on ℓ, we deal with two cases depending on whether or not $A/\mathrm{rad}(A)$ is quasiprimitive; the quasiprimitive case is considered in Lemma 3.2 and the non-quasiprimitive case is considered in Lemma 3.3.

Lemma 3.1. *Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 1$. Let $\varGamma$ be a connected cubic graph and let $G \leqslant X \leqslant \mathrm{Aut}(\varGamma)$ such that $\varGamma$ is G-vertex-transitive and X-arc-transitive. Then $\mathrm{rad}(X)G = \mathrm{rad}(X) \times G$.*

Proof. Since $G \cap \mathrm{rad}(X) \trianglelefteq G$, $G \cap \mathrm{rad}(X)$ is a direct product of copies of T. It follows that $\mathrm{rad}(X) \cap G = 1$ because $\mathrm{rad}(X)$ is solvable. Let $u \in V(\varGamma)$. By vertex-transitivity of G, $X = GX_u$ and $|G| = |V(\varGamma)|\,|G_u|$. By [**25**], $\varGamma$ is at most 5-arc-regular, and by the same reference $|X_u|$ is a divisor of 48. Since $\mathrm{rad}(X)G \leqslant GX_u$, $|\mathrm{rad}(X)|$ is a divisor of $|X_u|$ and hence a divisor of 48. To finish the proof, we use induction on $|\mathrm{rad}(X)|$.

The lemma is trivial if $|\mathrm{rad}(X)| = 1$. Assume that $\mathrm{rad}(X) \neq 1$ and let N be a minimal normal subgroup of X contained in $\mathrm{rad}(X)$. Then $G \cap N = 1$, $|N|$ is a divisor of 48 and N is elementary abelian. It follows that $N = \mathbb{Z}_3$ or $N = \mathbb{Z}_2^r$ for some $1 \leqslant r \leqslant 4$. First we prove the following claim.

Claim. $GN = G \times N$.

Consider the conjugate action of G on N and let K be the kernel of this action. Then $K = T^r$ for some $r \leqslant \ell$ and $G/K \leqslant \mathrm{Aut}(N)$. Clearly, $GN = G \times N$ if and only if $K = G$. To prove the claim we suppose on the contrary that $K \neq G$: that is, $r < \ell$. This implies that $G/K \cong T^{\ell - r}$ is non-solvable, and hence $\mathrm{Aut}(N)$ is non-solvable.

If $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$ or $\mathbb{Z}_3$, then $\mathrm{Aut}(N)$ is solvable, which is a contradiction. Thus, $N \cong \mathbb{Z}_2^4$ or $\mathbb{Z}_2^3$. Since $G \cap N = 1$, we have $|X| \geqslant |G|\,|N| = |V(\varGamma)|\,|G_u|\,|N|$. Since X is arc-transitive, $3|N|$ is a divisor of $|X_u|$. Then, by [**25**], X is 4-arc-transitive for $N \cong \mathbb{Z}_2^3$ and 5-arc-transitive for $N \cong \mathbb{Z}_2^4$. Moreover, $|X : GN| = 3$, 6 or 1.

Assume that $|X : GN| = 3$ or 6. Then $|X| = 3|V(\varGamma)|\,|G_u|\,|N|$ or $6|V(\varGamma)|\,|G_u|\,|N|$. Since X is at most 5-regular, $|G_u| = 1$ or 2 because $N \cong \mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$. Furthermore, if $N \cong \mathbb{Z}_2^4$, then $|X : GN| = 3$. Consider the action of X on the set $[X : GN]$ of right cosets of GN in X by right multiplication. The kernel of this action is $(GN)_X$, the largest normal subgroup of X contained in GN. Then $X/(GN)_X \lesssim S_3$ or S_6 depending on whether $N \cong \mathbb{Z}_2^4$ or $\mathbb{Z}_2^3$, respectively. Clearly, $N \leqslant (GN)_X$.

