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CONFORMALLY FLAT RIEMANNIAN MANIFOLDS AS HYPERSURFACES OF THE LIGHT CONE

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ABSTRACT. Simply connected conformally flat Riemannian manifolds are characterized as hypersurfaces in the light cone of a standard flat Lorentzian space, transversal to its generators. Some applications of this fact are given.

1. **Introduction**.

(1.1) A Riemannian manifold M^n is said to be conformally flat if every point of the manifold lies in a local coordinate system $(x_1, \ldots, x_n)$ with respect to which the Riemannian metric takes the form $ds^2 = \phi^2 \Sigma (dx_i)^2$, for some non-vanishing function ϕ. The main purpose of this note is to present, in a modern form, a classical result mostly due to Brinkmann ([1], [2]) which relates conformal flatness with Lorentzian geometry.

Let $V^{n+1} = \{x \in \mathbf{L}^{n+2} : (x, x) = 0, x \neq 0\}$ be the $(n+1)$-dimensional light cone with the induced semi-definite metric from the standard flat $(n+2)$-dimensional Lorentz space $\mathbf{L}^{n+2}$.

THEOREM 1.2. *Let $M^n, n \geqq 3$, be an n-dimensional simply connected Riemannian manifold. Then M^n is conformally flat if and only if M^n can be isometrically immersed into V^{n+1}.*

Let us consider the tensor L in M given by

$$L(\cdot\,,\cdot) = \frac{1}{(n-2)}\left\{ \mathrm{Ric}\,(\cdot\,,\cdot) - \frac{S}{2(n-1)}\langle \cdot\,,\cdot\rangle \right\}, \tag{1.3}$$

where $\langle\ ,\ \rangle$ is the inner product given by the Riemannian metric of M, Ric is the Ricci tensor and S is the scalar curvature of M. It is a well known result of Schouten [7] that, for $n \geqq 3$, a necessary and sufficient condition for M to be conformally flat is that the Weyl curvature tensor C, defined by

$$\begin{aligned}\langle C(X,Y)Z,W\rangle = \langle R(X,Y)Z,W\rangle - L(Y,Z)\langle X,W\rangle \\ + L(X,Z)\langle Y,W\rangle - L(X,W)\langle Y,Z\rangle + L(Y,W)\langle X,Z\rangle,\end{aligned} \tag{1.4}$$

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vanishes identically on M, and that L is a Codazzi tensor, i.e.,

$$(1.5) \qquad (\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z).$$

Here R is the curvature tensor and ∇ is the Riemannian connection of M. For a modern presentation in invariant notation of formulas (1.3) to (1.5) see [4] or [9].

(1.6) Furthermore, if $n \geqq 4$, by using the second Bianchi's identity one can show that (1.4) implies (1.5).

The idea of the proof of Theorem (1.2) is to show that equations (1.4) and (1.5) are precisely the Gauss and Codazzi equations of the immersion. In this context, (1.6) can be read as : for $n \geqq 4$, the Gauss equation implies the Codazzi equation (cf. [2], Appendix 22).

As a first application of Theorem (1.2), we present an alternative proof of the following well known result:

COROLLARY 1.7. (Kuiper's Theorem [3]). *Let $M^n, n \geqq 3$, be a simply connected conformally flat manifold. Then M^n can be conformally immersed (developed) into the Euclidean sphere S^n and this immersion is unique but for conformal transformations of S^n. In particular, if M^n is compact, then M^n is conformal to S^n.*

Let $S^n(c)$ and $\mathbf{H}^n(-c), c > 0$, be the n-dimensional Euclidean sphere and hyperbolic space, with radius $^1/\sqrt{c}$ and $-^1/\sqrt{c}$, respectively. Another application of (1.2) is:

COROLLARY 1.8. ([6]). *$S^n(c) \times \mathbf{H}^m(-c)$ is conformally flat.*

(1.9) We say that a Riemannian manifold $M^n, n \geqq 4$, satisfies *the axiom of conformally flat hypersurfaces* if for every point p in M and for every hyperplane section $W \subset T_pM$, there exists a conformally flat hypersurface N of M, passing through p, such that $T_pN = W$. As a final application of Theorem (1.2), we prove:

THEOREM 1.10. *Let $M^n, n \geqq 4$, be a Riemannian manifold which can be isometrically immersed as a (spacelike) hypersurface into $\mathbf{L}^{n+1}$. Then M^n satisfies the axiom of conformally flat hypersurfaces.*

It is easy to see that the necessary condition of Theorem (1.10) is not sufficient.

