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A NOTE ON A WIENER–PITT THEOREM ON WIENER GROUPS

BY

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ABSTRACT. It is shown that suitable conditions on the components of a measure in the algebra $M(G)$ ensure that this element lies outside of all maximal bi-ideals.

1. Let G be a locally compact non-discrete group. Let $M(G)$ be the convolution measure algebra of G. Let x be an element of $M(G)$ with a decomposition: $x = d + s + \alpha$, where α is the absolutely continuous component of x and d, s are two mutually singular measures; there is no reason to assume that d is the discrete part, and s is the singular (w.r.t. Haar measure) part of x. In this work we determine conditions on d, s and α to ensure that x is outside of every maximal bi-ideal (two-sided ideal) of $M(G)$, provided that G is a Wiener group. A group G is a Wiener group (G is in $[W]$) iff every closed bi-ideal in $L_1(G)$ is contained in the kernel of an irreducible continuous $\sim$-representation of $L_1(G)$.

As observed by Wiener and Pitt in [5], this problem in the case where $G = \mathbf{R}$, s is singular and d is a discrete measure can be handled, provided that s "is not too large". In fact they proved that if

(i)′ $\inf\{|\hat{x}(t)| : t \in \mathbf{R}\} > 0$

(ii)′ $\inf\{|\hat{d}(t)| : t \in \mathbf{R}\} > \|s\|$

where $\hat{x}$ and $\hat{d}$ are the Fourier transforms of x and d respectively, then x has an inverse in $M(G)$ i.e. x cannot be in a maximal ideal. For a significant extension of this Theorem on abelian groups, see N. Varopoulos [4], pp. 515.

Conditions (i)′ and (ii)′ can be reformulated as (i) and (ii) below:

(i) $0 < \inf\{\int_G h(t)\, dx^{\sim} * x(t) : h \in P(G)\}$

(ii) $\|s\| < \inf\{|t| : t \in \text{sp}(d)\}$

where $\text{sp}(d)$ is the spectrum of d as an element of $M(G)$ and $P(G)$ is the set of all continuous, normalized, pure positive definite functions on G ([1], 13.6). Our Theorem on Wiener groups contains the original W.P. Theorem as well the Varopoulos version of it, see Remarks 2–3. Note that there are several

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classes of non-abelian groups in $[W]$, for example: compact groups—motion groups—connected nilpotent Lie groups. The remarks at the end of the paper indicate examples of measures with and without inverses.

We hope that this work could give some light in the ideal structure of certain subalgebras of $M(G)$ of [W] group G.

2.

THEOREM. *Let G be a group in $[W]$ and $x = d + s + \alpha$ as above. Suppose that x, d and s satisfy* (i) *and* (ii), *then x cannot be in a bi-ideal of $M(G)$.*

LEMMA. *Let $d, s \in M(G)$ and suppose that d and s satisfy* (ii). *Then, if $y = d + s$, y cannot be in a bi-ideal of $M(G)$.*

Proof. We suppose that y is in a bi-ideal I of $M(G)$; hence there is a maximal left ideal L such that $I \subset L$. As in [3] II.2, we may consider the quotient algebra $M(G) - L$ and the space $B(M(G) - L)$ of all bounded operators on it. The operator $z \to U_z : M(G) \to B(M(G) - L)$, defined by $U_z v = (zu)' : v \in (M(G) - L)$, where $u' = v$ and $u \to u' : M(G) \to (M(G) - L)$ is called the left regular representation on $M(G) - L$. The norm of U_z is

$$\|U_z\| = \inf\{\|z + k\| : k \in M(G) \text{ and } k * M(G) \subset L\}$$

clearly $\|U_{d+s}\| = 0$ and so

$$(*) \qquad \|U_d\| = \|U_s\| \leq \|s\|$$

If $\text{sp}(U_d)$ is the spectrum of U_d as an element of $B(M(G) - L)$, then $\text{sp}(U_d) \subset \text{sp}(d)$. Since $\text{sp}(U_d) \neq \emptyset$, there is an element $t_0 \in \text{sp}(U_d)$ such that $|t_0| \leq \|U_d\|$. Thus by $(*)$ we have

$$\inf\{|t| : t \in \text{sp}(d)\} \leq |t_0| \leq \|U_d\| \leq \|s\|;$$

this contradicts our hypothesis (ii) and so y is as we claimed.

Note. This Lemma can be stated more generally: replace $M(G)$ by any Banach algebra.

Proof of the Theorem. We suppose that x is in maximal bi-ideal L of $M(G)$. According to Lemma $(d + s) \notin L$, hence $\alpha \notin L$ and L does not contain $L_1(G)$. Let $M = L \cap L_1(G)$, M is a bi-ideal of $L_1(G)$; since G is a Wiener group, M is in the kernel of an irreducible representation π of $L_1(G)$ on a Hilbert space H.

