ON THE DISCRETE SUBGROUPS AND HOMOGENEOUS SPACES OF NILPOTENT LIE GROUPS

YOZÔ MATSUSHIMA

Recently A. Malcev[1] has shown that the homogeneous space of a connected nilpotent Lie group G is the direct product of a compact space and an Euclidean space and that the compact space of this direct decomposition is also a homogeneous space of a connected subgroup of G. Any compact homogeneous space M of a connected nilpotent Lie group is of the form $M = \tilde{G}/D$, where $\tilde{G}$ is a connected simply connected nilpotent group whose structure constants are rational numbers in a suitable coordinate system and D is a discrete subgroup of G.

In this paper we first determine the "situations" of discrete subgroups of a connected simply connected nilpotent group. In making use of this result we may prove the results of Malcev in a different method. Then we make some considerations on the homological properties of a compact homogeneous space and show that the cohomology groups of dimensions 1 and 2 of a nilpotent Lie algebra $\mathfrak{G}_R$ over the field R of rational numbers are isomorphic to the corresponding rational cohomology groups of a compact homogeneous space of the connected simply connected nilpotent group corresponding to the Lie algebra $\mathfrak{G}$ obtained from $\mathfrak{G}_R$ by extending the ground field R to the field of real numbers. In the above discussions Hopf-Eilenberg-MacLane's theory[2] on the relations between homology and homotopy of a space will play an important rôle.

1. Let G be a Lie group. To every element L of its Lie algebra $\mathfrak{G}$ there corresponds a one-paramenter subgroup $g(t)$ such that L is the tangent vector at the unit element to the curve $g(t)$. We shall denote this one-parameter subgroup $g(t)$ by $\exp tL$ and $\exp L$ is the point of parameter 1 on this curve. If G is a connected simply connected solvable group, then G is homeomorphic to an Euclidean space and each Lie subgroup H of G corresponding to a subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ is closed and simply connected.[3]

THEOREM 1. *Let G be a connected simply connected nilpotent group with*

Received Oct. 26, 1950.

1) See Malcev [8].

2) See Hopf [5], [6], Eilenberg and MacLane [3], [4].

3) See Chevalley [1].

the Lie algebra $\mathfrak{G}$ and D a discrete subgroup of G. Then we may choose a basis $L_1, L_2, \ldots, L_n$ of $\mathfrak{G}$ which has the following properties:

1) *$\{L_{i+1}, \ldots, L_n\}$ is an ideal of $\{L_i, L_{i+1}, \ldots, L_n\}$ for $i = 1, \ldots, n-1$ and hence every element in G may be written uniquely in the form* $(\exp t_1L_1)(\exp t_2L_2)\ldots(\exp t_nL_n)$.

2) *There exists an integer m, $1 \leqq m \leqq n$, such that $\{L_m, L_{m+1}, \ldots, L_n\}$ is a subalgebra of $\mathfrak{G}$ and that, if $[L_j, L_k] = \sum_l c^l_{jk} L_l$ for $m \leqq j$, $k \leqq n$, then c^l_{jk} are rational numbers.*

3) *The elements $g_k = \exp L_k$ $(k = m, m+1, \ldots, n)$ constitute a system of generators of D and every element in D may be written uniquely in the form $g_m{}^{s_m} \ldots g_n{}^{s_n}$, where s_i are integers.*

For the proof of Theorem 1 several lemmas are necessary.

LEMMA 1. Let G be a connected nilpotent group with the Lie algebra $\mathfrak{G}$. Then to every element g in G there exists an $L \in \mathfrak{G}$ such that $g = \exp L$. Moreover, if G is simply connected, L is determined uniquely by g.

Proof. Since the center of G is not discrete, the existence of such an L may be proved easily by induction on the dimension of G. Now let G be simply connected and let $g = \exp L = \exp L'$. We denote by A_g, A_L and $A_{L'}$ the matrices corresponding to g, L, and L respectively in the adjoint representations of G and $\mathfrak{G}$.[4] Then $A_g = \exp A_L = \exp A_{L'}$. Since $\mathfrak{G}$ is nilpotent, A_L and $A_{L'}$ are nilpotent and hence $\log \exp A_L = A_L$ and $\log \exp A_{L'} = A_{L'}$. Thus $A_L = A_{L'}$. Therefore $L = L' + M$, where M is in the center of $\mathfrak{G}$ and $\exp L = \exp L' \cdot \exp M$. Hence $\exp M = e$, the unit element of G. This shows that the one-parameter subgroup $\exp tM$ is compact. But G can not contain any compact subgroup different from e and hence $\exp tM = E$. Thus $M = 0$ and $L = L'$.

LEMMA 2. Let G be a connected nilpotent Lie group and H a connected Lie subgroup of G. Then the normalizer $N(H)$ of H is connected.

Proof. Let $\mathfrak{H}$ be the subalgebra of the Lie algebra $\mathfrak{G}$ of G corresponding to H and $\mathfrak{N}(\mathfrak{H})$ the normalizer of $\mathfrak{H}$ in $\mathfrak{G}$. Let $g \in N(H)$. Then by Lemma 1 there exists an $L \in \mathfrak{G}$ such that $g = \exp L$. It is sufficient to show that $L \in \mathfrak{N}(\mathfrak{H})$. As $g \in N(H)$, $A_g\mathfrak{H} \subseteqq \mathfrak{H}$ and hence $\log A_g\mathfrak{H} \subseteqq \mathfrak{H}$. Since $\log A_g = A_L$, $A_L\mathfrak{H} = [L, \mathfrak{H}] \subseteqq \mathfrak{H}$. Thus $L \in \mathfrak{N}(\mathfrak{H})$.

By the similar argument we may prove the following

LEMMA 3. The center of a connected nilpotent Lie group is connected.

LEMMA 4. Let H be a subgroup of a connected nilpotent Lie group G and $N(H)$ the normalizer of H. Then $N(H) \supsetneqq H$.

[4] We use these notations throughout this paper.

Proof. By induction on the dimension of G. Let Z be the center of G. If $H \not\supset Z$, then clearly $N(H) \neq H$. If $H \supset Z$, we may use the assumption of the induction on G/Z and H/Z and we obtain $N(H) \neq H$.

