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ABSTRACT

For a formal scheme \mathfrak{X} of finite type over a complete rank-one valuation ring, we construct a specialization morphism

$$\pi_1^{\text{dJ}}(\mathfrak{X}_\eta) \rightarrow \pi_1^{\text{proét}}(\mathfrak{X}_k)$$

from the de Jong fundamental group of the rigid generic fiber to the Bhatt–Scholze pro-étale fundamental group of the special fiber. The construction relies on an interplay between admissible blowups of \mathfrak{X} and normalizations of the irreducible components of \mathfrak{X}_k , and employs the Berthelot tubes of these irreducible components in an essential way. Using related techniques, we show that under certain smoothness and semistability assumptions, covering spaces in the sense of de Jong of a smooth rigid space which are tame satisfy étale descent.

1. Introduction

Throughout the following we fix a non-archimedean field K with valuation ring \mathcal{O}_K , residue field k , and a pseudo-uniformizer ϖ . Let X be a proper scheme over \mathcal{O}_K with generic fiber X_K and special fiber X_k . In [Gro71, Exp. X], Grothendieck constructed a continuous homomorphism

$$\pi_1^{\text{ét}}(X_K, \bar{x}) \rightarrow \pi_1^{\text{ét}}(X_k, \bar{y})$$

between the étale fundamental groups, called the *specialization map*. This map is moreover surjective if X is normal or if X_k is reduced (see [Ray70, §6] for other criteria). Translated in terms of coverings, the existence of the specialization map amounts to the fact that the pullback functor $\mathbf{F}\acute{\text{E}}\mathbf{t}_X \rightarrow \mathbf{F}\acute{\text{E}}\mathbf{t}_{X_k}$ between the corresponding categories of finite étale coverings is an equivalence, so that by composing its inverse with the restriction functor $\mathbf{F}\acute{\text{E}}\mathbf{t}_X \rightarrow \mathbf{F}\acute{\text{E}}\mathbf{t}_{X_K}$ we obtain a functor

$$u: \mathbf{F}\acute{\text{E}}\mathbf{t}_{X_k} \rightarrow \mathbf{F}\acute{\text{E}}\mathbf{t}_{X_K}.$$

Surjectivity of the specialization map then corresponds to the statement that u maps connected coverings to connected coverings.

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In cases of geometric interest, such as semistable reduction, the special fiber X_k will often not be normal. For such schemes, the *pro-étale fundamental group* introduced by Bhatt and Scholze [BS15] contains more refined information than the étale fundamental group. The corresponding notion of a covering space, a *geometric covering*, is an étale morphism which satisfies the valuative criterion of properness. As connected geometric coverings may have infinite degree and may not admit Galois closures, the pro-étale fundamental group is not pro-finite (or even pro-discrete) in general, but is a Noohi topological group (see [BS15]). As it turns out though, there is no specialization map for the pro-étale fundamental group fitting inside the following commutative square.

$$\begin{array}{ccc}
 \pi_1^{\text{proét}}(X_K, \bar{x}) & \cdots \times \cdots \rightarrow & \pi_1^{\text{proét}}(X_k, \bar{y}) \\
 \downarrow & & \downarrow \\
 \pi_1^{\text{ét}}(X_K, \bar{x}) & \longrightarrow & \pi_1^{\text{ét}}(X_k, \bar{y})
 \end{array}$$

Example 1.1 (Tate elliptic curve). Suppose that K is algebraically closed of characteristic zero and let $X \subseteq \mathbf{P}_{\mathcal{O}_K}^2$ be a cubic hypersurface whose generic fiber X_η is smooth and whose special fiber X_k is nodal. Then X_k is isomorphic to \mathbf{P}_k^1 with 0 and ∞ identified, and has a (unique) geometric covering $Y \rightarrow X_k$ with Galois group \mathbf{Z} . Explicitly Y is given by an infinite chain of copies of \mathbf{P}_k^1 glued along the poles. In this case, the diagram of fundamental groups as above would have to be of the form

$$\begin{array}{ccc}
 \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}} & \cdots \times \cdots \rightarrow & \mathbf{Z} \\
 \wr \downarrow & & \downarrow \text{inclusion} \\
 \widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}} & \xrightarrow{\text{projection}} & \widehat{\mathbf{Z}}
 \end{array}$$

where there is no homomorphism making the square commute.

To clarify the issue, it is useful to note that Grothendieck’s specialization map may be understood in terms of a more general specialization map involving formal schemes and rigid spaces. Namely, for any formal scheme \mathfrak{X} locally of finite type over \mathcal{O}_K there is an equivalence $\mathbf{Ét}_{\mathfrak{X}_k} \xrightarrow{\sim} \mathbf{Ét}_{\mathfrak{X}}$ and, thus, one is able to form the functors

$$u: \mathbf{Ét}_{\mathfrak{X}_k} \simeq \mathbf{Ét}_{\mathfrak{X}} \rightarrow \mathbf{Ét}_{\mathfrak{X}_\eta} \quad \text{and} \quad u: \mathbf{FÉt}_{\mathfrak{X}_k} \simeq \mathbf{FÉt}_{\mathfrak{X}} \rightarrow \mathbf{FÉt}_{\mathfrak{X}_\eta},$$

where \mathfrak{X}_η is the *rigid generic fiber* in the sense of Raynaud, a rigid K -space (see our notation and conventions section for a precise definition). Consequently, we have a formal version of the specialization map $\pi_1^{\text{ét}}(\mathfrak{X}_\eta, \bar{x}) \rightarrow \pi_1^{\text{ét}}(\mathfrak{X}_k, \bar{y})$. To obtain Grothendieck’s specialization map, one uses Grothendieck’s existence theorem to show that if X is a proper \mathcal{O}_K -scheme and $\mathfrak{X} = \widehat{X}$ (the ϖ -adic completion of X), then all finite étale covers of \mathfrak{X} and $\mathfrak{X}_\eta = X_K^{\text{an}}$ are algebraizable (i.e. that $\mathbf{FÉt}_{\mathfrak{X}} \simeq \mathbf{FÉt}_X$ and $\mathbf{FÉt}_{\mathfrak{X}_\eta} \simeq \mathbf{FÉt}_{X_K}$). In contrast, in Example 1.1, if $\mathfrak{Y} \rightarrow \mathfrak{X}$ is the unique étale lifting of $Y \rightarrow X_k$, then the rigid generic fiber $u(Y) = \mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is the Tate uniformization $\mathbf{G}_m^{\text{an}} \rightarrow X_\eta^{\text{an}}$, which is patently non-algebraizable.

To accommodate coverings of \mathfrak{X}_η arising from geometric coverings of \mathfrak{X}_k , one needs a suitably general class of covering spaces in non-archimedean geometry. In the seminal paper [dJ95b] de Jong explored a class of morphisms, which we call *de Jong covering spaces*, defined as maps $Y \rightarrow X$ of rigid K -spaces such that for any point x of X there exists an overconvergent (i.e. Berkovich) open neighborhood U such that Y_U is a disjoint union of finite étale coverings of U .

This class of covering spaces contains finite étale coverings as well as more exotic examples, such as Tate’s uniformization of elliptic curves and period mappings for certain Rapoport–Zink spaces. In [dJ95b] it is shown that the category of de Jong covering spaces is sufficiently rich as to support a version of Galois theory and, thus, gives rise to a fundamental group which we call the *de Jong fundamental group* and denote $\pi_1^{\text{dJ}}(X, \bar{x})$.

Our first main result is then the following.

THEOREM 1.2 (See Theorem 3.7). *Let \mathfrak{X} be a quasi-paracompact formal scheme locally of finite type over \mathcal{O}_K , and let $Y \rightarrow \mathfrak{X}_k$ be a geometric covering. Then the induced étale morphism $u(Y) \rightarrow \mathfrak{X}_\eta$ is a de Jong covering space.*

This result is somewhat surprising, because it implies that $u(Y)$ is a disjoint union of finite étale coverings locally in a fairly coarse topology of \mathfrak{X}_η despite the fact that Y does not even necessarily split into finite étale coverings étale locally on \mathfrak{X}_k (see Remark 3.10). To address this apparent discrepancy, we first establish a criterion for the generic fiber of a morphism of formal schemes to be a de Jong covering space. Namely, for a map of formal schemes $\mathfrak{Y} \rightarrow \mathfrak{X}$ such that $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is étale, if its restriction to every irreducible component of \mathfrak{X}_k is the disjoint union of finite morphisms, then $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is a de Jong covering space (Proposition 3.5). In the second step, we reduce to the situation where this criterion applies using a delicate blowup procedure (see Proposition 3.13) and the fact that geometric coverings of a normal scheme are disjoint unions of finite étale coverings.