Note that $GN/N \cong G/G \cap N \cong G = T^\ell$ and $GN/(GN)_X \cong (GN/N)/((GN)_X/N)$. We have $GN/(GN)_X \cong T^s$ for some $s \geqslant 0$. If $s = 0$, then $GN = (GN)_X \trianglelefteq X$. Write $Y = GN$. By the transitivity of G, $Y = GY_u$ and hence $|Y| = |G|\,|N| = |G|\,|Y_u|/|G_u|$,

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implying that Y_u is a 2-group. On the other hand, since $Y \trianglelefteq X$, we have $Y_u \trianglelefteq X_u$. It follows that $3 \mid |Y_u|$ because the action of X_u on the neighbourhood of u in Γ is primitive, which is a contradiction. Thus, $s \geqslant 1$ and $X/(GN)_X$ is non-solvable. It follows that $N = \mathbb{Z}_2^3$ and $X/(GN)_X \lesssim S_6$. By [**4**], a non-abelian simple group with order dividing 720 must be A_5 or A_6, and hence $T = A_5$ or A_6. This is impossible because $G/K \leqslant \mathrm{Aut}(N) = \mathrm{PSL}(3,2)$ and $\mathrm{PSL}(3,2)$ has no subgroup isomorphic to A_5 or A_6 ($|\mathrm{PSL}(3,2)|$ is not divisible by 5).

Assume that $|X : GN| = 1$: that is, $X = GN$. Recall that $K = T^r$ and K is the kernel of G acting on N by conjugation. Set $M = KN$. Then $M \trianglelefteq GN = X$. Recall that $K \neq G$. Since $G/K \leqslant \mathrm{Aut}(N)$, we have $G/K \leqslant \mathrm{GL}(4,2)$ or $\mathrm{GL}(3,2)$ depending on whether $N = \mathbb{Z}_2^4$ or $\mathbb{Z}_2^3$, respectively, and since $|\mathrm{GL}(3,2)| = 2^3 \cdot 3 \cdot 7$ and $|\mathrm{GL}(4,2)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, we have $K \cong T^{\ell-1}$ and $X/M = GN/KN \cong G/K \cong T$.

If M is transitive on $V(\Gamma)$, then $X = MX_u$. Thus, $|M|\,|T| = |M|\,|X/M| = |X| = |M|\,|X_u|/|M_u|$, implying that $|T|$ is a divisor of $|X_u|$ and hence a divisor of 48, which is a contradiction. If M has two orbits, then Γ is a bipartite graph, and the two orbits of M are the partite sets of Γ. By the transitivity of G on $V(\Gamma)$, the subgroup of G fixing the two partite sets setwise has index 2 in G, which is a contradiction. It follows that M has more than two orbits. By Proposition 2.4, the quotient graph Γ_M is a cubic symmetric graph and X/M is 4-arc-transitive on Γ_M when $N = \mathbb{Z}_2^3$ and 5-arc-transitive when $N = \mathbb{Z}_2^4$.

Suppose that $N = \mathbb{Z}_2^4$. Then $T \cong G/K \leqslant \mathrm{GL}(4,2)$ and, since $|\mathrm{GL}(4,2)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, by [**4**, p. 239], T is one of the following groups:

$$A_5, \quad \mathrm{PSL}(3,2), \quad A_6, \quad \mathrm{PSL}(2,8), \quad A_7, \quad \mathrm{PSU}(3,3) \quad \text{or} \quad \mathrm{PSL}(4,2),$$

which have orders $2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3 \cdot 7$, $2^3 \cdot 3^2 \cdot 5$, $2^3 \cdot 3^2 \cdot 7$, $2^3 \cdot 3^2 \cdot 5 \cdot 7$, $2^5 \cdot 3^3 \cdot 7$ and $2^6 \cdot 3^2 \cdot 5 \cdot 7$, respectively. Note that $X/M \cong T$ is 5-arc-transitive on Γ_M. This means that $|T| = 3 \cdot 2^4 \cdot |V(\Gamma_M)|$. It follows that $T = \mathrm{PSU}(3,3)$ or $\mathrm{PSL}(4,2)$, forcing $|V(\Gamma_M)| = 126$ or 420. This is impossible because there is no connected cubic 5-arc-transitive graph of order 126 or 420 by [**3**]. Thus, $N = \mathbb{Z}_2^3$. It follows that $T \cong G/K \leqslant \mathrm{GL}(3,2)$, forcing $T = \mathrm{GL}(3,2)$. Since $|T| = 3 \cdot 2^3 \cdot |V(\Gamma_M)|$, Γ_M is a cubic symmetric graph of order 7, which is a contradiction. This completes the proof of the claim.