Proof of Theorem 1. We shall prove our theorem by induction on the dimension n of G. Let us assume that it has been proved already for all dimensions $< n$. Let $N(D)$ be the normalizer of D and A the connected component containing the unit element of $N(D)$. Since A contains the center of G, $\dim A \geqq 1$. If $A = G$, then D is normal and hence contained in the center of G. Since the center of G is connected by Lemma 3, every thing is clear in this case. Hence we may assume $A \neq G$. Now let K be the normalizer of A. Then K is connected by Lemma 2 and $N(D) \subset K$ and hence $D \subset K$. If $K \neq G$, then by our assumption of induction we may choose a basis $L_k, L_{k+1}, \ldots, L_n$ of the Lie algebra $\mathfrak{K}$ of K satisfying the conditions of our theorem. Then by Lemmas 4 and 2 we may add $L_1, \ldots, L_{k-1}$ to $L_k, L_{k+1}, \ldots, L_n$ so that $L_1, \ldots, L_n$ is a basis of $\mathfrak{G}$ satisfying our condition 1). Hence we may assume $K = G$. Then A is a normal subgroup of G. Now $D \cdot A \subset N(D)$ and hence $\overline{DA} \subset N(D)$. Therefore the connected component of the unit element of $\overline{DA}$ is A and hence the group DA/A is a discrete subgroup of G/A. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{G}$ corresponding to A. It follows that we may apply our theorem to G/A and DA/A: we obtain a basis $\{L_l^*, \ldots, L_l^*\}$ of $\mathfrak{G}/\mathfrak{A}$ satisfying the properties of our theorem. Let $\{t, t+1, \ldots, l\}$ be the subset of the set $\{1, \ldots, l\}$ satisfying our conditions 2) and 3) and let $g_k^* = \exp L_k^*$ $(k = t, t+1, \ldots, l)$. Let $g_k \in D$ be the representative of the element $g_k^* \in DA/A$ and $g_k = \exp L_k$ with $L_k \in \mathfrak{G}$. Let $L_k'^*$ de the class of L_k mod $\mathfrak{A}$ then $g_k^* = \exp L_k'^*$ and since G/A is simply connected, $L_k^* = L_k'^*$ by Lemma 1. Thus L_k is a representative of L_k^*. Further let L_j be a representative of L_j^* for $j < t$. Since $A \subset N(D)$, the group $D \cap A$ is a discrete normal subgroup of A. Then we may choose a basis $L_{l+1}, \ldots, L_n$ of $\mathfrak{A}$ such that $g_k = \exp L_k$ $(k = s, s+1, \ldots, n)$ constitute a system of free generators of the discrete central subgroup $D \cap A$ of A and $[L_k, L_j] = 0$ for $s \leqq k$, $j \leqq n$. Let $d \in D$. Then $d \equiv g_t^{u_t} \ldots g_l^{u_l}$ (A) and u_i are unique. Then $d = g_t^{u_t} \ldots g_l^{u_l} a$, $a \in D \cap A$, and $a = g_s^{u_s} \ldots g_u^{u_n}$ and u_i are unique. Hence $d = g_t^{u_t} \ldots g_l^{u_l} g_s^{u_s} \ldots g_n^{u_n}$ and this expression is unique. Now we show that $\{L_t, L_{t+1}, \ldots, L_l, L_s, L_{s+1}, \ldots, L_n\}$ is a subalgebra whose structure constants are rational. We set $\{L_t, \ldots, L_l, L_s, \ldots, L_n\} = \{M_m, M_{m+1}, \ldots, M_n\}$, i.e. $L_t = M_m$, $L_{t+1} = M_{m+1}, \ldots, L_n = M_n$. We shall show that, if $\mathfrak{H}_{k+1} = \{M_{k+1}, \ldots, M_n\}$ is such a subalgebra, $\mathfrak{H}_k = \{M_k, M_{k+1}, \ldots, M_n\}$ is also such a subalgebra $(k \leqq s-1)$. Let $(g_t, \ldots, g_l, g_s, \ldots,$

$g_n) = (h_m, \ldots, h_n)$ i.e. $g_t = h_m, g_{t+1} = h_{m+1}, \ldots, g_n = h_n$. Further let H_{k+1} and G_i be the subgroups of G corresponding to the subalgebras $\mathfrak{H}_{k+1}$ and $\mathfrak{G}_i = \{L_i, \ldots, L_n\}$ respectively. Then from our construction of the basis $\{L_1, \ldots, L_n\}$ we see that G_{i+1} is a normal subgroup of G_i. Now let $h = h_k h_u h_k^{-1}$ $(u \geqq k+1)$. Since $h_k = \exp M_k$ and $M_k = L_{l-(s-k)+1}$, $h_k \in G_{l-(s-k)+1}$ and $h_u \in G_{l-(s-k)+2}$. Hence $h = h_k h_u h_k^{-1} \in G_{l-(s-k)+2}$. As $h \in D$, $h = h_m^{u_m} \ldots h_n^{u_n} = \exp n_m M_m \ldots \exp u_n M_n$ and so $u_m = \ldots = u_k = 0$, i.e. $h_k h_u h_k^{-1} = h_{k+1}^{u_{k+1}} \ldots h_n^{u_n} \in H_{k+1}$. By Lemma 1 there exists $t_{k+1}M_{k+1} + \ldots + t_n M_n \in \mathfrak{H}_{k+1}$ such that $h = \exp(t_{k+1}M_{k+1} + \ldots + t_n M_n)$. But $h = h_k \exp M_u h_k^{-1} = \exp(A_{h_k} M_u)$. Hence we have $A_{h_k} \cdot M_u = t_{k+1}M_{k+1} + \ldots + t_n M_n$, whence $A_{h_k} \cdot \mathfrak{H}_{k+1} \subseteqq \mathfrak{H}_{k+1}$ and $\log A_{h_k} \cdot \mathfrak{H}_{k+1} \subseteqq \mathfrak{H}_{k+1}$. But $\log A_{h_k} = A_{M_k}$ and therefore $[M_k, \mathfrak{H}_{k+1}] \subseteqq \mathfrak{H}_{k+1}$. Hence $\{M_k, \mathfrak{H}_{k+1}\}$ is a subalgebra of $\mathfrak{G}$. Next we show that, if $[M_k, M_u] = r_{k+1}M_{k+1} + \ldots + r_n M_n$, r_j are rational numbers. Since $[M_k, M_u] = (\log A_{h_k}) \cdot M_u$, it is sufficient to show that $t_{k+1}, \ldots, t_n$ are rational. Now $h_{k+1}^{u_{k+1}} \ldots h_n^{u_n} = \exp(u_{k+1}M_{k+1}) \ldots \exp(u_n M_n) = \exp(t_{k+1}M_{k+1} + \ldots + t_n M_n)$. Since the structure constants of $\{M_{k+1}, \ldots M_n\}$ are assumed to be rational numbers and $u_{k+1}, \ldots, u_n$ are integers, we may easily see by the formula of Hausdorff that t_i are rational. Thus $\{M_k, \ldots, M_n\}$ is a rational subalgebra. Repeating this argument we may see that $\mathfrak{R} = \{M_m, \ldots, M_n\}$ is a rational subalgebra. Now we may add $M_1, \ldots, M_{m-1}$ to $M_m, \ldots, M_n$ so that $\{M_1, \ldots, M_n\}$ is a basis of $\mathfrak{G}$ satisfying the condition of Theorem 1, q.e.d.

2. We prove in this and next sections the theorems of Malcev in making use of the results in **1**.

THEOREM 2. *Let M be a homogeneous space of a connected nilpotent group G. There exists a compact subset C and a subset E, homeomorphic to an Euclidean space of some dimension, such that M is homeomorphic to the product space $C \times E$. A certain connected closed subgroup of G acts transitively on C.*

Proof. First let G be simply connected and $M = G/H$, where H is a closed subgroup of G and G/H is the right coset space of G mod H. Let H_0 be the component of the unit element of H and N the normalizer of H_0. Then N is connected by Lemma 2 and we may take the one-parameter subgroups $V_1, \ldots, V_r$ of G such that $G = V_1 \ldots V_r N$ and $V_{i+1} \ldots V_r N$ is a normal subgroup of $V_i \ldots V_r N$ by Lemma 4. Since $N \supset H$, $G/H \approx V_1 \times \ldots \times V_r \times N/H$, where $V_1 \times \ldots \times V_r$ is an Euclidean space. Further since H_0 is a normal subgroup of N and $N/H \approx N/H_0/H/H_0$ and N/H_0 is simply connected, we may consider only the case where H is a discrete subgroup. Hence let $H = D$ be a discrete subgroup of G. Let $\{L_1, \ldots, L_n\}$ be a basis of the Lie algebra $\mathfrak{G}$ of G satisfying

https://doi.org/10.1017/S0027763000010096 Published online by Cambridge University Press

the conditions of Theorem 1 and $\mathfrak{R} = \{L_m, \ldots, L_n\}$ and R the subgroup of G corresponding to $\mathfrak{R}$. $g_k = \exp L_k$ $(k = m+1, \ldots, n)$ form a system of generators of D and $D \subset R$. Hence $G/D \approx V_1 \times \ldots \times V_{m-1} \times R/D$ where $V_i = \exp tL_i$ $(i = 1, \ldots, m-1)$. We show by induction on the dimension of R that R/D is compact. Let R_1 be the normal subgroup of G corresponding to the ideal $\mathfrak{R}_1 = \{L_{m+1}, \ldots, L_n\}$ of $\mathfrak{R}$ and $V_1 = \exp tL_m$ and D_1 the normal subgroup of D generated by $g_{m+1}, \ldots, g_n$. Clearly $D_1 \subset R_1$. Now we define a continuous map f of the space $V_1 \times R_1/D_1$ onto R/D by $f(a, D_1b) = Dba$, where $a \in V_1$ and $b \in R_1$. This definition is independent of the choice of the representative b of the coset D_1b. Now we may easily verify that, if $f(a, D_1b) = f(a', D_1b')$, then

$$(a', D_1b') = (g_m{}^s a, D_1 g_m{}^s b g_m{}^{-s}) \ (s = \pm 1, \pm 2, \ldots).$$