In Corollary 3.9, we strengthen the link between de Jong covering spaces and geometric coverings by establishing a converse of Theorem 1.2. For an étale map of formal schemes, its generic fiber is a de Jong covering if and only if its special fiber is a geometric covering, and both are equivalent to the generic fiber being partially proper.

As a formal consequence of Theorem 1.2 we obtain the specialization homomorphism: assuming that \mathfrak{X}_η and \mathfrak{X}_k are both connected, for a compatible choice of geometric points \bar{x} of \mathfrak{X}_η and \bar{y} of \mathfrak{X}_k (see the discussion following the statement of Theorem 3.7), there is a continuous homomorphism

$$\pi_1^{\text{dJ}}(\mathfrak{X}_\eta, \bar{x}) \rightarrow \pi_1^{\text{proét}}(\mathfrak{X}_k, \bar{y})$$

fitting inside the following commutative diagram.

$$\begin{array}{ccc} \pi_1^{\text{dJ}}(\mathfrak{X}_\eta, \bar{x}) & \longrightarrow & \pi_1^{\text{proét}}(\mathfrak{X}_k, \bar{y}) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ét}}(\mathfrak{X}_\eta, \bar{x}) & \longrightarrow & \pi_1^{\text{ét}}(\mathfrak{X}_k, \bar{y}) \end{array}$$

As in the finite étale setting, we provide criteria for this homomorphism to have dense image (because the groups are no longer compact, density is not equivalent to surjectivity, and checking the latter could be more challenging). To this end, in Appendix A we define a property of formal schemes \mathfrak{X} over \mathcal{O}_K that we call η -normal. In essence this means that \mathfrak{X} is ‘integrally closed in \mathfrak{X}_η ’, and covers the cases when \mathfrak{X} is normal or \mathfrak{X}_k is reduced. We also show a version of Serre’s criterion for η -normality (see Theorem A.15).

We then have the following density result.

PROPOSITION 1.3 (See Corollary 3.8). *In the situation of Theorem 1.2, suppose that \mathfrak{X} is η -normal. Then the specialization map $\pi_1^{\text{dJ}}(\mathfrak{X}_\eta, \bar{x}) \rightarrow \pi_1^{\text{proét}}(\mathfrak{X}_k, \bar{y})$ has dense image.*

The criterion for being a de Jong covering space used in the proof of Theorem 1.2 can be useful even in the presence of ramification. In our second main result, we apply it to study covering spaces which are *tame* in the sense introduced by Hübner [Hüb21]. We warn the reader that these coverings are only ‘tame along the special fiber’, so to speak (for example, an Artin–Schreier covering of $\mathbf{A}_{\mathbf{F}_p}^1$ induces a tame covering of the affinoid unit disc over \mathbf{Q}_p). In [ALY21b], we studied the question whether de Jong coverings of a rigid K -space satisfy admissible or étale descent, and gave a negative answer by constructing a covering of an annulus in positive characteristic which is not a de Jong covering, but which becomes a de Jong covering space on a non-overconvergent open cover of the base. However, the construction relied on wild ramification phenomena, and it seems natural to expect that such behavior is avoided under a tameness assumption.

In order to state the result, we need to introduce some terminology used in [ALY21b]. Let us call a morphism of rigid K -spaces $Y \rightarrow X$ an *adm-covering space* (respectively, an *ét-covering space*) if there exists an open (respectively, étale) cover $\{U_i \rightarrow X\}$ such that $Y_{U_i} \rightarrow U_i$ is the disjoint union of finite étale coverings of U_i for all i .

THEOREM 1.4 (See Theorem 4.14 and Corollary 4.16). *Suppose that K is discretely valued. Let X be a smooth quasi-paracompact and quasi-separated rigid K -space, and suppose that generalized strictly semistable (gss) formal models of X are cofinal (see Definition 4.12). Then, every tame ét-covering space of X is a de Jong covering.*

In other words, tame de Jong coverings can be glued in the étale topology. The assumption on gss formal models is a technical condition that is automatically satisfied if K is of equal characteristic zero (see [Tem12]) and conjecturally satisfied in general. Moreover, because tameness is automatic when K has equal characteristic zero, we obtain the following unconditional corollary.

COROLLARY 1.5 (See Corollary 4.17). *Let K be a discretely valued non-archimedean field of equal characteristic zero. Then, for any smooth quasi-paracompact and quasi-separated rigid K -space X , the notions of de Jong, adm-covering spaces, and ét-covering spaces coincide.*

As stated previously, the proof of Theorem 1.4 contains in its kernel the same technique used in the proof of Theorem 1.2. Suppose for simplicity that $Y \rightarrow X$ is a tame étale map which splits into the disjoint union of finite étale coverings on an open covering of X . If \mathfrak{X} is a gss formal model of X such that this open covering of X extends to an open covering of \mathfrak{X} , we show using (a logarithmic version of) Abhyankar’s lemma the existence of a formal model $\mathfrak{Y} \rightarrow \mathfrak{X}$ which is Kummer étale. Using some basic toric geometry we show that $\mathfrak{Y} \rightarrow \mathfrak{X}$ satisfies the criterion of Proposition 3.5, and deduce that $\mathfrak{Y}_\eta = Y$ is a de Jong covering space of $\mathfrak{X}_\eta = X$ as desired. To extend this result from adm-covering spaces to ét-covering spaces we apply an étale bootstrap argument in the sense of [ALY21a, § 3.7] (see Corollary 4.16).

It seems believable that the smoothness and semistability assumptions in Theorem 1.4 are not necessary, but the authors were unable to remove them. In a different direction, the link with logarithmic geometry used in the proof strongly suggests the existence of a specialization map to a logarithmic variant of the pro-étale fundamental group of the special fiber of a semistable model, and we would expect such a map to be an isomorphism. Finally, in the companion paper [ALY21a], the authors have developed the notion of a ‘geometric covering’ of a rigid space which provides a generalization of ét-covering spaces. A natural question is whether in the setting of Corollary 1.5 every such tame geometric covering must be a de Jong covering space.

Notation and conventions

Throughout this paper, we use the following notation and terminology.

- By a *non-archimedean field* K we mean a field complete with respect to a rank-one valuation $|\cdot|: K \rightarrow [0, \infty)$.
- For a Huber ring A we denote by A° the subring of powerbounded elements and by $A^{\circ\circ}$ the set of topologically nilpotent elements. We abbreviate $\mathrm{Spa}(A, A^\circ)$ to $\mathrm{Spa}(A)$.
- By a *rigid K -space* we mean an adic space locally of finite type over $\mathrm{Spa}(K)$. Our conventions and notation concerning adic spaces is as in [ALY21a, §§ 2–3]. We denote by \mathbf{Rig}_K the category of rigid K -spaces, and by $\mathbf{Rig}_K^{\mathrm{qcs}}$ the full subcategory consisting of quasi-compact and quasi-separated rigid K -spaces.
- A topological space X is *quasi-compact* [Bos14, Definition 8.2/12] if it admits an open cover $X = \bigcup U_i$ by quasi-compact opens such that each U_i intersects only finitely many other U_j .
- An open subset U of a rigid K -space X is *overconvergent* if it is stable under specialization or, equivalently, if the inclusion $U \hookrightarrow X$ is partially proper [ALY21a, Proposition 3.3.3].
- By a *maximal point* of a rigid K -space X we mean a point which is maximal with respect to the generalization relation. By [Hub96, Lemma 1.1.10] this is equivalent to the corresponding valuation being rank 1.
- Let X be an adic space, formal scheme, or scheme. By $\mathbf{Ét}_X$ (respectively, $\mathbf{FÉt}_X$) we mean the category of objects étale (respectively, finite étale) over X .
- Closed subsets of (formal) schemes are implicitly treated as schemes endowed with the reduced scheme structure.
- We consistently use the term *cover* as in ‘open cover’ and *covering* as in ‘covering space’.

