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# A NOTE ON GROUP DIVISIBLE DESIGNS WITH MUTUALLY ORTHOGONAL RESOLUTIONS

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**Abstract**

In this note, we use a geometric construction to produce group divisible designs with mutually orthogonal resolutions.

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## 1. Introduction

A combinatorial design $D$ with replication number $r$ is said to be resolvable if the blocks of $D$ can be partitioned into classes (called resolution classes) $R_1, R_2, \ldots, R_r$ such that each element of $D$ is contained in precisely one block of each class. Two resolutions $R$ and $R'$ of $D$ are called orthogonal if $|R_i \cap R'_j| \leq 1$ for all $R_i \in R$, $R'_j \in R'$ and almost orthogonal if $R_1 = R'_1$ and $|R_i \cap R'_j| \leq 1$ for $2 \leq i, j \leq r$. (It should be noted that the blocks of the design are considered as being labelled so that if a subset of the element set occurs as a block more than once the blocks are treated as distinct.) A set $Q = \{R^1, R^2, \ldots, R^t\}$ of $t$ resolutions of $D$ is called a set of mutually orthogonal resolutions (MORs) if the resolutions of $Q$ are pairwise orthogonal. If $t = 2$, the design is called doubly resolvable. We are interested in the existence of designs with sets of $t$ mutually orthogonal resolutions for $t \geq 2$. The designs which have been most frequently investiaged are the $(v, k, \lambda)$-BIBDs. A summary of some of the results for BIBDs with sets of $t$ MORs for $t \geq 2$ can be found in [3], and we note that there are still very few constructions known for finding $t$ MORs for BIBDs, where $t > 3$. In [3], we considered $(r, \lambda)$-designs and group divisible designs and showed that it is possible to construct designs of these types with $t$ MORS where $t \geq 2$. We described several constructions for group divisible designs which used the theory of finite geometries. In this note, we extend our results on group divisible designs.

Let $V$ be a finite set of $v$ elements and let $K$ be some subset of positive integers. A group divisible design (GDD) is a collection $B$ of subsets (blocks) of size $k$ where $k \in K$ taken from $V$ along with a partition of $V$ into subsets (groups) $G_1, G_2, \ldots, G_n$ such that

(1) any two elements from distinct groups are contained in precisely $\lambda_2$ blocks of $B$,

(2) any two distinct elements from the same group are contained in exactly $\lambda_1$ blocks of $B$ $(\lambda_1 < \lambda_2)$.

We denote such a design by $\mathrm{GDD}(v; K; G_1, G_2, \ldots, G_n; \lambda_1\lambda_2)$. Let $M$ be a subset of positive integers such that $|G_i| \in M$ for $i = 1, 2, \ldots, n$. In this case, we denote the design by $\mathrm{GDD}(v; K; M; \lambda_1, \lambda_2)$. If an element of $K$ or $M$ is starred, this means that there is exactly one block or group of this size in the design. If $K = \{k\}$ or $M = \{m\}$, we denote the design by $\mathrm{GDD}(v; k; M; \lambda_1, \lambda_2)$ or $\mathrm{GDD}(v; K; m; \lambda_1, \lambda_2)$ respectively. A doubly resolvable GDD with replication number $r$ is denoted by $\mathrm{DRGDD}_r(v; K; M; \lambda_1, \lambda_2)$.

The group divisible designs that we described in [3] had only one block size and only one group size. In this note, we relax these parameters. We will use a geometric construction due to P. Gibbons and R. Mathon for a $\mathrm{GDD}_{q^2}(q^3 - q; q; q^2 - q; 0, 1)$ for $q$ a prime power [1]. In the next section, we describe this construction and our notation. In Section 3, we provide two new results for GDDs with $t$ MORs where $t \geq 2$. In one of these cases, we are able to show that we have constructed a largest possible set of MORs for the design.

## 2. Construction of a $\mathrm{PBD}((q^2 - q)(q + 1); \{q, q^2 - q\})$

A pairwise balanced design, denoted $\mathrm{PBD}(v; K)$, is a collection $B$ of subsets (called blocks) of a finite $v$-set of elements $V$ such that every pair of distinct elements of $V$ is contained in precisely one block of $B$ and for each $b \in B$, $|b| \in K$, where $K$ is some subset of positive integers. In this section, we describe a construction for a $\mathrm{PBD}((q^2 - q)(q + 1); \{q, q^2 - q\})$. This is the dual of the construction used by P. Gibbons and R. Mathon in [1] to find a $\mathrm{GDD}((q^2 - q)(q + 1); q; q^2 - q; 0, 1)$. We will need the terminology and details of this construction in the next section. Definitions and results about projective planes that are used in this note can be found in [2].