Let $\{Dc_k\}$ be an infinite subset of R/D. We may choose from $f^{-1}(Dc_k)$ a point (a_k, D_1b_k) such that the absolute value of the coordinate of a_k is $\leqq 1$. We may assume that the sequence $\{a_k\}$ converges to a point a of V_1. Since we have assumed that R_1/D_1 is compact, it is possible to choose a subsequence $\{D_1b_{k_i}\}$ of D_1b_k such that it converges to a point D_1b of R_1/D_1. Then the sequence $\{Dc_{k_i}\}$ converges to the point Dba. Hence R/D is compact. Next let G be an arbitrary connect nilpotent group and $\tilde{G}$ the universal covering group of G. Let f be the projection of $\tilde{G}$ onto G. M becomes a homogeneous space of $\tilde{G}$ by the formula $\tilde{g}(m) = f(\tilde{g})(m)$, where $\tilde{g} \in \tilde{G}$ and $m \in M$. Then $M = C \times E$, where C is a compact subset of M and E is a subset of M homeomorphic to an Euclidean space and a certain connected closed subgroup $\tilde{K}$ of $\tilde{G}$ operates transitively on C. Then $f(\tilde{K})$ is a connected subgroup of G which operates on C transitively. Then since C is compact, the closed connected subgroup $K = \overline{f(\tilde{K})}$ is also transitive on C, q.e.d.

We see from the above proof the following corollaries.

Corollary 1. *Let M be a compact homogeneous space of a connected nilpotent Lie group, then $M = G/D$, where G is a connected simply connected nilpotent rational Lie group*[5] *and D is a discrete subgroup of G.*

Corollary 2. *If a connected simply connected Lie group G acts on a compact space M transitively and almost effectively,*[6] *then G is rational and the isotropy subgroup*[6] *H of G is discrete.*

---

[5] A nilpotent group is said to be rational, if its structure constants are rational in a suitable coordinate system.

[6] A Lie group G is said to be almost effective on a homogeneous space M, if the closed normal subgroup of G consisting of all elements of G which leave fixed every point of M is discrete. The isotropy subgroup of G is the closed subgroup of G consisting of all elements of G which leave fixed a point of M.

For then $M = G/H$ and since M is compact the normalizer of the component group H_0 must be equal to G.

3. Now we consider the structure of the compact homogeneous spaces.[7] Let G be a connected simply connected nilpotent group. A discrete subgroup D of G is said to be *uniform* if the space G/D is compact. G contains a uniform discrete subgroup if and only if it is rational.[8]

LEMMA 5. Let H be a connected closed central subgroup of a connected Lie group G and D a discrete subgroup of G such that $H/D \cap H$ is compact. Then DH/H is a discrete subgroup of G/H.

Proof. Let $D_1 = D \cap H$. Then D_1 is central. Since H/D_1 is compact and D/D_1 is a closed subgroup of G/D_1, HD/D_1 is a closed subgroup of G/D_1, whence HD is a closed subgroup of G. Then since $DH \supset H$, DH/H is closed in G/H. But as DH/H is enumerable, it must be discrete.

Now let G be a connected simply connected group with the Lie algebra $\mathfrak{G}$ and $\mathfrak{G} \supset \mathfrak{C}(\mathfrak{G}) \supset \mathfrak{C}^2(\mathfrak{G}) \supset \ldots \supset \mathfrak{C}^l(\mathfrak{G}) \supset \mathfrak{C}^{l+1}(\mathfrak{G}) = \{0\}$ the descending central series of $\mathfrak{G}$ i.e. $\mathfrak{C}^i(\mathfrak{G}) = [\mathfrak{G}, \mathfrak{C}^{i-1}(\mathfrak{G})]$ and let $G \supset C(G) \supset C^2(G) \supset \ldots \supset C^l(G) \supset C^{l+1}(G) = \{e\}$ be the corresponding series of the subgroups of G. Then the following theorem holds.

THEOREM 3. *Let G be a connected simply connected nilpotent group with the Lie algebra $\mathfrak{G}$ and D a uniform discrete subgroup of G. Then we may choose a basis $\{L_1, \ldots, L_n\}$ of $\mathfrak{G}$ such that $\{L_{i_k}, L_{i_k+1}, \ldots, L_n\}$ is a basis of $\mathfrak{C}^k(\mathfrak{G})$ for $k = 1, \ldots, l$ and that $g_j = \exp L_j$ form a system of generator of D and every element of D may be written uniquely in the form $g_1^{u_1} \ldots g_n^{u_n}$. Let $D_i = \{g_i, \ldots, g_n\}$. Then D_i are normal subgroups of D*[9] *and $D_{i_k} = D \cap C^k(G)$.*

Proof. We shall prove this theorem by induction on the length l of the desecending central series of $\mathfrak{G}$. Let us assume that it has been proved already for groups whose length of the descending central series are $< l$. We first show that $C^l(D) \neq \{e\}$. For we may choose a basis $\{M_1, \ldots, M_n\}$ of $\mathfrak{G}$ such that $d_i = \exp M_i$ $(i = 1, \ldots, n)$ form a system of generators of D by Theorem 1. Then by the formula of Hausdorff $d_i \circ d_j = d_i d_j d_i^{-1} d_j^{-1} = \exp([M_i, M_j] + \phi(M_i,$

[7] As we remarked in the proof of Theorem 2, any homogeneous space of a connected Lie group may be considered as a homogeneous space of its universal covering group.

[8] If such a subgroup exists, then G is rational by Theorem 1. The existence of a uniform discrete subgroup in a rational nilpotent group has been proved independently by Malcev and Kuranishi. See Malcev [1] and Kuranishi [7].

[9] A system of generators $g_1, \ldots, g_n$ of D such that every element of D is written uniquely in the form $g_1^{u_1} \ldots g_n^{u_n}$ and $g_i, \ldots, g_n$ $(i = 1, \ldots, n)$ generate a normal subgroup of D will be called in the following the ***canonical basis*** of D.

$M_j))$, where $\phi(M_i, M_j)$ is a commutator polynomial of M_i, M_j such that $\phi(M_i, M_j) \in \mathfrak{C}^2(\mathfrak{G})$. By the repeated use of this formula, we have $d_{i_1} \circ (d_{i_2} \circ (\ldots \circ (d_{i_l} \circ d_{i_{l+1}}) \ldots)) = \exp([M_{i_1}[M_{i_2}[\ldots[M_{i_l}, M_{i_{l+1}}]\ldots]]) \in C^l(D)$. Since $\mathfrak{C}^l(\mathfrak{G}) \neq \{0\}$, we may choose $M_{i_1}, \ldots, M_{i_{l+1}}$ such that $[M_{i_1}[M_{i_2}[\ldots[M_{i_l}, M_{i_{l+1}}]\ldots]] \neq 0$. Then $d_{i_1} \circ (d_{i_2} \circ (\ldots (d_{i_l} \circ d_{i_{l+1}}) \ldots) \neq e$, whence $C^l(D) \neq \{e\}$. Since $C^l(D) \subset D \cap C^l(G)$, $D_1 = D \cap C^l(G) \neq \{e\}$. We show that D_1 is a uniform discrete subgroup of $C^l(G)$. Suppose that D_1 is not uniform in $C^l(G)$. Then there exists in $C^l(G)$ a closed connected subgroup H such that D_1 is uniform in H by Theorem 1. Since $C^l(G)$ is contained in the center of G, H and D_1 are also central. Then by Lemma 5 $DH/H = D'$ is a discrete subgroup of $G/H = G'$. Clearly D' is uniform in G' and $C^l(G)/H = C^l(G') \neq e$. We see easily that $D' \cap C^l(G') = \{e\}$. But we may prove as above that $D' \cap C^l(G') \neq \{e\}$ and this is a contradiction. Hence D_1 must be a uniform discrete subgroup of $C^l(G)$. Then $DC^l(G)/C^l(G)$ is a discrete subgroup of $G/C^l(G)$ by Lemma 5. The length of the central series of the group $G/C^l(G)$ is $l-1$ and we may use the assumption of induction and we may prove our theorem by the same way as in the proof of Theorem 1. The other part of our theorem is obvious, q.e.d.