2. Rigid generic fibers of formal schemes

In this section, we review the theory of formal schemes and their rigid generic fibers. As before, we fix a non-archimedean field K with ring of integers \mathcal{O}_K , residue field k , and a pseudo-uniformizer ϖ . We will need to work with certain non-adic formal schemes over \mathcal{O}_K , such as $\mathrm{Spf}(\mathcal{O}_K[[x]])$; for such formal schemes the rigid generic fiber has been constructed by Berthelot [Ber96b] (see also de Jong [dJ95a, § 7]) in case K is discretely valued. For an extension to K arbitrary, we use the approach of Fujiwara and Kato [FK18, Chapter II, § 9.6] (see also Scholze and Weinstein [SW13] for an alternative definition).

2.1 Formal schemes

We refer to [FK18, Chapter I] for the basic terminology regarding formal schemes. We only consider formal schemes which are locally of the form $\mathrm{Spf}(A)$ where A has the I -adic topology for a finitely generated ideal I , and is I -adically complete and separated. By an ideal of definition of A we shall always mean a finitely generated ideal of definition. Recall that a continuous homomorphism of such algebras $A \rightarrow B$ is *adic* if $I \cdot B$ is an ideal of definition of B for some (equivalently, every) ideal of definition I of A , and a map of formal schemes $\mathfrak{Y} \rightarrow \mathfrak{X}$ is *adic* if it is locally of the form $\mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A)$ for an adic homomorphism $A \rightarrow B$.

We first discuss a hypothesis on formal schemes over \mathcal{O}_K that is more flexible than locally topologically of finite type (in the discrete case such objects were considered in [Ber96a, § 1] under the terminology ‘special’).

PROPOSITION 2.1. *Let A be a topological \mathcal{O}_K -algebra which is complete and separated with respect to a finitely generated ideal. The following conditions are then equivalent.*

- (a) For every (equivalently, any) pseudo-uniformizer ϖ of \mathcal{O}_K and ideal of definition $I \subseteq A$ such that $\varpi A \subseteq I$, the ring A/I is a finitely generated \mathcal{O}_K/ϖ -algebra.
- (b) There is a continuous and adic \mathcal{O}_K -algebra surjection

$$\psi: P := \mathcal{O}_K\langle x_1, \dots, x_n \rangle[[y_1, \dots, y_m]] \rightarrow A$$

for some $n, m \geq 0$. (Here, P is the completion of $\mathcal{O}_K[x_1, \dots, x_n, y_1, \dots, y_m]$ with respect to the ideal¹ $(\varpi, y_1, \dots, y_m)$.)

Proof. Suppose that A/I is finitely generated over \mathcal{O}_K/ϖ and choose a surjection

$$\varphi: (\mathcal{O}_K/\varpi)[x_1, \dots, x_n] \rightarrow A/I.$$

Write $I = (f_1, \dots, f_m, \varpi)$, and let $\psi: P \rightarrow A$ be the unique continuous \mathcal{O}_K -algebra homomorphism sending x_i to prescribed lifts of $\varphi(x_i)$ and with $\psi(y_i) = f_i$. Such a homomorphism exists because each f_i is topologically nilpotent. As P is by definition the $(y_1, \dots, y_m, \varpi)$ -adic completion of $\mathcal{O}_K[x_1, \dots, x_n, y_1, \dots, y_m]$, the map ψ is adic. Therefore, by [FK18, Chapter I, Proposition 4.3.6] the map ψ is surjective, and hence condition (b) is satisfied.

For the other direction, note that there exists a $k \geq 1$ such that $(\psi(y_1^k), \dots, \psi(y_m^k), \varpi) \subseteq I$. Then $P/(\psi(y_1^k), \dots, \psi(y_m^k), \varpi) \simeq (\mathcal{O}_K/\varpi)[x_1, \dots, x_n, y_1, \dots, y_m]/(y_1^k, \dots, y_m^k)$ is a finitely generated \mathcal{O}_K/ϖ -algebra, and hence so is its quotient A/I . □

DEFINITION 2.2 (See [FK18, Chapter II, Definition 9.6.5]). Let A be a topological \mathcal{O}_K -algebra which is complete and separated with respect to a finitely generated ideal.

- (a) We say that A is *topologically formally of finite type* if it satisfies any of the equivalent conditions of Proposition 2.1, *topologically of finite type* if it is formally topologically of finite type and adic over \mathcal{O}_K (equivalently, an adic quotient of $\mathcal{O}_K\langle x_1, \dots, x_n \rangle$ for some $n \geq 0$), and *admissible* if it is topologically of finite type and flat (equivalently, torsion free) over \mathcal{O}_K .
- (b) We say that a formal scheme \mathfrak{X} over \mathcal{O}_K is *locally formally of finite type* (respectively, *locally of finite type*, respectively, *admissible*) if for all affine open covers $\{\mathrm{Spf}(A_i)\}$ (equivalently, for a single such open cover) each A_i is topologically formally of finite type (respectively, topologically of finite type, respectively, admissible).
- (c) We say that a formal scheme \mathfrak{X} over \mathcal{O}_K is *(formally) of finite type* if it is locally (formally) of finite type and quasi-compact.

We denote the category of formal schemes locally formally of finite type (respectively, locally of finite type, respectively, of finite type) over \mathcal{O}_K by $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{lff}}t$ (respectively, $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{ft}}$, respectively, $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{ft}}$). The notions of (locally) formally of finite type, (locally) of finite type, and admissible are preserved under base change along $\mathcal{O}_K \rightarrow \mathcal{O}_{K'}$ for K' a non-archimedean extension of K .

Admissible ideals. For a formal scheme \mathfrak{X} we say that an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ is an *ideal sheaf of definition* if it is adically quasi-coherent (see [FK18, Chapter I, Definition 3.1.3]) and if for all affine open formal subschemes U we have that $\mathcal{J}(U)$ is an ideal of definition of $\mathcal{O}_{\mathfrak{X}}(U)$. If \mathfrak{X} is quasi-compact and quasi-separated, then the set of ideal sheaves of definition is cofiltering, and in particular non-empty (cf. [FK18, Chapter I, Corollary 3.7.12]). We say that an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ is *admissible* if it is adically quasi-coherent, of finite type, and open (i.e. locally

¹ Note that we have $\mathcal{O}_K\langle x_1, \dots, x_n \rangle[[y_1, \dots, y_m]] \simeq \mathcal{O}_K[[y_1, \dots, y_m]]\langle x_1, \dots, x_n \rangle$ where the latter is the ring of restricted power series as in [FK18, Chapter 0, § 8.4].

contains an ideal of definition). For an admissible ideal sheaf \mathcal{J} we denote by $\mathfrak{X}(\mathcal{J})$ the scheme cut out by \mathcal{J} .

Admissible blowups. Let \mathfrak{X} be a formal scheme and \mathcal{J} an admissible ideal sheaf of \mathfrak{X} . There then exists a final object amongst morphisms $\pi: \mathfrak{X}' \rightarrow \mathfrak{X}$ which are adic, proper, and satisfy $\mathcal{J}\mathcal{O}_{\mathfrak{X}'}$ is invertible. We call this universal object the *admissible blowup* of \mathfrak{X} relative to \mathcal{J} and denote it by $\pi_{\mathcal{J}}: \mathfrak{X}_{\mathcal{J}} \rightarrow \mathfrak{X}$. We call the subscheme cut out by \mathcal{J} the *center* of the admissible blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$. One can give an explicit description of this admissible blowup as in classical algebraic geometry (cf. [FK18, Chapter II, §§ 1.1.(a) and 1.1.(b)]). As admissible blowups are finite type and do not introduce torsion for an ideal of definition (see [FK18, Chapter II, Corollary 1.1.6] for this latter claim), one sees that the properties defined in Definition 2.2(bc) are stable under admissible blowups.