THEOREM 2.1. *Let $q$ be a prime power. There exists a* $\mathrm{PBD}((q^2-q)(q+1); \{q, q^2-q\})$.

PROOF. Let $P$ be a projective plane of order $q^2$ and let $B$ be a Baer subplane of $P$. Let $L_\infty$ be a line of $P$ which meets $B$ in a line. We will denote the points of $l_\infty$ which belong to $B$ by $S = \{a_1, a_2, \ldots, a_{q+1}\}$. The remaining points of $l_\infty$ will be denoted by $U = \{u_1, u_2, \ldots, u_{q^2-q}\}$. We define $l(a_i)$ to be the set of $q^2-q$ lines of $P$ which contain $a_i$ and no other points of $B$,

$$l(a_i) = \{l \in P \mid \ a_i \in l \text{ and } |l \cap B| = 1\}.$$

Let $\{l_1, l_2, \ldots, l_{q^2+q}\} = \{l \in P \mid l \neq l_\infty \text{ and } |l \cap B| = q+1\}$. We note that $(l_i - B) \cap (l_j - B) = \varnothing$ for $i \neq j$. Let $W = \bigcup_{i=1}^{q^2+1}(l_i - B)$. $W$ contains $(q^2-q)(q^2+q)$ distinct points of $P$.

We can now construct a $\mathrm{GDD}((q^2-q)(q+1); q; q^2-q; 0; 1)$. The elements of the GDD will be the lines $\bigcup_{i=1}^{q+1} l(a_i)$. There are $(q^2-q)(q+1)$ elements. The groups will be $l(a_i)$ for $i = 1, 2, \ldots, q+1$. There are $q+1$ groups of size $q^2-q$. Let $x \in W$. $L(x) = \{l \in \bigcup_{i=1}^{q+1} l(a_i) \mid x \in l\}$ will be a block of the GDD. The number of blocks is $|W| = (q^2-q)(q^2+q)$. It is straightforward to verify that two elements from distinct groups occur together in precisely one block and that two elements from the same group do not occur together in any block.

By adding the groups as $q+1$ new blocks of size $q^2-q$ to the GDD described above, we construct a $\mathrm{PBD}((q^2-q)(q+1); \{q, q^2-q\})$.

P. Gibbons and R. Mathon observed that the GDD constructed above is highly resolvable. We state without proof their result.

THEOREM 2.2. [1] *There exists a* $\mathrm{GDD}_{q^2}(q^3-q; q; q^2-q; 0, 1)$ *with a set of* $q^2-q$ *MORs for $q$ a prime power.*

## 3. Main results

In this section, we use the $\mathrm{PBD}((q^2-q)(q+1); \{q, q^2-q\})$ to construct GDDs with $t$ MORs for $t \geq 2$.

THEOREM 3.1. *For $q$ a prime power, there exists a* $\mathrm{DRGDD}_{q^2}((q^2-q)(q+1); \{q, q^2-q); q; 0, 1)$.

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PROOF. We define a pair of almost orthogonal resolutions for the

$$\mathrm{PBD}((q^2-q)(q+1); \{q, q^2-q\}).$$

RESOLUTION 1. Let $b \in B$, $b \notin S$, Then $b$ occurs in $q+1$ lines in $P$ which meet $B$ in a line; denote these lines by $B'_1, B'_2, \ldots, B'_{q+1}$ and suppose that $a_i \in B'_i$. Let $B_i = B'_i - B$. Then $|B_i| = q^2 - q$.

Let $C'_1, C'_2, \ldots, C'_{q^2-q}$ denote the remaining $q^2-q$ lines in $P$ which contain $b$. Each of these lines contains one element of $U = \{u_1, u_2, \ldots, u_{q^2-q}\}$; suppose $u_i \in C'_i$. Let $C_i = C'_i - (U \cup B)$. The $|C_i| = q^2 - 1$.