Remark. $D \cap C(G)$ contains the commutator group $C(D)$ of D. They are not always equal, as we may show by an example. We shall show in the next section that $D \cap C(G)/C(D)$ is a finite group. Since $D/D \cap C(G)$ is a free abelian group with r ($= \dim G - \dim C(G)$) free generators, $D/C(D)$ is the direct product of the finite group $D \cap C(G)/C(D)$ and the free abelian group of rank r.

The following theorem is a slight generalization of a theorem in [8] and will be used later.

THEOREM 4. *Let G_1 and G_2 be the connected simply connected nilpotent groups and D_1 and D_2 the uniform discrete subgroups of G_1 and G_2 respectively. Let φ be a homomorphism of D_1 onto D_2. Then we may extend φ to a continuous homomorphism ψ of G_1 onto G_2.*

Proof. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be the Lie algebras of G_1 and G_2 respectively. Let $\{L_1, \ldots, L_n\}$ be a basis of $\mathfrak{G}_1$ satisfying the condition of Theorem 1. Let $h_i = \varphi(g_i)$ and $h_i = \exp M_i$. Suppose that $\mathfrak{M}_{i+1} = \{M_{i+1}, \ldots, M_n\}$ is a subalgebra of $\mathfrak{G}_2$ such that the subalgebra $\mathfrak{L}_{i+1} = \{L_{i+1}, \ldots, L_n\}$ is mapped homomorphically onto $\mathfrak{M}_{i+1}$ by the correspondence f: $L_j \to M_j$ $(j = i+1, \ldots, n)$. We show that $\mathfrak{M}_i = \{M_i, \ldots, M_n\}$ is also a subalgebra of $\mathfrak{G}_2$ and $\mathfrak{L}_i = \{L_i, \ldots, L_n\}$ is homomorphic to $\mathfrak{M}_i$ by the correspondence $L_j \to M_j$ $(j = i, \ldots, n)$. Since $g = g_i g_j g_i^{-1} = g_{i+1}^{m_{i+1}} \ldots g_n^{m_n}$ for $j > i$, we have $h = h_i h_j h_i^{-1} = h_{i+1}^{m_{i+1}} \ldots h_n^{m_n}$. Hence $\exp(A_{g_i} L_j)$

https://doi.org/10.1017/S0027763000010096 Published online by Cambridge University Press

$= (\exp m_{i+1}L_{i+1}) \ldots (\exp m_n L_n)$ and $\exp(A_{h_i}M_j) = (\exp m_{i+1}M_{i+1}) \ldots (\exp m_n M_n)$. Let $G_1^{(i+1)}$ and $G_2^{(i+1)}$ be the subgroups of G_1 and G_2 corresponding to $\mathfrak{L}_{i+1}$ and $\mathfrak{M}_{i+1}$ respectively. Then by our assumption there exists a continuous homomorphism ψ of $G_1^{(i+1)}$ onto $G_2^{(i+1)}$ such that $\psi(g_j) = h_j$ for $j > i$. Now let $(\exp m_{i+1}L_{i+1}) \ldots (\exp m_n L_n) = \exp(t_{i+1}L_{i+1} + \ldots + t_n L_n)$. Then, since $\psi(g) = \varphi(g) = h$ and $\psi(\exp L) = \exp f(L)$, $L \in \mathfrak{L}_{i+1}$, we have $(\exp m_{i+1}M_{i+1}) \ldots (\exp m_n M_n) = \exp(t_{i+1}M_{i+1} + \ldots + t_n M_n)$. Hence it follows that $A_{g_i} \cdot L_j = t_{i+1}L_{i+1} + \ldots t_n L_n$ and $A_{h_i} \cdot M_j = t_{i+1}M_{i+1} + \ldots + t_n M_n$. Then since $A_{L_i} = \log A_{g_i}$ and $A_{M_i} = \log A_{h_i}$, it follows that $[L_i, L_j] = \sum_{k=i+1}^{n} s_k L_k$ and $[M_i, M_j] = \sum_{k=i+1}^{n} s_k M_k$. Thus our assertion is proved. Then repeating this argument we verify that $\mathfrak{G}_1$ is mapped homomorphically onto the subalgebra $\mathfrak{M}_1 = \{M_1, \ldots, M_n\}$ of $\mathfrak{G}_2$. If $\mathfrak{G}_2 \neq \mathfrak{M}_1$, then the subgroup of G_2 corresponding $\mathfrak{M}_1$ contains D_2 and hence D_2 is not uniform. Therefore $\mathfrak{G}_2 = \{M_1, \ldots, M_n\}$ and our theorem is thus proved.

Remark. If φ is an isomorphism, then ψ is also an isomorphism.

The following corollary is contained in [8].

COROLLARY. *If M_1 and M_2 are compact homogeneous spaces with isomorphic fundamental groups, then they are homeomorphic.*

Proof. As we have already seen, there exist the connected simply connected rational nilpotent groups G_1 and G_2 such that $M_1 = G_1/D_1$ and $M_2 = G_2/D_2$, where D_1 and D_2 are the uniform discrete subgroups of G_1 and G_2 respectively. Then D_1 and D_2 are the fundamental groups of M_1 and M_2 respectively and $M_1 \approx M_2$ by Theorem 4.

4. Let M be a compact homogeneous space and $M = G/D$ as above. Since G is homeomorphic to an Euclidean space, the homotopy groups $\pi_i(M)$ $(i>1)$ of M vanish. Hence by Hopf-Eilenberg-MacLane's theory the i-th integral cohomology group of M is isomorphic to the i-th integral cohomology group of D.[2] We shall consider the 1-st and 2-nd cohomology group of M. The 1-st integral cohomology group $H_1(D)$ of D is the group of all homomorphisms of D into the additive group of integers I: $H_1(D) = \mathrm{Hom}(D, I)$.[10] Since I is an abelian group without element of finite order, $\mathrm{Hom}(D, I) = \mathrm{Hom}(D/D_1, I)$, where D_1 is a normal subgroup of D containing the commutator group $C(D)$ of D such that $D_1/C(D)$ is the torsion group (i.e. the maximal finite subgroup) of $D/C(D)$. Let $\varphi \in \mathrm{Hom}(D, I)$ and $\varphi \neq 0$. Then $\varphi(D)$ is a free cyclic subgroup of I and hence a uniform

[10] See Eilenberg and MacLane [4]. The fact that Hom (D, I) is isomorphic to the 1-st Betti group of M may be seen also from the fact that $D/C(D)$ is isomorphic to the 1-st integral homology group of M.