Strict transform. If $\mathfrak{X}_{\mathcal{J}} \rightarrow \mathfrak{X}$ is an admissible blowup and $\mathfrak{Y} \rightarrow \mathfrak{X}$ an adic morphism, then the map $\mathfrak{Y}_{\mathfrak{X}_{\mathcal{J}}} \rightarrow \mathfrak{Y}$ will, in general, not be an admissible blowup. That said, one can show (see [FK18, Chapter II, § 1.2]) that the \mathcal{J} -torsion ideal sheaf $\mathcal{K} = \mathcal{O}_{\mathfrak{Y}_{\mathfrak{X}_{\mathcal{J}}}, \mathcal{J}\text{-tors}}$ is of finite type. One then considers the closed formal subscheme \mathfrak{Y}' of $\mathfrak{Y}_{\mathfrak{X}_{\mathcal{J}}}$ cut out by \mathcal{K} which is called the *strict transform* of $\mathfrak{Y} \rightarrow \mathfrak{X}$ along $\mathfrak{X}_{\mathcal{J}} \rightarrow \mathfrak{X}$. The map $\mathfrak{Y}' \rightarrow \mathfrak{Y}$ is isomorphic to the admissible blowup $\mathfrak{Y}_{\mathcal{J}\mathcal{O}_{\mathfrak{Y}}}$ (see [FK18, Chapter II, Proposition 1.2.9]).

Underlying reduced subscheme. For a formal scheme \mathfrak{X} there exists a unique ideal sheaf $\mathcal{O}_{\mathfrak{X}}^{\circ\circ} \subseteq \mathcal{O}_{\mathfrak{X}}$ such that $\mathcal{O}_{\mathfrak{X}}^{\circ\circ}(\text{Spf}(A)) = A^{\circ\circ}$ for all affine open $\text{Spf}(A) \subseteq \mathfrak{X}$. We call $\mathcal{O}_{\mathfrak{X}}^{\circ\circ}$ the *ideal sheaf of topologically nilpotent elements*. The pair $\underline{\mathfrak{X}} := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^{\circ\circ})$ defines a closed subscheme of \mathfrak{X} called the *underlying reduced subscheme* of \mathfrak{X} . For any ideal sheaf of definition \mathcal{J} of \mathfrak{X} the map $\underline{\mathfrak{X}} \rightarrow \mathfrak{X}$ factorizes through $\mathfrak{X}(\mathcal{J})$ and identifies $\underline{\mathfrak{X}}$ with the underlying reduced subscheme of $\mathfrak{X}(\mathcal{J})$.

Completions. Let \mathfrak{X} be a formal scheme and let $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ be an admissible ideal sheaf. Consider the colimit² $\mathfrak{X}(\mathcal{J}^{\infty}) = \varinjlim \mathfrak{X}(\mathcal{J}^n)$ of locally ringed spaces. Let us note that the underlying topological space of $\mathfrak{X}(\mathcal{J}^n)$ is independent of $n \geq 1$. Let us call it X . We then consider the sheaves $\mathcal{O}_{\mathfrak{X}(\mathcal{J}^n)}$ on X for all n . We have obvious surjections $\mathcal{O}_{\mathfrak{X}(\mathcal{J}^i)} \rightarrow \mathcal{O}_{\mathfrak{X}(\mathcal{J}^j)}$ for $i \geq j \geq 1$. Using [FK18, Chapter I, Proposition 1.4.2] one shows that $\mathfrak{X}(\mathcal{J}^{\infty})$ is a formal scheme. If $\mathfrak{X} = \text{Spf}(A)$ and $J \subseteq A$ is the ideal corresponding to \mathcal{J} , then $\mathfrak{X}(\mathcal{J}^{\infty})$ is in fact equal to $\text{Spf}(B)$ with B the J -adic completion of A .

The proof of the following representability result is left to the reader.

PROPOSITION 2.3. *Let \mathfrak{X} be a formal scheme such that $\underline{\mathfrak{X}}$ is locally Noetherian. Then, for any closed subset Z of $|\mathfrak{X}|$ the functor*

$$\widehat{\mathfrak{X}}_Z: \mathbf{FSch}^{\text{op}} \rightarrow \mathbf{Set}, \quad \mathfrak{X}' \mapsto \{f \in \text{Hom}_{\mathbf{FSch}}(\mathfrak{X}', \mathfrak{X}) : f(|\mathfrak{X}'|) \subseteq Z\}$$

is representable. Moreover, if $|\mathfrak{X}(\mathcal{J})| = Z$ for an admissible ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$, then it is represented by $\mathfrak{X}(\mathcal{J}^{\infty})$.

We call the formal scheme $\widehat{\mathfrak{X}}_Z$ the *completion of \mathfrak{X} along Z* . It is clear that if \mathfrak{U} is an open formal subscheme of \mathfrak{X} , then $\widehat{\mathfrak{U}}_{\mathfrak{U} \cap Z}$ is the preimage of \mathfrak{U} in $\widehat{\mathfrak{X}}_Z$. By reducing to the affine case using [FK18, Chapter I, Proposition 3.7.11] one can show that if \mathfrak{X} is moreover quasi-compact and quasi-separated, then every closed subset Z is of the form $|\mathfrak{X}(\mathcal{J})|$ for some admissible ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$.

² Here we consider $\mathfrak{X}(\mathcal{J}^n)$ as a *pseudo-discrete formal scheme* as in [SP21, Tag 0AHY].

The following result will play an important role in the proof of Proposition 3.5. In its statement, we use the notion of *adically locally of finite presentation* as in [FK18, Chapter I, Definition 5.3.1]. Note that by [FK18, Chapter I, Corollary 2.2.4] and [FK18, Chapter 0, Corollary 9.2.9] this condition is automatic for a morphism of admissible formal schemes over \mathcal{O}_K .

PROPOSITION 2.4. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism adically locally of finite presentation of formal schemes and assume that the underlying reduced scheme $\underline{\mathfrak{X}}$ of \mathfrak{X} is locally Noetherian. Let Z be a closed subset of \mathfrak{X} and \mathfrak{Z} any closed subscheme of \mathfrak{X} with $|\mathfrak{Z}| = Z$. Then if $\mathfrak{Y}_{\mathfrak{Z}} \rightarrow \widehat{\mathfrak{Z}}$ is finite (respectively, a disjoint union of finite morphisms), then the same is true for $\mathfrak{Y}_{\widehat{\mathfrak{X}}_Z} \rightarrow \widehat{\mathfrak{X}}_Z$.*

Proof. Suppose first that the map is finite. Without loss of generality, we may assume that $\mathfrak{Z} = Z$ with the reduced scheme structure. By working locally on the target, we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ and that $Z_{\mathrm{red}} = \mathrm{Spec}(A/J)_{\mathrm{red}}$ for some admissible ideal J of A . Set $T = \mathrm{Spec}(A/J)$. Note then that $\mathfrak{Y}_T \rightarrow T$ is, by assumption, a morphism of schemes locally of finite presentation. By assumption, we have that the pullback of this map along $T_{\mathrm{red}} \rightarrow T$ is finite, and thus it is itself finite by [SP21, Tag 0BPG]. By [FK18, Chapter I, Proposition 4.2.3] to verify $\mathfrak{Y}_{\widehat{\mathfrak{X}}_Z} \rightarrow \widehat{\mathfrak{X}}_Z$ is finite, it suffices to check this after base change along $T \rightarrow \widehat{\mathfrak{X}}_Z$ from where the claim follows. The second claim is clear by the previous discussion because $\mathfrak{Y}_{Z_{\mathrm{red}}} \rightarrow \mathfrak{Y}_{\widehat{\mathfrak{X}}_Z}$ is a homeomorphism, and so if $\mathfrak{Y}_{Z_{\mathrm{red}}}$ is a disjoint union of clopen subsets finite over Z_{red} , then $\mathfrak{Y}_{\widehat{\mathfrak{X}}_Z}$ is a disjoint union of clopen subsets which, by the above, must be finite over $\widehat{\mathfrak{X}}_Z$. \square

2.2 Generic fibers of formal schemes

In this subsection we discuss the notion of the *rigid generic fiber* of a formal scheme locally formally of finite type over \mathcal{O}_K , which is a rigid K -space. We then note several properties of the generic fiber construction.