We can now list the resolution classes of Resolution $1, R$.

$$R_i = \{L(x) \mid x \in B_i\} \cup l(a_i) \quad \text{for } i = 1, 2, \ldots, q+1.$$
$$R_{q+1+i} = \{L(x) \mid x \in C_i\} \quad \text{for } i = 1, 2, \ldots, q^2-q.$$
$$R = \{R_1, R_2, \ldots, R_{q^2+1}\}.$$

RESOLUTION 2. Let $\{f'_1, f'_2, \ldots, f'_{q^2}\} = \{l \in P \mid u_1 \in l, |l \cap B| = 1, l \neq l_\infty\}$. Suppose that $b \in f'_1$. Let $f_i = f'_i - (B \cup \{u_1\})$. Then $|f_i| = q^2 - 1$. We list the resolution classes of Resolution $2, T$.

$$T_i = \{L(x) \mid x \in f_i\} \quad \text{for } i = 1, 2, \ldots, q^2 \text{ and}$$
$$T_{q^2+1} = \{l(a_1), l(a_2), \ldots, l(a_{q+1})\}.$$

Since the resolution classes $R_{q+1}$ and $T_1$ are the same, $R$ and $T$ are almost orthogonal. By using this resolution to define the groups, we can construct a $\mathrm{DRGDD}_{q^2}((q^2-q)(q+1); \{q^2-q, q\}; 0, 1)$.

THEOREM 3.2. *For $q$ a prime power, there is a*

$$\mathrm{GBDD}_{q^2}((q^2-q)(q+1); \{q, q^2-q\}$$

*with a set of $q$ MORs.*

PROOF. We use the $\mathrm{PDB}((q^2-q)(q+1); \{q, q^2-q\})$ to construct the GDD and to define a set of $q$ MORs.

Let $f_1$ be a line of $P$ which meets $B$ in a line, where $f_1 \neq l_\infty$ and $a_1 \in f_1$. Let $\{b_1, b_2, \ldots, b_q, a_1\}$ be the points of $f_1$ which belong to $B$. Each $b_i$ occurs in $q$ other lines of $P$ which meet $B$ in a line. We denote these lines by $f'_2(b_i), f'_3(b_i), \ldots, f'_{q+1}(b_i)$ and we suppose $a_j \in f'_j(b_i)$. Let $f_j(b_i) = f'_j(b_i) - B$. Then $|f_j(b_i)| = q^2 - q$.

There are $q^2 - q$ remaining lines in $|P$ which contain $b_i$. We denote these lines by $C'_1(b_i), C'_2(b_i), \ldots, C'_{q^2-q}(b_i)$. Each of these lines contains one element of $U$. Let $C_j(b_i) = C'_j(b_i) - (B \cup U)$. Then $|C_j(b_i)| = q^2 - 1$.

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We now define $q$ resolutions: $R^1, R^2, \ldots, R^q$. Let $R^k = \{R_1^k, R_2^k, \ldots, R_{q^2}^k\}$ where the resolution classes are $R_i^k = \{L(x) \mid x \in f_{i+1}(b_k)\} \cup l(a_{i+1})$ for $i = 1, 2, \ldots, q$ and $R_{q+i}^k = \{L(x) \mid x \in C_i(b_k)\}$ for $i = 1, 2, \ldots, q^2 - q$.

$\{R^1, R^2, \ldots, R^q\}$ is a set of $q$ MORs for a $\mathrm{GDD}_{q^2}((q^2 - q)(q + 1); \{q, q^2 - q\}; \{q, (1^2 - q)\}; 0, 1)$ where the groups are defined as follows. Let $f_1 - B = \{d_1, d_2, \ldots, d_{q^2-q}\}$. There is one group of size $q^2 - q$; it is $l(a_1)$. There are $q^2 - q$ groups of size $q$; they are $\{L(d_i) \mid i- = 1, 2, \ldots, q^2 - q\}$.

We can show that this construction produces a largest possible set of MORs for a $\mathrm{GDD}((q^2 - q)(q+1); \{q, q^2 - q\}; \{q, (q^2 - q)^*\}; 0, 1)$. The counting argument is analogous to the arguments used to provide bounds on the size of the largest set of MORs for a design in [3].

THEOREM 3.3. *Let $D$ be a* $\mathrm{GDD}((q^2-q)(q+1); \{q, q^2-q\}; \{q, (q^2-q)^*\}; 0, 1)$. *If $t$ is the size of a largest set of MORs for $D$, then $t \le q$.*

PROOF. Let $B$ be a block of size $q^2 - q$ in $D$. Let $x$ be an element of the group of size $q^2 - q (x \notin B)$. Then $t$ is bounded by the number of blocks in $D$ which contain $x$ and no element of $B$. So $t \le q^2 - (q^2 - q) = q$.

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