discrete subgroup of the additive group of real numbers V_1. Hence we may extend φ to a continuous homomorphism ψ of G onto V_1. Then $f(g) = \exp\psi(g)$ is an one-dimensional continuous representation of G and $f(d) = \exp m$, where $d \in D$ and m is an integer. Now take a basis $\{L_1, \ldots, L_n\}$ of the Lie algebra $\mathfrak{G}$ of G as in Theorem 3. Then $f(\exp L_i) = \exp m_i$, where m_i are integers. Since $f(C(G)) = 1$, if $L_i \in \mathfrak{C}(\mathfrak{G})$, $m_i = 0$. Let $L_1, \ldots, L_r$ be the elements of this basis which is not contained in $\mathfrak{C}(\mathfrak{G})$. Then we obtain a representation g of $\mathfrak{G}/\mathfrak{C}(\mathfrak{G})$ such that $g(L_i) = m_i$ $(i = 1, \ldots, r)$ and $g(L_j) = 0$ $(j > r)$. Coversely to every one dimensional representation of $\mathfrak{G}/\mathfrak{C}(\mathfrak{G})$ such that L_i correspond to the integers m_i, there corresponds an element $\varphi \in \mathrm{Hom}(D, I)$, where $\varphi(\exp L_i) = m_i$ for $i \leqq r$ and $\varphi(\exp L_j) = 0$ for $j > r$. Hence we see that $\mathrm{Hom}(D, I)$ and the group of the such representations of $\mathfrak{G}/\mathfrak{C}(\mathfrak{G})$ are isomorphic and the subgroup of D generated by $\exp L_{r+1}, \ldots, \exp L_n$, i.e. $D \cap C(G)$ coincides with the above mentioned group D_1. Since $D/D \cap C(G)$ is a free abelian group with the free generators $\exp L_1, \ldots, \exp L_r$, $\mathrm{Hom}(D, I) = \mathrm{Hom}(D/D \cap C(G), I)$ is a free abelian group of rank r and $r = \dim \mathfrak{G} - \dim \mathfrak{C}(\mathfrak{G})$. Thus $H_1(D)$ is a free abelian group of rank r. Since $H_1(D)$ is isomorphic to the 1-st integral cohomology group $H_1(M)$ of M and $H_1(M)$ is isomorphic to the 1-st Betti group $B^1(M)$ of M. On the other hand the 1-st homology group $H^1(M)$ of M is isomorphic to $D/C(D)$. Hence the 1-st torsion group of M is isomorphic to $D \cap C(G)/C(D)$. Thus we have proved the following.

THEOREM 5. *Let M be a compact homogeneous space and $M = G/D$, where G is a connected simply connected rational nilpotent group with the Lie algebra $\mathfrak{G}$ and D is a uniform discrete subgroup of G. Further let $C(G)$, $C(D)$ and $\mathfrak{C}(\mathfrak{G})$ be the commutator groups and algebra of G, D and $\mathfrak{G}$ respectively. Then the 1-st Betti number of M is equal to* $\dim \mathfrak{G} - \dim \mathfrak{C}(\mathfrak{G})$ *and the 1-st torsion group of M is isomorphic to $D \cap C(G)/C(D)$. Therefore the 1-st cohomology group of M with rational coefficients is isomorphic to the 1-st cohomology group of the rational Lie algebra $\mathfrak{G}$ with rational coefficients.*[11]

5. Next we consider the 2-nd cohomology group of M. We propose to show that the 2-nd cohomology group of M is isomorphic to the 2-nd cohomology group of the rational Lie algebra $\mathfrak{G}$ with rational coefficients. For this purpose we resume here the connections between the 2-nd cohomology groups of groups

[11] Let $L_1, \ldots, L_n$ be a basis of $\mathfrak{G}$ such that its structure constants are rational. Then $L_1, \ldots, L_n$ span a Lie algebra $\mathfrak{G}_R$ over rational numbers. If $L_1', \ldots, L_n'$ is another basis of $\mathfrak{G}$ possessing the above property and if $\mathfrak{G}_R'$ is defined analogously, then $\mathfrak{G}_R$ and $\mathfrak{G}_R'$ are not always isomorphic but have the isomorphic cohomology groups. Cf. the proof of Theorem 6.

https://doi.org/10.1017/S0027763000010096 Published online by Cambridge University Press

and Lie algebras and their central extensions.[12] The group $C^q(D)$ of the q-dimensional integral cochain of the discrete group D is defined as the group of all functions f of q variables in D with values in the additive group I of integers. Addition in $C^q(D)$ is defined by

$$(f_1+f_2)(x_1,\ldots,x_q)=f_1(x_1,\ldots,x_q)+f_2(x_1,\ldots,x_q).$$

The coboundary operator $\delta: C^q(D)\to C^{q+1}(D)$ is defined by the formula

$$(\delta f)(x_1,\ldots,x_{q+1})=f(x_2,\ldots,x_{q+1})+\sum_{i=1}^{q}(-1)^i f(x_1,\ldots,x_ix_{i+1},\ldots,x_{q+1}) + (-1)^{q+1}f(x_1,\ldots,x_q).$$

Cocycles, coboundaries and cohomology groups are defined as usual. Let D^*, D and E be three groups and $\phi: D^*\to D$ a homomorphism of D^* onto D such that E is the kernel of ϕ. If E is contained in the center of D^*, the pair (D^*,ϕ) is called a central extension of D^* by the kernel E. Two extension (D_1^*,ϕ_1) and (D_2^*,ϕ_2) with the kernel E are called equivalent if there is an isomorphism τ of D_1^* onto D_2^* such that $\phi_2\tau=\phi_1$ and every element of E is fixed under τ. We consider here only the central extensions with infinite cyclic groups as kernels and hence we call for simplicity such extensions simply the extensions of D. Now let (D^*,ϕ) be such an extension of D. For each $x\in D$ select a representative $u(x)\in D^*$ such that $\phi(u(x))=x$. Since ϕ is a homomorphism, $u(x_1)u(x_2)$ and $u(x_1x_2)$ have the same image x_1x_2 in D under ϕ, so

$$u(x_1)u(x_2)=g(x_1,x_2)u(x_1x_2),\quad x_1,x_2\in D,$$

where $g(x_1,x_2)\in E$, the kernel of the extension. $g(x_1,x_2)$ is called the factor set corresponding to the given set of representatives $\{u(x)\}$. It satisfies the relation $g(x_1,x_2)g(x_1x_2,x_3)=g(x_2,x_3)g(x_1,x_2x_3)$ for $x_1,x_2,x_3\in D$. Since E is infinite cyclic, let e be a fixed free generator of E and $g(x_1,x_2)=e^{f(x_1,x_2)}$. Then $f(x_1,x_2)$ is an integral 2-cochain of D and satisfies the relation

$$f(x_1,x_2)+f(x_1x_2,x_3)=f(x_2,x_3)+f(x_1,x_2x_3).$$

Hence $f(x_1,x_2)$ is a cocycle. Now let $\{v(x)\}$ be another set of representatives and $h(x_1,x_2)$ the 2-cocycle corresponding to $\{v(x)\}$ in the above manner. Then we can show that $f(x_1,x_2)$ and $h(x_1,x_2)$ are cohomologous. We conclude that each extension (D^*,ϕ) determines uniquely an element of the 2-nd cohomology group $H_2(D)$ of D. Conversely it can be shown that to each element of $H_2(D)$ there corresponds an extension (D^*,ϕ) which is unique within an equivalence class. Thus there is a $1-1$ correspondence between the elements of $H_2(D)$ and the equivalence classes of extensions. Now let $\mathfrak{G}$ be a Lie algebra over a field K. A q-linear alternating mapping of $\mathfrak{G}$ into K will be called a q-chain and

[12] For the details see Eilenberg and MacLane [4] and Chevalley and Eilenberg [2].

they form a linear space $C^q(L)$ over K. The coboundary operator δ is a linear mapping of $C^q(L)$ into $C^{q+1}(L)$ and is defined by the formula

$$(\delta f)(x_1, \ldots, x_{q+1}) = \sum_{i<j} (-1)^{i+j+1} f([x_i, x_j], x_1, \ldots, \check{x}_i, \ldots, \check{x}_j, \ldots, x_{q+1}).$$