We first recall Raynaud’s equivalence as developed by Fujiwara–Kato in [FK18], which, in particular, gives the construction for formal schemes locally of finite type over \mathcal{O}_K . Let A be a topologically finite type \mathcal{O}_K -algebra. The algebra $A_K = A[1/\pi]$ is then an affinoid K -algebra, i.e. an algebra topologically of finite type over K . The subring $A_K^\circ \subseteq A_K$ of powerbounded elements coincides with the integral closure of (the image of) A in A_K (see [FK18, Chapter II, Corollary A.4.10]). Combining [FK18, Chapter II] and [FK18, Chapter II, Theorem A.5.1], there exists a unique functor

$$(-)_\eta: \mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{ft}} \rightarrow \mathbf{Rig}_K^{\mathrm{qcqs}}$$

such that $\mathrm{Spf}(A)_\eta = \mathrm{Spa}(A_K)$ for every topologically finite type \mathcal{O}_K -algebra A , and which respects open immersions and open covers. This functor naturally extends to a functor

$$(-)_\eta: \mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{lft}} \rightarrow \mathbf{Rig}_K^{\mathrm{qs}},$$

and for \mathfrak{X} locally of finite type over \mathcal{O}_K , the rigid K -space \mathfrak{X}_η is called the *rigid generic fiber* of \mathfrak{X} . Furthermore, $(-)_\eta$ sends the class W of admissible blowups to isomorphisms and induces equivalences of categories

$$\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{adm, qc}}[W^{-1}] \xrightarrow[\mathrm{incl}]{} \mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{ft}}[W^{-1}] \xrightarrow[(-)_\eta]{} \mathbf{Rig}_K^{\mathrm{qcqs}}. \tag{2.2.1}$$

Here $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{adm, qc}}$ consists of quasi-compact admissible formal schemes over \mathcal{O}_K , and $(-)[W^{-1}]$ denotes the localization with respect to W , i.e. the category obtained by formally inverting all admissible blowups. By a *formal model* of a rigid K -space X we shall mean a formal scheme \mathfrak{X} locally of finite type over \mathcal{O}_K together with an isomorphism $\mathfrak{X}_\eta \simeq X$. By [FK18, Chapter II,

Proposition 2.1.10], for any quasi-compact and quasi-separated rigid K -space, the category of *admissible* formal models of X is (equivalent to) a cofiltering poset.

If X is a quasi-compact and quasi-separated rigid K -space, then the construction of the rigid generic fiber allows one to identify the locally topologically ringed space (X, \mathcal{O}_X^+) as $\varprojlim (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ where \mathfrak{X} runs over admissible formal models of X . In particular, for each admissible formal model \mathfrak{X} of X we have a morphism of locally topologically ringed spaces

$$\mathrm{sp}_{\mathfrak{X}}: (X, \mathcal{O}_X^+) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

called the *specialization map* for \mathfrak{X} . For $\mathfrak{X} = \mathrm{Spf}(A)$ affine, the underlying map of topological spaces sends a valuation $\nu: A_K \rightarrow \Gamma \cup \{0\}$ to the (open) prime ideal $\{x \in A : \nu(x) < 1\}$ and the map on global sections is the natural map $\mathcal{O}_{\mathfrak{X}}(\mathfrak{X}) = A \rightarrow A_K^{\circ} = \mathcal{O}_X(X)^+$.

As $|\mathfrak{X}| = |\mathfrak{X}_k|$ we shall often implicitly treat the specialization map $\mathrm{sp}_{\mathfrak{X}}$ as a map of topological spaces $|X| \rightarrow |\mathfrak{X}_k|$. This map is continuous, quasi-compact, closed, and surjective (see [FK18, Chapter II, Theorem 3.1.2] and [FK18, Chapter II, Proposition 3.1.5]). If $\varphi: \mathfrak{X}' \rightarrow \mathfrak{X}$ is a morphism in $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{ft}}$ then the diagram

$$\begin{array}{ccc} (X', \mathcal{O}_{X'}^+) & \xrightarrow{\mathrm{sp}_{\mathfrak{X}'}} & (\mathfrak{X}', \mathcal{O}_{\mathfrak{X}'}) \\ \varphi_{\eta} \downarrow & & \downarrow \varphi \\ (X, \mathcal{O}_X^+) & \xrightarrow{\mathrm{sp}_{\mathfrak{X}}} & (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \end{array}$$

commutes. Moreover, for any open subset \mathfrak{U} of \mathfrak{X} the induced map $\mathfrak{U}_{\eta} \rightarrow \mathfrak{X}_{\eta}$ is an open immersion with image $\mathrm{sp}_{\mathfrak{X}}^{-1}(\mathfrak{U})$ (see [FK18, Chapter II, Proposition 3.1.3]). It follows that the definition of the specialization map may be extended to formal schemes \mathfrak{X} locally of finite type over \mathcal{O}_K , and that this extension enjoys also the property of being continuous, quasi-compact, closed, and surjective if \mathfrak{X} is admissible.

In order to extend the definition of the rigid generic fiber to the category $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{lft}}$, we need the following result.

PROPOSITION 2.5 (See [FK18, Chapter II, § 9.6.(b)]). *Let \mathfrak{X} be a formal scheme locally formally of finite type over \mathcal{O}_K . Then, the functor*

$$(\mathbf{Rig}_K^{\mathrm{qcqs}})^{\mathrm{op}} \rightarrow \mathbf{Set}, \quad Z \mapsto \varinjlim \mathrm{Hom}_{\mathcal{O}_K}(\mathfrak{Z}, \mathfrak{X}),$$

where \mathfrak{Z} runs over the admissible formal models of Z , is representable by a rigid K -space \mathfrak{X}_{η} .

We call \mathfrak{X}_{η} the *rigid generic fiber* of \mathfrak{X} . If \mathfrak{X} is locally of finite type over \mathcal{O}_K , this agrees with the previous definition of \mathfrak{X}_{η} . The association $\mathfrak{X} \mapsto \mathfrak{X}_{\eta}$ is functorial in \mathfrak{X} and commutes with finite limits and disjoint unions. Moreover, it sends admissible blowups to isomorphisms.

The following construction makes the rigid generic fiber more explicit in the affine case.

CONSTRUCTION 2.6 (The rigid generic fiber of an affine formal scheme). Let $\mathfrak{X} = \mathrm{Spf}(B)$ for a topologically formally of finite type \mathcal{O}_K -algebra B , and let $J = (b_1, \dots, b_m)$ be an ideal of definition of B . We denote by $B[J/\varpi]$ the affine blowup algebra [SP21, Tag 052Q], i.e. the image of $B[x_1, \dots, x_m]/(\pi x_i - b_i)$ in $B[1/\varpi]$; it is independent of the choice of generators of J . Let $B\langle J/\varpi \rangle$ be the J -adic completion of $B[J/\varpi]$, which is an admissible \mathcal{O}_K -algebra on which the J -adic topology coincides with the ϖ -adic topology. The map $B \rightarrow B\langle J/\varpi \rangle$ is continuous, and

the morphism $\mathrm{Spf}(B\langle J/\varpi \rangle) \rightarrow \mathrm{Spf}(B)$ induces a map $\mathrm{Spf}(B\langle J/\varpi \rangle)_\eta \rightarrow \mathrm{Spf}(B)_\eta$ of rigid K -spaces. Writing $B(J) = B\langle J/\varpi \rangle[1/\varpi]$, we have $\mathrm{Spf}(B\langle J/\varpi \rangle)_\eta = \mathrm{Spa}(B(J))$.³

For $J' \subseteq J$, we have an inclusion $B[J'/\varpi] \subseteq B[J/\varpi]$ and consequently a morphism $\mathrm{Spf}(B[J/\varpi]) \rightarrow \mathrm{Spf}(B[J'/\varpi])$ over $\mathrm{Spf}(B)$. The induced morphism $\mathrm{Spa}(B(J)) \rightarrow \mathrm{Spa}(B(J'))$ is an isomorphism onto a rational open domain of $\mathrm{Spa}(B(J'))$. Thus, the inductive system $\{\mathrm{Spa}(B(J))\}$ indexed by all finitely generated ideals of definition of B gives a well-defined adic space $\varinjlim_J \mathrm{Spa}(B(J))$. As this system admits compatible maps to \mathfrak{X}_η , we obtain a morphism of rigid K -spaces

$$\varinjlim_J \mathrm{Spa}(B(J)) \rightarrow \mathfrak{X}_\eta. \tag{2.2.2}$$

One then has the following concrete description of the generic fiber \mathfrak{X}_η .