We are now ready to finish the proof. Consider the quotient graph Γ_N corresponding to the orbits of N. Since $|N|$ is a power of 2 or 3, N cannot be transitive on $V(\Gamma)$. Also, N cannot have two orbits because Γ is not bipartite; otherwise, G has a subgroup of index 2, which is a contradiction. Thus, N has more than two orbits and, by Proposition 2.4, Γ_N is a cubic symmetric graph. Furthermore, $GN/N \cong T^\ell$ is transitive and X/N is arc-transitive on Γ_N. Note that $\mathrm{rad}(X/N) = \mathrm{rad}(X)/N$. By inductive hypothesis, $\mathrm{rad}(X/N) \cdot GN/N = \mathrm{rad}(X/N) \times GN/N$. Thus, $\mathrm{rad}(X)G/N = \mathrm{rad}(X)/N \times GN/N$, implying that $GN \trianglelefteq \mathrm{rad}(X)G$. By the claim above, G is characteristic in GN and hence normal in $\mathrm{rad}(X)G$. It follows that $\mathrm{rad}(X)G = \mathrm{rad}(X) \times G$. □

Lemma 3.2. *Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 2$. Let Γ be a connected cubic symmetric graph and let $G \leqslant X \leqslant \mathrm{Aut}(\Gamma)$ such that G is transitive and X is quasiprimitive on $V(\Gamma)$. Then G is normal in X.*

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Proof. Let $N = \mathrm{soc}(X)$. Since X is quasiprimitive, N is transitive on $V(\Gamma)$ and, by Proposition 2.5, one may assume that $N = S^t$ for a simple group S and positive integer t. By the transitivity of G, we have $X = GX_u$ for $u \in V(\Gamma)$. Since Γ is a connected cubic symmetric graph, by [**25**], $|X_u|$ is a divisor of 48.

Consider the group NG. Since $N \cap G \lhd G$, we may assume that $N \cap G = T^r$ for some $0 \leqslant r \leqslant \ell$, implying that $|NG| = |N|\,|G\colon G \cap N| = |N|\,|T^{\ell-r}|$. Set $Y = NG$. Since N is transitive, we have $Y = NY_u$. Thus, $|Y_u| = |N \cap Y_u|\,|T^{\ell-r}|$. Since Y_u is a subgroup of X_u, $|T^{\ell-r}|$ is a divisor of 48, and since T is a non-abelian and simple, $\ell = r$: that is, $N \cap G = G$. It follows that $G \leqslant N$. In particular, N is non-abelian and $N = N_uG$.

By Proposition 2.5, X has one of the following types: HS, HC, AS, TW, SD, CD or PA. Suppose that X has two minimal normal subgroups, say N_1 and N_2. By Proposition 2.5, $N = N_1 \times N_2$, and both N_1 and N_2 are direct products of the non-abelian simple group S. Since X is quasiprimitive, N_1 is transitive on $V(\Gamma)$. It follows that $|N_1|\,|N_2| = |N| = |N_1|\,|N_u|$ and hence $|N_2|$ is a divisor of 48. This is impossible because N_2 is non-solvable. Thus, X cannot be of types HS and HC and X has a unique minimal normal subgroup: that is, $N = \mathrm{soc}(X)$. Since $|N_u|$ is a divisor of 48, N_u cannot be non-solvable; thus X cannot be of type CD or type SD. The remaining types of X are AS, TW and PA.

Let X be of type AS. Then $N = \mathrm{soc}(X)$ is a non-abelian simple group and $N \leqslant X \leqslant \mathrm{Aut}(N)$. Recall that $N = N_uG$. Clearly, $N \neq N_u$ and hence N_u is a proper subgroup of N. By Proposition 2.6, $N = A_n$ and $N_u = N \cap N_u = A_{n-1}$, where $n = |N\colon N_u| \geqslant 10$. Thus, N is 2-transitive on $V(\Gamma)$, implying $\Gamma = K_n$, which is a contradiction.

Let X be of type TW. Then N is regular on $|V(\Gamma)|$ and hence $|N| \leqslant |G|$. Since $G \leqslant N$, we have $G = N$. Thus, $G \trianglelefteq X$, as required.

Finally, let X be of type PA. In this case, $N_u \neq 1$. Recall that $G \leqslant N$, $G = T^\ell$ and $N = S^t$, where $\ell \geqslant 2$ and $t \geqslant 2$. Since Γ has valency 3, X_u is primitive on the neighbourhood $N(u)$ of u in Γ, and since $N_v = N \cap X_v \trianglelefteq X_v$, N_v is transitive on $N(u)$, implying that Γ is N-arc-transitive. Let $H \trianglelefteq N$ and $H \cong S^{t-1}$. Then $N/H \cong S$.