It holds the relation $\delta\delta f = 0$ and cocycles, coboundaries and cohomology groups are defined as usual. Now let $\mathfrak{G}^*$, $\mathfrak{G}$ and $\mathfrak{V}$ be three Lie algebras and $\theta: \mathfrak{G}^* \to \mathfrak{G}$ a homomorphism of $\mathfrak{G}^*$ onto $\mathfrak{G}$ and $\mathfrak{V}$ the kernel of θ. If $\mathfrak{V}$ is contained in the center of $\mathfrak{G}^*$, the pair $(\mathfrak{G}^*, \theta)$ is called a central extension of $\mathfrak{G}$ by the kernel $\mathfrak{V}$. The equivalence of two extensions are defined as in the case of the group extensions. We consider in the following only the central extensions with one-dimensional kernel and so we call such extensions simply the extensions of $\mathfrak{G}$. Given such an extension $(\mathfrak{G}^*, \theta)$ of $\mathfrak{G}$ we select a linear mapping $v: \mathfrak{G} \to \mathfrak{G}^*$ with $\theta v(x) = x$. The set $\{v(x)\}$ is called a system of representatives of the extension $(\mathfrak{G}^*, \theta)$. Since $\theta([v(x), v(y)]) = [x, y] = \theta v([x, y])$, there is an element $g(x, y) \in \mathfrak{V}$ such that

$$[v(x), v(y)] = g(x, y) + v([x, y]).$$

Since $\mathfrak{V}$ is one-dimensional, we may consider $g(x, y)$ as an element of $C^2(L)$ and is called the factor set corresponding the representatives $\{v(x)\}$. We may prove that the factor set $g(x, y)$ is a cocycle and if $\{\bar{v}(x)\}$ is a different system of representatives, the factor set $\bar{g}(x, y)$ corresponding to $\{\bar{v}(x)\}$ is cohomologous to $g(x, y)$. Thus to each extension $(\mathfrak{G}^*, \theta)$ there corresponds a definite element of the 2-nd cohomology group $H_2(\mathfrak{G})$ of $\mathfrak{G}$. Coversely it can be shown that to each element of $H_2(\mathfrak{G})$ there corresponds an extension $(\mathfrak{G}^*, \theta)$ which is unique up to equivalence. Hence there is a 1-1 correspondence between the elements of $H_2(\mathfrak{G})$ and the equivalence classes of extensions.

Now let D be a discrete subgroup of a connected simply connected nilpotent group G. Then, since G is nilpotent and contains no element of finite order, the same holds for D and by Theorem 1 D has finite generators. Let, conversely, D be a (discrete) finitely generated nilpotent group without element of finite order. It has been proved by Malcev[13] that there exists a connected simply connected nilpotent group G such that D is a uniform discrete subgroup of G. For the sake of convenience and completeness we sketch here a proof of this theorem of Malcev. Let D be as above. There exists a series of normal subgroups of D: $D \supset D_1 \supset \ldots \supset D_r \supset D_{r+1} = \{e\}$, such that D_i/D_{i+1} are infinite cyclic. We call it a normal series of D. The length r of the normal series is an invariant of D and is called the length of D. Let us assume that the theorem has been proved already for all groups whose length

[13] See Malvec [8].

are $< r$. Let D be the group of length r. Then $D = \{d, D_1\}$, where D/D_1 is infinite cyclic and d mod D_1 is a generator of D/D_1. Let G_1 be a connected simply connected nilpotent group which contain D_1 as a uniform discrete subgroup. Let $\varphi(x) = dxd^{-1}$ be the automorphism of D_1. Then we can extend φ to a continuous automorphism $\tilde{\varphi}$ of G_1 (Theorem 4). $\tilde{\varphi}$ induces an automorphism θ of the Lie algebra $\mathfrak{G}_1$ of G_1 and we can show without difficulties that the eigen-values of θ are 1. Then $\log\theta = \eta$ is a derivation of $\mathfrak{G}_1$ whose eigen values are all 0 and whence the extesion of $\mathfrak{G}_1$ by η is a nilpotent Lie algebra $\mathfrak{G}$ and the simply connected group G corresponding to $\mathfrak{G}$ contains D as a uniform discrete subgroup.

6. Now we prove the following

THEOREM 6. *Let $\mathfrak{G}_R$ be a nilpotent Lie algebra over the field R of rational numbers and $\mathfrak{G}$ the Lie algebra obtained from $\mathfrak{G}_R$ by extending the ground filed R to the field P of real numbers. Further let G be the simply connected nilpotent group corresponding to $\mathfrak{G}$ and M the compact homogeneous space of G[14] on which G is almost effective.*[6] *Then the 2-nd cohomology group $H_2(\mathfrak{G}_R)$ of $\mathfrak{G}_R$ is isomorphic to the 2-nd cohomology group with rational coefficients of M.*

Proof. Let $M = G/D$, where D is a uniform discrete subgroup of G. Let $g_1, \dots, g_n$ be a fixed canonical basis[9] of D and $g_i = \exp L_i$. Then the set G_R' of all linear combinations of $L_1, \dots, L_n$ with rational coefficients forms a Lie algebra over R[15] which is not always isomorphic to $\mathfrak{G}_R$. But $H_2(\mathfrak{G}_R) \cong H_2(\mathfrak{G}_R')$, for by extending ground field R to P $\mathfrak{G}_R$ and $\mathfrak{G}_R'$ yield the same Lie algebra $\mathfrak{G}$, and $H_2(\mathfrak{G})$ regarded as a vector space over P is obtained from $H_2(\mathfrak{G}_R)$ and $H(\mathfrak{G}_R')$ respectively by extending R to P. Hence we may assume $\mathfrak{G}_R = \mathfrak{G}_R'$. The 2-nd integral cohomology group $H_2(M)$ of M is isomorphic to the 2-nd cohomology group $H_2(D)$ of D.[2] Let us take an element of $H_2(D)$ or equivalently an extension (D^*, ϕ) of D. Since D^* is a central extension of D with an infinite cyclic group E as the kernel, D is also a finitely generated nilpotent group without element of finite order. Let $\phi(g_i^*) = g_i$ for $i = 1, \dots, n$ and g_{n+1}^* a fixed generator of E. Then g_i^* form a canonical basis of D^*. Let now G^* be the simply connected nilpotent group containing D^* as a uniform discrete subgroup and let $g_i^* = \exp L_i^*$. Then $L_1^*, \dots, L_{n+1}^*$ span the Lie algebra $\mathfrak{G}_R^*$ over R.[15] We can extend ϕ to a continous homomorphism of G^* onto G and we get a homomorphism θ of $\mathfrak{G}_R^*$ onto $\mathfrak{G}_R$. θ is defined by $\theta(L_i^*) = L_i$ for

[14] There exists always such a homogeneous space. Cf. [8].

[15] We may show as in the proof of Theorem 1 that the structure constants of $L_1, \dots, L_n$ are rational.