LEMMA 2.7 (See [FK18, Chapter II, Remark 9.6.3]). *With notation as in Construction 2.6 the map (2.2.2) is an isomorphism.*

Let us note that for any given ideal of definition J_0 of B , the set $\{J_0^n\}$ is cofinal in the set of all ideals of definition of B . In particular, in Lemma 2.7 we may replace $\varinjlim_J \mathrm{Spa}(B(J))$ with $\varinjlim_n \mathrm{Spa}(B(J_0^n))$. We often do this without comment in the following.

With this, we can give a more concrete description of the functor of points of the generic fiber in the situation dictated in Construction 2.6. It will enable us to show easily that an étale morphism of formal schemes induces an étale morphism on rigid generic fibers.

Construction 2.8. Let $\mathfrak{X} = \mathrm{Spf}(B)$ and $\mathfrak{X}' = \mathrm{Spf}(B')$ be affine objects of $\mathbf{FSch}_{\mathcal{O}_K}^{\mathrm{lft}}$ and let $\mathfrak{X}' \rightarrow \mathfrak{X}$ be a morphism over \mathcal{O}_K . Let J be an ideal of definition of B and let J' be an ideal of definition of B' such that $JB \subseteq J'$ (which exists because $B \rightarrow B'$ is continuous). It is then easy to see that the morphism $\mathfrak{X}'_\eta \rightarrow \mathfrak{X}_\eta$ maps $\mathrm{Spa}(B'((J')^n))$ into $\mathrm{Spa}(B(J^n))$ for all n .

Let (R, R^+) be a Huber (K, \mathcal{O}_K) -algebra. Then, any morphism $\mathrm{Spa}(R, R^+) \rightarrow \mathfrak{X}_\eta$ of adic spaces over (K, \mathcal{O}_K) must factorize through $\mathrm{Spa}(B(J^n))$ for some n and, thus, defines a map of Huber pairs $(B(J^n), B(J^n)^\circ) \rightarrow (R, R^+)$ which, in turn, defines a continuous map $B \rightarrow R^+$ of \mathcal{O}_K -algebras independent of the choice of n . Moreover, this map must send ϖ to an element of R^\times . Note that one has a natural map

$$\mathrm{Hom}_{\mathfrak{X}_\eta}(\mathrm{Spa}(R, R^+), \mathfrak{X}'_\eta) \rightarrow \mathrm{Hom}_B(B', R^+).$$

Indeed, any map $\mathrm{Spa}(R, R^+) \rightarrow \mathfrak{X}'_\eta$ must factorize through $\mathrm{Spa}(B'((J')^m))$ for some $m \geq n$ and so defines a map of Huber pairs $(B'((J')^m), B'((J')^m)^\circ) \rightarrow (R, R^+)$. This then defines a continuous map $B' \rightarrow R^+$ of B -algebras which is independent of m .

LEMMA 2.9. *With notation as in Construction 2.8, the map*

$$\mathrm{Hom}_{\mathfrak{X}_\eta}(\mathrm{Spa}(R, R^+), \mathfrak{X}'_\eta) \rightarrow \mathrm{Hom}_B(B', R^+)$$

is a functorial bijection. Moreover,

$$\mathrm{Hom}_B(B', R^+) = \varinjlim_{R_0} \mathrm{Hom}_{\mathfrak{X}}(\mathrm{Spf}(R_0), \mathfrak{X}')$$

*as R_0 runs over the open and bounded B -subalgebras of R^+ .*⁴

³ Note that $\mathrm{Spa}(B(J))$ may also be described as the adic spectrum of a fairly simple non-complete Huber pair. Namely, $\mathrm{Spa}(B(J))$ is equal to $\mathrm{Spa}(B[1/\varpi], B[J/\varpi]^\sim)$ where $B[1/\varpi]$ has the unique structure of a Huber ring so that $B[J/\varpi]$ equipped with the J -adic topology is an open subring, and where $B[J/\varpi]^\sim$ denotes the integral closure of $B[J/\varpi]$ in $B[1/\varpi]$.

⁴ A B -subalgebra R_0 of R^+ is *bounded* if there is a ring of definition $R_1 \subseteq R$ such that $R_0 \subseteq \varpi^{-n}R_1$ for $n \gg 0$.

With this, we can prove the following.

PROPOSITION 2.10. *Let $f: \mathfrak{X}' \rightarrow \mathfrak{X}$ be an étale (respectively, finite) morphism in $\mathbf{FSch}_{\mathcal{O}_K}^{\text{lft}}$. Then, $f_\eta: \mathfrak{X}'_\eta \rightarrow \mathfrak{X}_\eta$ is étale (respectively, finite).*

Proof. We may assume that $\mathfrak{X} = \text{Spa}(B)$ and $\mathfrak{X}' = \text{Spa}(B')$. Suppose first that $\mathfrak{X}' \rightarrow \mathfrak{X}$ is étale. We deduce from Lemma 2.9 that for any Huber (K, \mathcal{O}_K) -algebra (R, R^+) and square-zero ideal N of R one has a commutative square

$$\begin{CD} \text{Hom}_{\mathfrak{X}_\eta}(\text{Spa}(R, R^+), \mathfrak{X}'_\eta) @>>> \varinjlim_{R_0} \text{Hom}_{\mathfrak{X}}(\text{Spf}(R_0), \mathfrak{X}') \\ @VVV @VVV \\ \text{Hom}_{\mathfrak{X}_\eta}(\text{Spa}(R/N, R^+/(R^+ \cap N)), \mathfrak{X}'_\eta) @>>> \varinjlim_{R_0} \text{Hom}_{\mathfrak{X}}(\text{Spf}(R_0/(R_0 \cap N)), \mathfrak{X}') \end{CD}$$

where the horizontal maps are bijections. However, the right vertical arrow is a bijection because $\mathfrak{X}' \rightarrow \mathfrak{X}$ is étale and, thus, the left vertical arrow must also be a bijection. As $f_\eta: \mathfrak{X}'_\eta \rightarrow \mathfrak{X}_\eta$ is locally of finite type (because both are locally of finite type over $\text{Spa}(K)$) we deduce that f_η is étale as desired.

Now suppose that f is finite, and let J be an ideal of definition of B . As f is adic, the ideal $J' = JB'$ is an ideal of definition of B' . As $B \rightarrow B'$ is finite, $B' \langle (J')^n / \varpi \rangle = B \langle J^n / \varpi \rangle \otimes_B B'$ and it follows that $B' \langle (J')^n \rangle = B \langle J^n \rangle \otimes_B B'$. Since $\mathfrak{X}'_\eta \rightarrow \mathfrak{X}_\eta$ is the colimit of the maps $\text{Spa}(B' \langle (J')^n \rangle) \rightarrow \text{Spa}(B \langle J^n \rangle)$, which are finite by [Hub96, Lemma 1.4.5 iv)], it follows that this map is finite, as desired. \square

We end this subsection by showing that the generic fiber preserves the partial properness of the special fiber.⁵ Recall here that a morphism of schemes is called *partially proper* if it is locally of finite type, quasi-separated, and satisfies the valuative criterion for properness (see [SP21, Tag 03IX]). A morphism of rigid K -spaces is called *partially proper* if it is separated and universally specializing (see [Hub96, Definition 1.3.3]).

LEMMA 2.11. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of admissible formal schemes over \mathcal{O}_K , and assume that \mathfrak{X} is quasi-paracompact. If $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is partially proper, then so is $\mathfrak{Y}_k \rightarrow \mathfrak{X}_k$.*

Before proving the lemma, we recall some facts about taut spaces. A locally spectral topological space X is *taut* if it is quasi-separated and the closure of every quasi-compact open is quasi-compact [Hub96, Definition 5.1.2]. It is easy to check that a locally topologically Noetherian space is taut if and only if its irreducible components are quasi-compact. Moreover, if X is quasi-paracompact and quasi-separated, then X is taut by [Hub96, Lemma 5.1.3(ii)]. If $Y \rightarrow X$ is a partially proper morphism of rigid K -spaces and X is taut, then so is Y , by [Hub96, Lemma 5.1.4]. Finally, for an admissible formal scheme \mathfrak{Y} over \mathcal{O}_K , because the specialization map $\text{sp}_{\mathfrak{Y}}$ is closed, quasi-compact, and surjective, we deduce that \mathfrak{Y}_η is taut if and only if \mathfrak{Y}_k is taut.