Let $(u_j)_{j \in J}$ be an approximate identity of $L_1(G)$, it is clear that $\pi(u_j) \to I$, $j \in J$, strongly, where I is the identity operator on H. As in [1] 2.9, π can be extended to an irreducible representation φ of $M(G)$ on the same Hilbert space. We observe that $\|\varphi(x)\| = 0$, in fact $\pi(u_j * x) = \pi(u_j)\varphi(x) \to I\varphi(x)$, strongly, and $\| = (u_j * x)\| = 0$, $j \in J$. Thus for all $h \in P(G)$ associated with π, $\langle h, x^{\sim} * x\rangle = 0$; this contradicts (i) and x is as required.

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3. **Remarks**

1. We suppose that the component d of x is in a closed subalgebra A of $M(G)$ containing the identity; where A is the range of a projection P, such that $P: M(G) \to A$ is an algebra homomorphism and $P(d) = d$, $d \in A$. If $\mathrm{sp}_A(d)$ is the spectrum of d as an element of A, it is trivial that $\mathrm{sp}(d) = \mathrm{sp}_A(d)$.

Next we examine some algebras A such that for $d \in A$, we have: $\inf(|t| : t \in \mathrm{sp}_A(d)) = \inf(\langle f, d^{\sim} * d\rangle^{1/2} : f \in P(G))$, therefore we may replace (ii) by the following:

(ii)″ $\|s\| < \inf(\langle f, d^{\sim} * d\rangle^{1/2} : f \in P(G))$

2. Let G_d be the abstract group G with the discrete topology and $P(G_d)$ be the set of all normalized pure positive definite functions on G; we shall see that if G_d is amenable then

(iii) $\inf(\langle f, d^{\sim} * d\rangle^{1/2} : f \in P(G)) = \inf(\langle f, d^{\sim} * d\rangle^{1/2} : f \in P(G_d))$

If $f \in P(G_d)$ has finite support F, then there is a continuous function g with compact support, which agrees with f on F. As in [1] 13.6.5 the linear span of $P(G)$ approximates g in the uniform topology on compact sets of G and so as in [1] 3.4.4 f is in the weak $*$ closure of $P(G)$ in $L^{\infty}(G_d)$; this implies (iii).

Now if G is an abelian group, $P(G)$ is the dual group $G^{\wedge}$ of G. $L_1(G_d)$ is the range of a projection whose kernel is the ideal of continuous measures; if $d \in L_1(G_d)$, $\mathrm{sp}(d) = \{\langle f, d\rangle : f \in P(G_d) = G_d^{\wedge}\}$. Hence (ii) and (iii) imply (ii)″ and the W.P. Theorem follows.

3. N. Varopoulos in [4] considers elements d in a subalgebra A of $M(G)$, such that $G^{\wedge}$ is dense in the maximal ideal space of A. For all these d's $\inf(|t| : t \in \mathrm{sp}(d)) = \inf(|\hat{d}(f)| : f \in G^{\wedge})$ and so this work contains also the Varopoulos W. P. Theorem.

4. It is natural to ask whether a measure $x = d + s + \alpha$ satisfying (i) and (ii) is invertible. It is shown in [2] Theorem 6.3, that there exists a compact group G and $x \in L_1(G_d)$ such that x satisfies (i) but x is not invertible. Choose a measure d with finite support: $2\|s\| < \inf(\langle f, x^{\sim} * x\rangle^{1/2} : f \in P(G))$, where $s = x - d$. It is clear that

$$2\|s\| < \inf_{f \in P(G)} \{\langle f, d^{\sim} * d\rangle + 2\langle f, d^{\sim} * d\rangle^{1/2} \langle f, s^{\sim} * s\rangle^{1/2} + \|s^{\sim} * s\|\}^{1/2}$$

and this implies (ii)″. Thus our Theorem does not provide invertible measures.

5. Let x, d, s and α be as in Theorem, suppose that the inverse $(d+s)^{-1}$ of $(d+s)$ exists in $M(G)$ and G is an Hermitian group, i.e. for any self-adjoint $y \in L_1(G)$, $\mathrm{sp}(y) \subseteq \mathbf{R}$. Then the inverse x^{-1} of x exists in $M(G)$. Observe that if x has no inverse in $M(G)$, then $1 + (d+s)^{-1} * \alpha$ has no inverse. Since $(d+s)^{-1} * \alpha \in L_1(G)$ and G is Hermitian, there exists an $f \in P(G)$, such that $\langle f, (1 + y * \alpha)^{\sim} * (1 + y * \alpha)\rangle = 0$, where $y = (d+s)^{-1}$. One can show that this contradicts (i) and so x is as we claimed.

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