$i = 1, \ldots, n$ and $\theta(L^*_{n+1}) = 0$ (cf. the proof of Theorem 4.). Since the kernel of θ is one dimensional and is contained in the center of $\mathfrak{G}_R^*$, $(\mathfrak{G}_R^*, \theta)$ is an extension of $\mathfrak{G}_R$ and hence determines an element of $H_2(\mathfrak{G}_R)$. We shall show that the correspondence $(D^*, \phi) \to (\mathfrak{G}_R^*, \theta)$ is a homomorphism of $H_2(D)$ into $H_2(G_R)$. Let, for simplicity, $g_i^* = h_i$ and $L_i^* = M_i$. $u(g_1^{s_1} \ldots, g_n^{s_n}) = h_1^{s_1} \ldots h_n^{s_n}$ (s_i being integers) and $v(\sum_{i=1}^{n} \alpha_i L_i) = \sum_{i=1}^{n} \alpha_i M_i$ (α_i being rational numbers) form a set of representatives of (D^*, ϕ) and $(\mathfrak{G}_R^*, \theta)$ respectively. Let $(h_1^{s_1} \ldots h_n^{s_n})(h_1^{t_1} \ldots h_n^{t_n}) = h_1^{u_1} \ldots h_n^{u_n} h_{n+1}^{u_{n+1}}$ and $[\sum_{i=1}^{n} \alpha_i M_i, \sum_{i=1}^{n} \beta_i M_i] = \sum_{k=1}^{n+1} \gamma_k M_k$. Then $u_{n+1} = d(g_1^{s_1} \ldots g_n^{s_n}, g_1^{t_1} \ldots g_n^{t_n}) = \phi(s_1, \ldots, s_n, t_1, \ldots, t_n)$ and $\gamma_{n+1} = g(\sum_{i=1}^{n} \alpha_i L_i, \sum_{i=1}^{n} \beta_i L_i) = \sum_{i,j=1}^{n} \alpha_i \beta_j$, c_{ij}^{n+1} are the 2-cocycles corresponding to (D^*, ϕ) and $(\mathfrak{G}_R, \theta)$ respectively. We consider $\psi(s_1, \ldots, s_n, t_1, \ldots, t_n)$ as a function of $s_1, \ldots, s_n, t_1, \ldots, t_n$ and show that it is a polynomial of s and t with rational co-efficients and that ψ is the form

$$\psi(s_1, \ldots, s_n, t_1, \ldots, t_n) = \frac{1}{2} \sum_{i,j=1}^{n} s_i t_j c_{ij}^{n+1} + \ldots, \tag{1}$$

where $\ldots$ denote the terms of degree > 2.

By the formula of Hausdorff $(\exp M)(\exp N) = \exp(M + N + \frac{1}{2}[M, N] + \ldots)$ we have, $\prod_{i=1}^{n} \exp s_i M_i = \exp(\sum_{i=1}^{n} s_i M_i + \frac{1}{2}(\sum_{i,j} s_i s_j [M_i, M_j]) + \ldots)$ and $\prod_{i=1}^{n} \exp s_i M_i \prod_{i=1}^{n} \exp t_i M_i = \exp(\sum_{i=1}^{n} (s_i + t_i) M_i + \frac{1}{2}(\sum_{i,j} (s_i s_j + t_i t_j + s_i t_j)[M_i, M_j]) + \ldots) = \exp \sum_{k=1}^{n+1} \varphi_k(s, t) M_k$, where

$$\varphi_k(s, t) = s_k + t_k + \frac{1}{2} \sum_{i,j} (s_i s_j + t_i t_j + s_i t_j) c_{ij}^k + \ldots \tag{2}$$

are the polynomials with rational coefficients. Moreover since $\{M_i, M_{i+1}, \ldots, M_{n+1}\}$ are the ideals of $\mathfrak{G}^*$ for $i = 1, 2, \ldots, n+1$ the terms of degree $\geqq 2$ of φ_k are the polynomials of $s_1, \ldots, s_{k-1}$ and $t_1, \ldots, t_{k-1}$. Let $\exp(\sum_{i=1}^{n+1} \lambda_i M_i) = \prod_{i=1}^{n+1} \exp u_i M_i$. Then since $\prod_{i=1}^{n+1} \exp u_i M_i = \exp \sum_{i=1}^{n+1} (u_i + \alpha_i(u)) M_i$, we have

$$\lambda_i = u_i + \alpha_i(u),$$

where
$$\alpha_i(u) = \frac{1}{2} \sum_{k,l} u_k u_l c_{kl}^i + \ldots$$

are the polynomials of $u_1, \ldots, u_{i-1}$ with rational coefficients and $\ldots$ denotes the terms of degree > 2.

Then we can easily show inductively that

$$u_k = \lambda_k + \beta_k(\lambda_1, \ldots, \lambda_{k-1}),$$

where
$$\beta_k(\lambda_1, \ldots, \lambda_{k-1}) = -\frac{1}{2} \sum_{i,j} \lambda_i \lambda_j c_{ij}^k + \ldots$$

are the polynomials of $\lambda_1, \ldots, \lambda_{k-1}$ with rational coefficients. Then, substituting φ_i in λ_i, we have

$$\psi(s, t) = \varphi_{n+1} - \frac{1}{2}\sum_{i,j} \varphi_i \varphi_j c_{ij}^{n+1} + \ldots$$

and by a simple calculation we may verify the desired properties of $\psi(s, t)$. Then we obtain the relation:

(3) $g(\sum_{i=1}^{n} \alpha_i L_i, \sum_{i=1}^{n} \beta_i L_i) = 2 \times$ (The term of degree 2 of $\psi(\alpha, \beta)$).

Let now (D_1^*, ϕ_1) be another extension of D and let G_1^*, $\mathfrak{G}_{1R}^*$, $k_1^*, \ldots, k_{n+1}^*$, $N_1, \ldots, N_n$, N_{n+1}, $(\mathfrak{G}_{1R}^*, \theta_1)$ be defined as in the case of (D^*, ϕ). Then $k_i^* = \exp N_i$, $\mathfrak{G}_{1R}^* = RN_1 + \ldots + RN_{n+1}$ and $\theta_1(N_i) = L_i$, $i = 1, \ldots, n$, $\theta_1(N_{n+1}) = 0$. We take the representatives of (D_1^*, ϕ_1) and $(\mathfrak{G}_{1R}^*, \theta_1)$ respectively as in the case of (D^*, ϕ) and $(\mathfrak{G}_R^*, \theta)$ and obtain the cocycles $d_1(g_1^{s_1}, \ldots g_n^{s_n}, g_1^{t_1} \ldots g_n^{t_n}) = \psi_1(s, t)$ and $g_1(\sum_{i=1}^{n} \alpha_i L_i, \sum_{i=1}^{n} \beta_i L_i)$ of D and $\mathfrak{G}$ respectively. Then the relation (3) holds also for g_1 and ψ_1. Let $d_2(g_1^{s_1} \ldots g_n^{s_n}, g_1^{t_1} \ldots g_n^{t_n}) = \psi_2(s, t) = d + d_1 = \psi(s, t) + \psi_1(s, t)$. Then $d_2 = \psi_2(s, t)$ is a cocycle of D. Now we construct an extension (D_2^*, ϕ_2) of D as follows: Let D_2^* be the set of symbols $h_1^{*s_1} \ldots h_n^{*s_n} h_{n+1}^{*s_{n+1}}$, where s_i are integers and we define the product in D_2^* by the formula

$$(h_1^{*s_1} \ldots h_n^{*s_n} h_{n+1}^{*s_{n+1}})(h_1^{*t_1} \ldots h_n^{*t_n} h_{n+1}^{*t_{n+1}}) = h_1^{*u_1} \ldots h_n^{*u_n} h_{n+1}^{s_{n+1}+t_{n+1}+\psi_2(s,t)},$$

where $u_1, \ldots, u_n$ are determined by the relation

$$(g_1^{s_1} \ldots g_n^{s_n})(g_1^{t_1} \ldots g_n^{t_n}) = g_1^{u_1} \ldots g_n^{u_n}.$$