LEMMA 2.12. *Let $f: Y \rightarrow X$ be a separated morphism of schemes locally of finite type over k where Y is taut. Then f is partially proper if and only if the maps*

$$f_n = f \times \text{id}_{\mathbf{A}_k^n}: Y \times \mathbf{A}_k^n \rightarrow X \times \mathbf{A}_k^n$$

are specializing for all $n \geq 0$.

Proof. The only if direction is clear, because a partially proper map is specializing and being partially proper is stable under base change. To prove the converse, we first claim that f is partially proper if and only if for all irreducible components Z of Y the map $Z \rightarrow X$ is proper.

⁵ Although we do not need the converse, we still expect this implication to hold.

As Z is quasi-compact, evidently if $Y \rightarrow X$ is partially proper, then $Z \rightarrow X$ is partially proper and, thus, proper (see [SP21, Tag 0BX5]). The converse is clear by thinking about the valuative criterion of properness for $Y \rightarrow X$ because for any valuation ring V with fraction field K , a map $\text{Spec}(K) \rightarrow Y$ must factorize through a unique irreducible component of Y .

From the above we see that it suffices to show that $Z \rightarrow X$ is proper for all irreducible components Z of Y . However, because $Z \rightarrow X$ is of finite type, by [SP21, Tag 0205] it suffices to show that $Z \times \mathbf{A}_k^n \rightarrow X \times \mathbf{A}_k^n$ is specializing for all $n \geq 0$. Since Z is a closed subscheme of Y , a moment's thought reveals that this is implied by the fact that the maps f_n are specializing for all $n \geq 0$. \square

Proof of Lemma 2.11. As f_η is separated, so is f_k , by [BL93, Proposition 4.7]. As \mathfrak{X}_η is quasi-paracompact and quasi-separated, it is taut, and hence so is \mathfrak{Y}_η and therefore also \mathfrak{Y}_k . By Lemma 2.12, it suffices to show that $\mathfrak{Y}_k \times \mathbf{A}_k^n \rightarrow \mathfrak{X}_k \times \mathbf{A}_k^n$ is specializing for all $n \geq 0$. Replacing $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ with $\mathfrak{Y} \times \widehat{\mathbf{A}}_{\mathcal{O}_K}^n \rightarrow \mathfrak{X} \times \widehat{\mathbf{A}}_{\mathcal{O}_K}^n$, we are reduced to showing that f_k itself is specializing. We have the following commutative square of topological spaces.

$$\begin{array}{ccc} \mathfrak{Y}_\eta & \xrightarrow{\text{sp}_\mathfrak{Y}} & \mathfrak{Y}_k \\ f_\eta \downarrow & & \downarrow f_k \\ \mathfrak{X}_\eta & \xrightarrow{\text{sp}_\mathfrak{X}} & \mathfrak{X}_k \end{array}$$

Here, the horizontal arrows are specializing (because they are closed) and surjective. If the left arrow is specializing, then so is the diagonal composition, and then it is straightforward to check that the right arrow must be specializing as well. \square

2.3 Tube open subsets

Let \mathfrak{X} be a formal scheme of finite type over \mathcal{O}_K and let $Z \subseteq |\mathfrak{X}|$ be a closed subset. We then define the *tube open subset* of \mathfrak{X}_η associated with Z , denoted $T(\mathfrak{X}|Z)$, to be the open subset $\text{sp}_\mathfrak{X}^{-1}(Z)^\circ$, i.e. the topological interior of $\text{sp}_\mathfrak{X}^{-1}(Z)$. With the notation $]Z[$, such opens were first considered by Berthelot in the context of rigid cohomology [Ber96b]. The following topological properties of tube open subsets of \mathfrak{X}_η are important for the proof of Theorem 3.7.

PROPOSITION 2.13. *Let notation be as before.*

- (i) *The tube open subset $T(\mathfrak{X}|Z)$ is an overconvergent open subset.*
- (ii) *The tube open subset $T(\mathfrak{X}|Z)$ contains every maximal point of $\text{sp}_\mathfrak{X}^{-1}(Z)$.*
- (iii) *For any cover $\{Z_i\}$ of $|\mathfrak{X}|$ by closed subsets, the set of corresponding tube open subsets $\{T(\mathfrak{X}|Z_i)\}$ forms an overconvergent open cover of \mathfrak{X}_η .*

Proof. The proof of the first statement follows by combining [FK18, Chapter 0, Proposition 2.3.15] and [FK18, Chapter II, Proposition 4.2.5]. The second statement follows by combining [FK18, Chapter II, Proposition 4.2.9] and [FK18, Chapter II, Proposition 4.2.10 (2)]. The final statement is clear because $\bigcup_i T(\mathfrak{X}|Z_i)$ is an overconvergent open subset of \mathfrak{X}_η by the first statement, but also clearly contains every maximal point by the second statement, and thus must be all of \mathfrak{X}_η as desired. \square

We now relate the tube open subset $T(\mathfrak{X}|Z)$ with the generic fiber $(\widehat{\mathfrak{X}}_Z)_\eta$ of the formal completion of \mathfrak{X} along Z .

PROPOSITION 2.14. *Let \mathfrak{X} be a formal scheme of finite type over \mathcal{O}_K and let $Z \subseteq |\mathfrak{X}|$ be a closed subset. The map of rigid K -spaces $(\widehat{\mathfrak{X}}_Z)_\eta \rightarrow \mathfrak{X}_\eta$ induced by $\widehat{\mathfrak{X}}_Z \rightarrow \mathfrak{X}$ is an open immersion with image $T(\mathfrak{X}|Z)$.*

Proof. Let us first verify that $(\widehat{\mathfrak{X}}_Z)_\eta \rightarrow \mathfrak{X}_\eta$ is an open immersion. By working on an affine open cover we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$ and that J is an admissible ideal of A such that $\mathrm{Spf}(A/J) = Z$. However, we know that $\widehat{\mathfrak{X}}_Z = \mathrm{Spf}(B)$ where B is the J -adic completion of A . One can then directly check that, in the notation of Construction 2.6 that each $(B(J^n), B(J^n)^+)$ is a rational localization of (A_K, A_K°) and, thus, we see from Lemma 2.7 that the map $\mathfrak{X}_\eta \rightarrow \mathrm{Spa}(A_K) = \mathrm{Spf}(A)_\eta$ is an open immersion.

Thus, to show our desired claim it suffices to show that for any quasi-compact and quasi-separated open subset U of \mathfrak{X}_η that the map $j: U \rightarrow \mathfrak{X}_\eta$ factorizes through $T(\mathfrak{X} | Z)$ if and only if it factorizes through $(\widehat{\mathfrak{X}}_Z)_\eta$. However, j factorizes through $T(\mathfrak{X} | Z)$ if and only if there exists a morphism of admissible formal schemes $j: \mathfrak{U} \rightarrow \mathfrak{X}$ such that $j = j_\eta$ and $j(|\mathfrak{U}|) \subseteq Z$. However, by combining Propositions 2.3 and 2.5 this is also precisely the condition for j to factorize through $(\widehat{\mathfrak{X}}_Z)_\eta$. \square

3. The specialization map

In this section we produce the specialization map from the de Jong fundamental group of the rigid generic fiber of a formal scheme to the pro-étale fundamental group of its special fiber.

3.1 Geometric coverings

We briefly recall the theory of the pro-étale fundamental group [BS15, § 7], which is based on the following notion of a covering space.

DEFINITION 3.1 (See [BS15, Definition 7.3.1(3)]). Let X be a locally topologically Noetherian scheme. A morphism of schemes $Y \rightarrow X$ is a *geometric covering* if it is étale and satisfies the valuative criterion of properness (see [SP21, Tag 03IX]) or equivalently if it is étale and partially proper (see [BS15, Remark 7.3.3]). We denote by \mathbf{Cov}_X the category of geometric coverings of X , treated as a full subcategory of $\mathbf{Ét}_X$.

