LIFTING DISJOINT SETS IN VECTOR LATTICES

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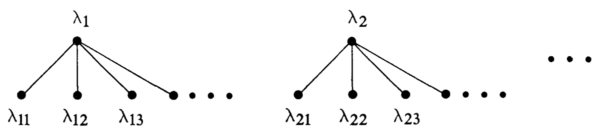
A subset $\{s_\alpha | \alpha \in \Delta\}$ of a lattice-ordered group (l-group) is *disjoint* if $s_\alpha \wedge s_\beta = 0$ for all $\alpha \neq \beta$ in Δ. An l-group G has the *lifting property* if for each l-ideal S of G and each countable disjoint subset $X_1, X_2, \ldots$ of G/S one can choose elements $0 \leqq x_i \in X_i$ so that $x_1, x_2, \ldots$ is a disjoint subset of G. In (**2**) Topping showed by an example, that uncountable sets of disjoint elements cannot necessarily be lifted and asserted (Theorem 8) that each vector lattice has the lifting property. His proof is valid for finite disjoint subsets of G/S, but we show by an example that this is, in general, all that one can establish. Moreover, using this example, it is easy to show that the result does not hold for free vector lattices or for free abelian l-groups.

Let Λ be the po-set of elements λ_i and λ_{ij}, where $i, j = 1, 2, \ldots$ and where $\lambda_i > \lambda_{ij}$ for all i and j (this being the only relation between the elements in Λ).



For each $\lambda \in \Lambda$ let R_λ be the naturally ordered additive group of real numbers and let G be the (unrestricted) direct sum of the R_λ. For $g = (\ldots, g_\lambda, \ldots)$ in G we say that g_λ is a maximal component if $g_\lambda \neq 0$ and either $\lambda = \lambda_i$ or else $\lambda = \lambda_{ij}$ and $g_{\lambda_i} = 0$, and we define g to be positive if each maximal component is positive. Then G is an l-group (**1**, Theorem 2.2) and with the natural scalar multiplication, it is a vector lattice. Let

$$S = \{g \in G | \, g_{\lambda_i} = 0 \text{ for all } i \text{ and all but a finite number of the } g_{\lambda_{ij}} \text{ are zero}\}.$$

Then S is an l-ideal of G, the basis group. For each positive integer n we define an element $a(n)$ in G as follows:

$$a(n) = \begin{cases} 1 & \text{if } \lambda = \lambda_n \text{ or } \lambda = \lambda_{in} \text{ for } i = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_n = S + a(n)$ for $n = 1, 2, \ldots$. One can picture the $a(n)$'s as follows:

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$$\begin{array}{llllll}
 & \quad 1 & \quad 0 & \quad 0 & \quad 0 & \\
a(1) & & & & & \cdots \\
 & 10000\ldots & 10000\ldots & 10000\ldots & 10000\ldots & \\
 & \quad 0 & \quad 1 & \quad 0 & \quad 0 & \\
a(2) & & & & & \cdots \\
 & 01000\ldots & 01000\ldots & 01000\ldots & 01000\ldots & \\
 & \quad 0 & \quad 0 & \quad 1 & \quad 0 & \\
a(3) & & & & & \cdots \\
 & 00100\ldots & 00100\ldots & 00100\ldots & 00100\ldots & \\
\cdots\cdots\cdots & & & & &
\end{array}$$

(1) *$X_m \wedge X_n = S + a(m) \wedge a(n) = S$ for $m \neq n$.*

Proof. $(a(m) \wedge a(n))_\lambda = 0$ except when $\lambda = \lambda_{mn}$ or λ_{nm}. Thus $a(m) \wedge a(n) \in S$.

(2) *If $0 < x_n \in X_n$ for $n = 1, 2, \ldots$, then $x_1, x_2, \ldots$ is not a disjoint subset of G.*

Proof. x_n has a non-zero component in all but a finite number of the blocks $\Lambda_i = \{\lambda_i, \lambda_{ij} | j = 1, 2, \ldots\}$ and the λ_n component of x_n is 1. Thus x_n is disjoint from at most a finite number of the other x_m.

Therefore G does not have the lifting property. Next let π be an l-homomorphism of a vector lattice F onto G and let η be the natural l-homomorphism of G onto G/S. If there exists a disjoint subset $f_1, f_2, \ldots$ in F so that $f_i\pi\eta = X_i$, then $0 < f_i\pi \in X_i$ and $f_1\pi, f_2\pi, \ldots$ is a disjoint subset of G, which is impossible by (2). Therefore F does not have the lifting property. In particular, a free vector lattice and hence an Archimedean vector lattice need not have the lifting property.

G contains a countable l-subgroup H without the lifting property. Simply, let Z_λ be the group of integers for each $\lambda \in \Lambda$ and let H be the l-subgroup of G that is generated by the small sum of the Z_λ and the elements $a(n)$. If we use $S \cap H$ instead of S, then the above argument shows that H does not have the lifting property. Now H is an l-homomorphic image of a countable free abelian l-group C and hence it follows that C does not have the lifting property.

Let A be the subgroup of G generated by all the scalar multiples of the $a(n)$. Then $K = S \oplus A$ is a vector sublattice of G. Each $k \in K$ has a unique representation $k = s + r_1a(1) + \ldots + r_ka(k)$, where $s \in S$ and the r_i are real numbers and it follows that the mapping $k \rightarrow (r_1, r_2, \ldots)$ is an l-homomorphism of K onto the restricted cardinal sum $\sum R_i$ of a countable number of copies of the reals. Moreover, S is the kernel of this homomorphism and hence $K/S \cong \sum R_i$. As above, the countable disjoint set

$$S + a(1), S + a(2), \ldots$$

cannot be lifted to K. Therefore, $\sum R_i$ is not projective and this contradicts the last part of Theorem 10 in (**2**).

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Finally, we observe that Theorem 9 of (**2**), for which no proof is given, is true. Namely, if π is an l-homomorphism of a complete vector lattice G that preserves all suprema and $\{a_\alpha \pi | \alpha \in \Delta\}$ is a disjoint subset of $G\pi$, not necessarily countable, then there exists a disjoint subset $\{b_\alpha | \alpha \in \Delta\}$ of G such that $b_\alpha \pi = a_\alpha \pi$ for all $\alpha \in \Delta$. For, in this case, the kernel K of π is a closed l-ideal of G and hence a cardinal summand. Thus $G = B \oplus K$ and hence $a_\alpha = b_\alpha + c_\alpha$, where $0 \leqq b_\alpha \in B$ and $0 \leqq c_\alpha \in K$. Thus $a_\alpha \pi = b_\alpha \pi$ and $\{b_\alpha | \alpha \in \Delta\}$ is a disjoint subset of G.

References

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