Suppose that H is transitive on Γ. Then $N = HN_u$, implying that $|S|$ is a divisor of $|N_u| = 48$, which is a contradiction. Suppose that H has two orbits. Since N is arc-transitive, Γ is bipartite with the orbits of H as its two partite sets. The subgroup of N fixing each partite set of Γ has index 2 in N. This is impossible because $N = S^t$.

Thus, H has more than two orbits. By Proposition 2.4, the quotient graph Γ_H corresponding to the orbits of H is a cubic N/H-arc-transitive graph. Since $G \leqslant N$, we have $G \cap H \trianglelefteq G$, implying that $G \cap H = T^m$ for some non-negative integer m. Since G is transitive, $G \nleqslant H$, forcing $m < \ell$. Note that $T^{\ell-m} \cong G/G \cap H \cong GH/H \leqslant N/H \cong S$. Let Δ be an orbit of H on $V(\Gamma)$. Since GH/H is transitive on $V(\Gamma_H)$, we have $N/H = GH/H(N/H)_\Delta$, where $(N/H)_\Delta$ is the stabilizer of Δ in N/H. If $\ell - m \geqslant 2$, by Proposition 2.6, $N/H = A_n$ and $(N/H)_\Delta = A_{n-1}$, where $n = |N/H\colon (N/H)_\Delta| = |V(\Gamma_H)| \geqslant 10$. Thus, Γ_H is the complete graph K_n with $n \geqslant 10$, which is a contradiction. It follows that $GH/H \cong T$. Noting that $|(N/H)_\Delta|$ is a divisor of 48 and that $GH/H \leqslant N/H$, we have $|S| = n|T|$ with $n|48$, and one may view T as a subgroup of S.

Recall that $N = N_uG$. Then $|N|/|G||48$, implying that $|T|^{t-\ell}n^t|48$. It follows that $t = \ell$ because $|T|^{t-\ell}n^t$ is a positive integer and $|T|$ has at least three distinct prime

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factors. This means that $n^t|48$. Since $t \geqslant 2$, we have $n = 1$, 2 or 4. If $n = 2$ or 4, then $|S\colon T| = 2$ or 4, contrary to the simplicity of S. Thus, $n = 1$. In this case, $G = N \trianglelefteq X$, as required. □

Lemma 3.3. *Let T be a non-abelian simple group and let $G = T^\ell$ with $\ell \geqslant 1$. Let Γ be a connected cubic graph and let $G \leqslant X \leqslant \mathrm{Aut}(\Gamma)$ such that Γ is G-vertex-transitive and X-arc-transitive. Suppose that X is non-quasiprimitive on $V(\Gamma)$ and that every minimal normal subgroup of X is non-abelian. Then X has a non-trivial normal subgroup contained in G.*

Proof. Assume that M is a normal subgroup of X such that $M = S^t$ for a non-abelian simple group S and positive integer t. Since G is transitive on $V(\Gamma)$, we have $MG \leqslant X = GX_u$, and since Γ is X-arc-transitive, $|M\colon M \cap G|$ is a divisor of $|X_u|$ and hence a divisor of 48. Thus, $M \cap G \trianglelefteq G$ implies that $M \cap G = T^r$ for some $r \geqslant 1$.

Claim. If $S \ncong T$, then M is simple.

Suppose that $S \ncong T$. Assume to the contrary that M is not simple. Then $t \geqslant 2$. Choose $T_1 \trianglelefteq M \cap G$ such that $T_1 \cong T$. Since the intersection of all normal subgroups of M isomorphic to S^{t-1} is trivial, one may assume that $M = K \times H$ such that $K \cong S^{t-1}$, $H \cong S$ and $T_1 \nleqslant K$. Since $K \trianglelefteq M$ and $T_1 \leqslant M$, we have $T_1 \cap K \trianglelefteq T_1$. It follows that $T_1 \cap K = T_1$ or 1. If $T_1 \cap K = T_1$, then $T_1 \leqslant K$, which is a contradiction. Thus, $T_1 \cap K = 1$ and $T \cong T_1 \cong T_1/T_1 \cap K \cong T_1K/K \leqslant M/K \cong H \cong S$, implying that $|T|$ is a divisor of $|S|$.