We may easily verify that D_2^* forms a group and $\{h_{n+1}^{*s_{n+1}}\}$ is contained in the center of D_2^* and $\phi_2(h_1^{*s_1} \ldots h_n^{*s_n} h_{n+1}^{*s_{n+1}}) = g_1^{s_1} \ldots g_n^{s_n}$ is a homomorphism of D_2^* onto D with the kernel $E = \{h_{n+1}^{*s_{n+1}}\}$ and hence (D_2^*, ϕ_2) is an extension of D. Clearly h_i^* form a canonical basis of D_2^* such that $\phi_2(h_i^*) = g_i$ and $\psi_2(s, t)$ is the cocycle of D corresponding to the representatives $u_2(g_1^{s_1} \ldots g^{s}{}_n{}^{n}) = h_1^{*s_1} \ldots h_n^{*s_n}$. Then we can construct an extension $(\mathfrak{G}_{R2}^*, \theta_2)$ of $\mathfrak{G}_R$ such that the cocycle g_2 of $\mathfrak{G}_R$ corresponding to this extension satisfies the relation anologous to (3). Since $\psi_2(s, t) = \psi(s, t) + \psi_1(s, t)$, we get by (3) $g_2(X, Y) = g(X, Y) + g_1(X, Y)$, where $X, Y \in \mathfrak{G}_R^*$. Thus the correspondence $(D^*, \phi) \to (\mathfrak{G}_R^*, \theta)$ defines a homomorphism ξ of $H_2(D)$ into $H_2(\mathfrak{G}_R)$. We show that the kernel of ξ is the group T of all elements of finite order of $H_2(D)$. Since $H_2(\mathfrak{G}_R)$ contains no element of finite order T is contained in the kernel of ξ. Let conversely $\xi(\psi(s, t)) = g(X, Y)$ and $g \sim 0$ in $H_2(\mathfrak{G}_R)$. Then there exitss an element $f(x) \in C^1(\mathfrak{G}_R)$ such that $g(X, Y) = \delta f(X, Y) = f([X, Y])$. Then $g(\sum_{i=1}^{n} \alpha_i L_i, \sum_{i=1}^{n} \beta_i L_i) = \sum_{i,j=1}^{n} c_{ij}^{n+1} \alpha_i \beta_j = f(\sum_{i,j=1}^{n} \alpha_i \beta_j [L_i L_j]) = \sum_{i,j,k=1}^{n} \alpha_i \beta_j c_{ij}^k f(L_k)$. Hence c_{ij}^{n+1}

$= \sum_{k=1}^{n} c_{ij}^k f(L_k)$. Let $L_i' = L_i^* + f(L_i)L_{n+1}^*$ for $i = 1, \ldots, n$ and $L_{n+1}' = L_{n+1}^*$. Then $\mathfrak{G}' = \{L_1', \ldots, L_n'\}$ is an ideal of $\mathfrak{G}^*$ isomorphic to $\mathfrak{G}$ under the correspondence $L_i' \to L_i$ and $\mathfrak{G}^* = \mathfrak{G}' + \{L_{n+1}'\}$. If $f(L_i)$ are all integers, then $\psi(s, t) \sim 0$. For then $g_i' = g_i^* g_{n+1}^{*f(L_i)} = \exp L_i'$ $(i = 1, \ldots, n)$ and $g_{n+1}' = g_{n+1}^*$ form a canonical basis of D^* and $g_1', \ldots g_n'$ generate a subgroup isomorphic to D and hence D^* splits over $E = \{g_{n+1}'\}$. If $f(L_i)$ are rational, take an integer m such that $mf(L_i)$ are the integers. Then since $\xi(m\psi(s, t)) = mg((X, Y) = m\delta f(X, Y)$ and $mf(L_i)$ are integers, $m\psi(s, t) \sim 0$. Thus the cohomology class containing $\psi(s, t)$ is of finite order. Hence the kernel of ξ is equal to T. Now take an element of $H_2(\mathfrak{G}_R)$ or equivalently an extension $(\mathfrak{G}_R^*, \theta)$ of $\mathfrak{G}_R$. Let $\{L_1^*, \ldots L_n^*, L_{n+1}^*\}$ be a basis of $\mathfrak{G}_R^*$ such that $\theta(L_i^*) = L_i$ $(1 \leqq i \leqq n)$ and $\theta(L_{n+1}^*) = 0$ and $g(X, Y) = g(\sum_{i=1}^{n} \alpha_i L_i, \sum_{i=1}^{n} \beta_i L_i) = \sum_{i,j=1}^{n} \alpha_i \beta_j c_{ij}^{n+1}$ be the cocycle ($=$ factor set) corresponding to the representatives $v(\sum_{i=1}^{u} \alpha_i L_i) = \sum_{i=1}^{n} \alpha_i L_i^*$, where c_{ij}^{n+1} are the structure constants of the basis $\{L_i^*\}$. We show that there exist an integer m and $\psi(s, t)$ such that $\xi(\psi(s, t)) = mg(X, Y)$. Let G^* be the simply connected group corresponding to $\mathfrak{G}^*$ and $\prod_{i=1}^{n+1} \exp s_i L_i^* \prod_{i=1}^{n+1} \exp t_i L_i^* = \prod_{i=1}^{n+1} \exp u_i L_i^*$. Since $\exp L_1^*, \ldots, \exp L_n^*$ generate a group isomorphic to D mod. $\exp tL_{n+1}^*$, if s_i, t_j are integers, $u_1, \ldots, u_n$ are also integers. Now

$$u_{n+1} = s_{n+1} + t_{n+1} + \psi(s_1 \ldots s_n, t_1, \ldots, t_n),$$

where $\psi(s, t)$ is, as proved before, a polynomial with rational coefficients of s, t. Then there exists an interger m such that the value of $m\psi(s, t)$ is an integer for integers s, t. This shows that $\exp L_1^*, \ldots, \exp L_n^*$, $\exp \dfrac{L^*_{n+1}}{m}$ generate a uniform discrete subgroup D^* of G^* which is clearly an extension of D and the cocycle $\psi'(s, t)$ corresponding to the representatives $\prod_{i=1}^{n} \exp s_i L_i^* = u\,(g_1^{s_1} \ldots g_n^{s_n})$ is $m\psi(s, t)$. Then $\xi(m\psi(s, t)) = mg(X, Y)$. Thus for each element $z \in H_2(\mathfrak{G}_R)$ there exists an integer m such that $mz \in \xi(H_2(D))$. $\xi(H_2(D)$ is a free abelian group of rank r and r is equal to the rank of $H_2(D)$. We may easily see that r is equal to the dimension q_2 of $H_2(\mathfrak{G}_R)$ over R. On the other hand, since $H_2(D) \cong H_2(M)$, r is equal to the 2-nd Betti number p_2 of M and hence $p_2 = q_2$. Thus $H_2(\mathfrak{G}_R)$ and the 2-nd cohomology group $H_2(M, R)$ with rational coefficients are the vector spaces over R of the same dimensions and hence they are isomorphic.

Remark. The 2-nd cohomology group of any Lie algebra $\mathfrak{G}$ is the dual space of the full exterior center of $\mathfrak{G}$. It is a theorem of Ado that, if $\mathfrak{G}$ is nilpotent, then $H_2(\mathfrak{G}) \neq 0$. See Chevalley and Eilenberg [2].

REFERENCES

[1] Chevalley, C., *On the topological structure of solvable groups*, Ann. of Math. **42** (1941).

[2] Chevalley. C., and Eilenberg, S., *Cohomology theory of Lie groups and Lie algebras*, Trans. of Amer. Math. Soc. **63** (1948).

[3] Eilenberg, S. and MacLane, S., *Relations between homology and homotopy groups of spaces*, Ann. of Math. **46** (1945).

[4] ——, *Cohomology theory in abstract groups*. I, Ann. of Math. **48** (1947).

[5] Hopf, H., *Fundamental Gruppe und zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1942).

[6] ——, *Ueber die Bettische Gruppen, die zu einer beliebigen Gruppe gehören*, Commet. Math. Helv. **17** (1945).

[7] Kuranishi, M., *On everywhere dense imbedding of free groups in Lie groups*, Nagoya Math. J., this volume.

[8] Malcev, A., *On a class of homogeneous spaces*, Izvestiya Akad. Nauk SSSR. Ser. Math. **13** (1949) (in Russian).

Mathematical Institute,
Nagoya University

https://doi.org/10.1017/S0027763000010096 Published online by Cambridge University Press