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2. **Proofs of the Results.** Through this note we assume the knowledge of the basic theory of submanifolds with indefinite (semi-Riemannian) metrics. A recent reference for this material is Chapter 4 of [5]. The complete statement and proof of the fundamental theorem of submanifolds, in its Euclidean version, can be found in [8], pp. 61–73. We remark here that the (statement and) proof of this theorem, in its Lorentzian version, follows the same lines of the Euclidean case.

(2.1) PROOF OF THEOREM 1.2. Suppose that there exists an isometric immersion $f : M^n \longrightarrow V^{n+1}$ and denote by $\delta = i \circ f$, where $i : V^{n+1} \longrightarrow \mathbf{L}^{n+2}$ is the inclusion map. For any tangent vector $X \in T_qM$, we have $0 = X\langle\delta, \ \delta\rangle = \langle\delta_*X, \delta(q)\rangle$. So we may think of $\delta(q)$ itself as a vector of the normal space of the immersion δ at $q \in M$. Furthermore, since $\tilde{\nabla}_X\delta = \delta_*X = X$, where $\tilde{\nabla}$ is the connection in $\mathbf{L}^{n+2}$, then $\nabla^{\perp}_X\delta = 0$, i.e., δ is parallel in the normal connection $\nabla^{\perp}$ of the normal bundle $TM^{\perp}$, and the second fundamental form A_δ of the immersion δ satisfies.

$$A_\delta = -Id. \tag{2.2}$$

Fix $p \in M$. Then there exist orthogonal vectors ξ_p, η_p in $T_pM^{\perp}$ so that $\delta(p) = \xi_p - \eta_p, -|\xi_p|^2 = |\eta_p|^2 = 1$. From (2.2) we get $A_{\delta(p)} = A_{\xi_p} - A_{\eta_p} = -Id$, and consequently the curvature tensor $R^{\perp}$ of the normal connection vanishes identically, by the Ricci equation. This implies that there exist local extensions of ξ_p and η_p to parallel vector fields ξ and η, which can be taken globally defined if M is simply connected. Since δ and $\xi - \eta$ are both null vector fields in a Lorentzian vector bundle with 2-dimensional fibres, it follows that $\delta = \phi(\xi - \eta)$, for some smooth positive function ϕ with $\phi(p) = 1$. But, for any field X tangent to $M, 0 = \nabla^{\perp}_X\delta = X(\phi)(\xi - \eta) + \phi\nabla^{\perp}_X(\xi - \eta) = X(\phi)$ $(\xi - \eta)$. So $X(\phi) = 0$, hence $\phi \equiv 1$. Then

$$\delta = \xi - \eta. \tag{2.3}$$

At an arbitrary point q in M, choose a tangent orthonormal frame $e_1, \cdots, e_n$ which diagonalizes A_ξ and A_η simultaneously, that is, $A_\xi e_i = \lambda_i e_i, A_\eta e_i = \mu_i e_i$, for some real numbers $\lambda_i, \mu_i, i = 1, \cdots, n$. Then from (2.2), (2.3) and the Gauss equation, we have $\kappa(e_i, e_j) = 1 + \lambda_i + \lambda_j$, where κ is the sectional curvature of the tensor R. Now a straightforward calculation shows that $L(e_i, e_j) = (1 + 2\lambda_i)\delta_{ij}/2$. Therefore

$$L(X, Y) = \langle X, Y\rangle/2 + \langle A_\xi X, Y\rangle. \tag{2.4}$$

From (2.2) and (2.4) it follows that

$$\begin{aligned} \langle A_\xi X, Y\rangle &= L(X, Y) - \langle X, Y\rangle/2, \\ \langle A_\eta X, Y\rangle &= L(X, Y) + \langle X, Y\rangle/2. \end{aligned} \tag{2.5}$$

By using (2.5) it is now easy to see that the Gauss and Codazzi equations are equivalent to equations (1.4) and (1.5), respectively. Then M is conformally flat.