Set $\Omega = [M\colon M \cap G]$, the set of right cosets of $M \cap G$ in M. Consider the action of M on Ω by right multiplication. The kernel of this action is $(M \cap G)_M$, the largest normal subgroup of M contained in $M \cap G$. Since $(M \cap G)_M \trianglelefteq M \cap G$, $(M \cap G)_M$ is a direct product of the non-abelian simple group T, and since $(M \cap G)_M \trianglelefteq M$, $(M \cap G)_M$ is a direct product of the non-abelian simple group S. It follows that $(M \cap G)_M = 1$ because $S \ncong T$, and hence the action of M on Ω is faithful. Recall that $M = K \times H$ with $K \cong S^{t-1}$ and $H \cong S$. Then $H \leqslant C_M(K)$, where $C_M(K)$ is the centralizer of K in M.

If K is transitive on Ω, then, by Proposition 2.1, H is semiregular on Ω. Thus, $|H|$ is a divisor of $|\Omega|$ and hence a divisor of 48, contrary to the fact that H is non-abelian and simple. Let K have n orbits of length m. Then $m, n \geqslant 2$ and $|\Omega| = mn|48$. Since a non-solvable group cannot have a faithful action of degree less than 5, we have $m, n \geqslant 5$. It follows that $\{m, n\} = \{6, 8\}$. Since M is transitive on Ω, we have $H \lesssim S_n$ and $K \lesssim S_m \times \cdots \times S_m = S_m^n$. Thus, $S \lesssim A_6$ because $\{m, n\} = \{6, 8\}$. Since $|T|$ is a divisor of $|S|$ and $S \ncong T$, we have $T \cong A_5$ and $S \cong A_6$, which is impossible because $|\Omega| = |M|/|M \cap G| = |S|^t/|T|^r$ is a divisor of 48 for some $t \geqslant 2$. It follows that $t = 1$, as claimed.

In what follows we prove that X has a minimal normal subgroup that is a direct product of the non-abelian simple group T.

Suppose on the contrary that every minimal normal subgroup of X is not a direct product of copies of T. Since X is non-quasiprimitive, X has a minimal normal subgroup, say N_1, which is intransitive on $V(\Gamma)$. By hypothesis, N_1 is non-abelian, and, by the claim

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above, N_1 is non-abelian and simple. Set $C = C_X(N_1)$, the centralizer of N_1 in X. If $C = 1$, then $X/N_1 \lesssim \mathrm{Out}(N_1)$. By Proposition 2.3, X/N_1 is solvable and hence GN_1/N_1 is solvable. Note that $N_1 \cap G \trianglelefteq G$ implies that $G \cap N_1 \cong T^s$ for some integer s. Since $GN_1/N_1 \cong G/G \cap N_1 \cong T^{\ell-s}$, we have $\ell = s$: that is, $G \cap N_1 = G$. Thus, $G \leqslant N_1$. This is impossible because G is transitive and N_1 is intransitive on $V(\Gamma)$.

Thus $C \neq 1$. Let $N_2 \leqslant C$ be a minimal normal subgroup of X. By hypothesis and the claim above, N_2 is non-abelian and simple and $N_2 \ncong T$. Set $N = N_1N_2 = N_1 \times N_2$. Again by the claim above, $N_1 \ncong N_2$. Since $NG \leqslant X = GX_u$, $|N\colon N \cap G|$ is a divisor of 48. Moreover, $N \cap G \trianglelefteq G$ implies that $N \cap G = T^t$ for some $t \geqslant 1$.

Set $\Omega_1 = [N\colon N \cap G]$. Consider the action of N on Ω_1 by right multiplication and the kernel of this action is $(N \cap G)_N$, which is the largest normal subgroup of N contained in $N \cap G$. It follows that $(N \cap G)_N = 1$ because $(N \cap G)_N$ is normal in both $N \cap G$ and N. This means that the action of N on Ω_1 is faithful. Since $N_1 \leqslant C_N(N_2)$ and $|N_1|$ is not a divisor of 48, N_2 is intransitive on Ω_1 by Proposition 2.1. Assume that N_2 has n orbits of length m. Then $m, n \geqslant 2$ and $|\Omega_1| = mn$, which is a divisor of 48. Since N_1 and N_2 are non-abelian and simple, we have $m, n \geqslant 5$ and $\{m, n\} = \{6, 8\}$, implying $|\Omega_1| = 48$. Since N is transitive on Ω_1, we have $N_1 \lesssim S_n$ and $N_2 \lesssim S_m \times \cdots \times S_m = S_m^n$. It follows that either $N_1 \lesssim A_6$ and $N_2 \lesssim A_8$ or $N_1 \lesssim A_8$ and $N_2 \lesssim A_6$. If $G \cap N_i = 1$ for $i = 1$ or 2, then $|N_i|$ is a divisor of 48, which is a contradiction. Thus $G \cap N_i \neq 1$, implying $|T|\,\big|\,|N_i|$. It follows that $T \cong A_5$ because $T \ncong N_i$ for $i = 1$ or 2. Furthermore, either $N_1 \cong A_6$ or $N_2 \cong A_6$. However, $48 = |\Omega_1| = |N_1|\,|N_2|/|N \cap G| = |N_1|\,|N_2|/|A_5|^t$, which is impossible.

Thus, X has a minimal normal subgroup isomorphic to a direct product of copies of T, say $L = T^p$ ($p \geqslant 1$). Let $L \cap G = T^q$. Then $|L|/|G \cap L| = |T|^{p-q}$. Since $GL \leqslant X = GX_u$, $|L|/|G \cap L|$ is a divisor of 48. It follows that $p = q$: that is, $L \cap G = L$. Thus $L \leqslant G$: that is, X contains a non-trivial normal subgroup L contained in G. $\square$

Proof of Theorem 1.1. Recall that T is a non-abelian simple group and that $G = T^\ell$ with $\ell \geqslant 1$. Assume that every connected T-vertex-transitive cubic symmetric graph is T-normal. Let Γ be a connected G-vertex-transitive cubic symmetric graph and let $A = \mathrm{Aut}(\Gamma)$. We aim to prove that $G \trianglelefteq A$.

If $\ell = 1$, then the theorem is true by hypothesis. Assume that $\ell \geqslant 2$. To prove the theorem, we apply induction on ℓ. Note that Γ is not bipartite because it is G-vertex-transitive and G has no subgroup of index 2.

Set $R = \mathrm{rad}(A)$. By Lemma 3.1, $RG = R \times G$. Consider the quotient graph Γ_R of Γ corresponding to the orbits of R. Let $u \in V(\Gamma)$. If R is transitive on $V(\Gamma)$, then $A = RA_u$. It follows that $|RG|$ is a divisor of $|RA_u|$, implying that $|G|$ is a divisor of 48, which is a contradiction. If R has two orbits on $V(\Gamma)$, then Γ is bipartite, which is a contradiction. Thus, R has more than two orbits and, by Proposition 2.4, Γ_R is a connected cubic symmetric graph, which is A/R-arc-transitive and GR/R-vertex-transitive. Note that $GR/R \cong G = T^\ell$. It is easy to see that every minimal normal subgroup of A/R is non-abelian.

Assume that A/R is quasiprimitive on $V(\Gamma_R)$. By Lemma 3.2, $GR/R \trianglelefteq A/R$: that is, $GR \trianglelefteq A$. Since $GR = G \times R$ and R is solvable, G is characteristic in GR. It follows that G is normal in A, as required.

Assume that A/R is non-quasiprimitive on $V(\Gamma_R)$. By Lemma 3.3, A/R has a non-trivial normal subgroup M/R contained in GR/R. Thus $M \trianglelefteq A$, $M \leqslant GR$, $R \leqslant M$ and $R \neq M$. It follows that $M = M \cap RG = R(M \cap G)$ and $M \cap G \neq 1$. Since $GR = G \times R$ and $M \cap G \trianglelefteq G$, we have $M = R \times M \cap G$ and $M \cap G = T^r$ for some $1 \leqslant r \leqslant \ell$. The subgroup $M \cap G$ is characteristic in M and hence $M \cap G \trianglelefteq A$. If $r = \ell$, then $G = G \cap M$ and G is normal in A, as required. Suppose that $r < \ell$. Consider the quotient graph $\Gamma_{G \cap M}$ corresponding to the orbits of $G \cap M$. If $G \cap M$ is transitive on Γ, then $A = (G \cap M)A_u$, implying that $|T|$ is a divisor of 48, which is a contradiction. If $G \cap M$ has two orbits, then Γ is bipartite, which is a contradiction. Thus, $G \cap M$ has more than two orbits and, by Proposition 2.4, $\Gamma_{G \cap M}$ is a connected cubic symmetric graph with $G/M \cap G$ as a vertex-transitive group and $A/M \cap G$ as an arc-transitive group on $\Gamma_{G \cap M}$. Note that $G/M \cap G \cong T^{\ell - r}$ with $1 \leqslant \ell - r < \ell$. By inductive hypothesis, $G/M \cap G \lhd \mathrm{Aut}(\Gamma_{G \cap M})$. Thus $G/M \cap G \trianglelefteq A/M \cap G$: that is, $G \trianglelefteq A$. This completes the proof. □

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