Assume now that M is conformally flat and consider the trivial Lorentzian vector bundle $T = M^n \times \mathbf{L}^2$ over M. Endow this bundle with a compatible connection, say ∇', which makes the sections $\xi(q) = (q, e_1)$ and $\eta(q) = (q, e_2)$ parallel, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Now define a symmetric bilinear form $\beta : TM \times TM \longrightarrow T$ by $\beta(X, Y) = -\langle A_\xi X, Y\rangle\xi + \langle A_\eta X, Y\rangle\eta$, where A_ξ and A_η are given by (2.5). In view of the fundamental theorem of submanifolds in the Lorentzian version (see the beginning of this section), one has to verify the Gauss, Codazzi and Ricci equations for β and ∇' in order to define an isometric immersion $G : M^n \longrightarrow \mathbf{L}^{n+2}$. In that case β is the second fundamental form of G and ∇' is the induced connection in the normal bundle. But

Gauss and Codazzi equations follow from (1.4) and (1.5), and the Ricci equation from the fact that A_ξ and A_η can be simultaneously diagonalized. To conclude the proof of the theorem we only have to show that $G(M)$ is contained in a light cone of $\mathbf{L}^{n+2}$. For this, let $h = G - (\xi - \eta)$. For any tangent field Z we have $Z(h) = Z + A_{\xi-\eta}(Z) = 0$ by (2.5). Therefore $G = h + (\xi - \eta)$, where h is a constant vector, $\|G - h\|^2 = 0$ and $G - h \neq 0$ everywhere. □

(2.6) PROOF OF COROLLARY 1.7. We need the following

LEMMA . *Let $M^n, n \geqq 3$, be a Riemannian manifold. Then there exists an isometric immersion $f : M^n \longrightarrow V^{N+1}, N \geqq n$, if and only if there exists a conformal immersion $F : M^n \longrightarrow S^N$. Moreover, f is unique up to rigid motions of V^{N+1} if and only if F is unique up to conformal transformations of S^N.*

PROOF OF THE LEMMA. Assume that there exists an isometric immersion $f : M^n \longrightarrow V^{N+1} \subset \mathbf{L}^{n+2}$ and denote by $e_1, \ldots, e_{N+2}$ the canonical basis of $\mathbf{L}^{n+2}$, where $(e_{N+2}, e_{N+2}) = -1$. Define $F : M^n \longrightarrow V^{N+1}$ by

$$F(x) = \frac{1}{\Psi(x)} f(x),$$

where $\psi : M^n \longrightarrow \mathbf{R}$ is the non-vanishing function given by $\psi(x) = (f(x), e_{N+2})$. Clearly $F(M)$ is contained in some Euclidean sphere $S^N \subset V^{N+1}$. Moreover F is a conformal immersion because $(F_*X, F_*Y) = (f_*X, f_*Y)/\psi^2 = \langle X, Y \rangle / \psi^2$ for any pair X, Y of vector fields on M. The converse is similar and the rest of the proof of the Lemma is straightforward.

The first part of Corollary (1.7) now follows from the Lemma. For the compact case, use a standard argument on covering spaces. □

(2.7) Notice that in case $N = n$, the above lemma gives a proof of the *local* version of Theorem (1.2). Thus this observation jointly with (2.5) and the argument following it, gives a short proof of Schouten's Theorem (cf [1], p. 2).

(2.8) PROOF OF COROLLARY 1.8. Just observe that the standard inclusion of $S^n(c) \times H^m(-c)$ into $\mathbf{R}^{n+1} \oplus \mathbf{L}^{m+1} = \mathbf{L}^{n+m+2}$ gives a hypersurface of the light cone $V^{n+m+1} \subset \mathbf{L}^{n+m+2}$ and use (1.2).

(2.9) PROOF OF THEOREM (1.10). Given a point p in M and a subspace $W^{n-1} \subset T_pM$, take an n-dimensional subspace $U^n \subset T_p\mathbf{L}^{n+1}$ such that $U = W \oplus \operatorname{span}\{Z\}$, where $Z \neq 0$ is an arbitrary null vector in $T_p\mathbf{L}^{n+1}$. Take a light cone $H \subset \mathbf{L}^{n+1}$ such that $T_pH = U$. Since M is transversal to $H, M \cap H$ is a hypersurface of both M and H, whose tangent plane at p is W and which is conformally flat, by Theorem (1.2) □

Since $\mathbf{L}^{n+1}$ can be foliated by light cones, it follows from Theorem (1.2) that any M^n as in Theorem (1.10) can be locally foliated by conformally flat hypersurfaces.

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