The following lemma plays a pivotal role in the construction of the specialization map.

LEMMA 3.2 (See [BS15, Lemma 7.4.10] and its proof). *Let X be a locally topologically Noetherian scheme. If X is geometrically unibranch ([SP21, Tag 0BQ2], e.g. normal), then every geometric covering of X is the disjoint union of finite étale coverings of X .*

By [BS15, Lemma 7.4.1], for a connected and locally topologically Noetherian scheme X with a geometric point \bar{x} , the pair $(\mathbf{Cov}_X, F_{\bar{x}})$ where $F_{\bar{x}}: \mathbf{Cov}_X \rightarrow \mathbf{Set}$ is the fiber functor at \bar{x} , is a tame infinite Galois category (see [BS15, § 7.2]). Following [BS15, Definition 7.4.2] we denote the fundamental group of $(\mathbf{Cov}_X, F_{\bar{x}})$ by $\pi_1^{\mathrm{proét}}(X, \bar{x})$ and call it the *pro-étale fundamental group* of X . By definition, it is a Noohi topological group (see [BS15, § 7.1]) for which the induced functor

$$F_{\bar{x}}: \mathbf{Cov}_X \xrightarrow{\sim} \pi_1^{\mathrm{proét}}(X, \bar{x})\text{-Sets}$$

is an equivalence, where the target is the category of (discrete) sets with a continuous action of $\pi_1^{\mathrm{proét}}(X, \bar{x})$.

3.2 Coverings of rigid-analytic spaces

Next, we recall various notions of ‘covering spaces’ of a rigid space, as defined in [ALY21b]. Here, we only use de Jong covering spaces, whereas the more general adm-covering spaces and ét-covering spaces are needed in the next section.

Let X be a rigid K -space, and let $\tau \in \{\text{oc}, \text{adm}, \text{ét}\}$. By a τ -cover of X we shall mean a surjective morphism $U \rightarrow X$ such that:

- (i) if $\tau = \text{oc}$, then $U \rightarrow X$ is the disjoint union of overconvergent open immersions into X ;
- (ii) if $\tau = \text{adm}$, then $U \rightarrow X$ is the disjoint union of open immersions into X ;
- (iii) if $\tau = \text{ét}$, then $U \rightarrow X$ is étale.

DEFINITION 3.3. Let X be a rigid K -space, and let $\tau \in \{\text{oc}, \text{adm}, \text{ét}\}$. By a τ -covering space of X we mean a morphism $Y \rightarrow X$ for which there exists a τ -cover $U \rightarrow X$ such that $Y_U \rightarrow U$ is the disjoint union of finite étale coverings of U . We denote by \mathbf{Cov}_X^τ the full subcategory of $\mathbf{Ét}_X$ consisting of τ -covering spaces, and by \mathbf{UCov}_X^τ the full subcategory of $\mathbf{Ét}_X$ consisting of arbitrary disjoint unions of objects of \mathbf{Cov}_X^τ .

Remark 3.4. It is not difficult to see, and already pointed out by de Jong, that de Jong covering spaces are not closed under disjoint unions. Let us indicate an example for adm-covering spaces, using a variant of the construction in [ALY21b, §2] (see also Example 4.7). Let X be the annulus $\{1 \leq |x| \leq |\varpi|^{-1}\} \subseteq \mathbf{A}_K^{1,\text{an}}$ where K is of equal characteristic p . For $n \geq 1$, let U_n^- be the subannulus $\{1 \leq |x| \leq |\varpi|^{-1/n}\}$ and let $U_n^+ = \{|\varpi|^{-1/n} \leq |x| \leq |\varpi|^{-1}\}$. For each n , we perform a construction analogous to that of [ALY21b, §2] using the affinoid cover $X = U_n^- \cup U_n^+$, obtaining an adm-cover $Y_n \rightarrow X$ which is the disjoint union of finite étale coverings on U_n^\pm . Then $Y = \coprod_{n \geq 1} Y_n \rightarrow X$ is not an adm-covering space, as the type 5 point η_+ in the closure of the Gauss point η of the circle $C = \{|x| = 1\}$ does not have a required neighborhood. We leave the details to the reader.

When X is a taut rigid K -space (see [Hub96, Definition 5.1.2]), then $\mathbf{Cov}_X^{\text{oc}}$ is naturally equivalent (via the functor in [Hub96, §8.3]) to the category of étale covering spaces of the associated Berkovich space X^{Berk} considered in [dJ95b]. Thus, we refer to oc-covering spaces as *de Jong covering spaces*.

For a geometric point \bar{x} let $F_{\bar{x}}: \mathbf{Ét}_X \rightarrow \mathbf{Set}$ be the fiber functor $F_{\bar{x}}(Y) = \pi_0(Y_{\bar{x}})$. For a connected rigid K -space X and $\tau = \text{oc}$, de Jong essentially showed that the pair $(\mathbf{UCov}_X^{\text{oc}}, F_{\bar{x}})$ is a tame infinite Galois category (see [dJ95b, Theorem 2.9]). More generally, for a connected rigid K -space X one knows from [ALY21b, Theorem 2] that the pair $(\mathbf{UCov}_X^\tau, F_{\bar{x}})$ is a tame infinite Galois category for each $\tau \in \{\text{oc}, \text{adm}, \text{ét}\}$. In particular, we have an equivalence of categories

$$F_{\bar{x}}: \mathbf{UCov}_X^\tau \xrightarrow{\sim} \pi_1(\mathbf{UCov}_X^\tau, F_{\bar{x}})\text{-Sets}.$$

We denote the fundamental group of $(\mathbf{UCov}_X^{\text{oc}}, F_{\bar{x}})$ by $\pi_1^{\text{dJ}}(X, \bar{x})$ and call it the *de Jong fundamental group*.

One of the upshots of the material discussed and developed in §2 is the ability to prove the following criterion for when the rigid generic fiber of an étale map of formal schemes is a de Jong covering space.

PROPOSITION 3.5 (Overconvergence criterion). *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of admissible formal schemes over \mathcal{O}_K with \mathfrak{X} quasi-compact and such that $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is étale. Suppose that for each irreducible component Z of \mathfrak{X}_k one has that \mathfrak{Y}_Z is a disjoint union of finite Z -schemes. Then, $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is a de Jong covering space.*

Proof. Let Z_1, \dots, Z_r be the irreducible components of \mathfrak{X}_k and let $\widehat{\mathfrak{X}}_i$ for $i = 1, \dots, r$ be the formal completion of \mathfrak{X} along Z_i . As the pullback of $\mathfrak{Y}_{\widehat{\mathfrak{X}}_i} \rightarrow \widehat{\mathfrak{X}}_i$ to Z_i is the disjoint union of finite maps, we know by Proposition 2.4 that the same holds for $\mathfrak{Y}_{\widehat{\mathfrak{X}}_i} \rightarrow \widehat{\mathfrak{X}}_i$. We deduce from Propositions 2.10 and 2.14 that the pullback of \mathfrak{Y}_η to $T(\mathfrak{X} | Z_i)$ is the disjoint union of finite étale coverings (étale

by our assumption on $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ for all i . As $\{T(\mathfrak{X} | Z_i)\}$ is an overconvergent open cover of \mathfrak{X}_η by Proposition 2.13 we are done. \square

3.3 The étale site of a formal scheme

We now recall the topological invariance of the étale site of a formal scheme.

PROPOSITION 3.6. *Let $\mathfrak{X}' \rightarrow \mathfrak{X}$ be a universal homeomorphism of formal schemes. Then, the induced functor $\mathbf{Ét}_{\mathfrak{X}'} \rightarrow \mathbf{Ét}_{\mathfrak{X}}$ is an equivalence. In particular, the functor $\mathbf{Ét}_{\mathfrak{X}} \rightarrow \mathbf{Ét}_{\underline{\mathfrak{X}}}$ is an equivalence, and if \mathfrak{X} is an adic formal scheme over \mathcal{O}_K , then $\mathbf{Ét}_{\mathfrak{X}} \rightarrow \mathbf{Ét}_{\mathfrak{X}_k}$ is an equivalence.*

Proof. Note that because $\mathfrak{X}' \rightarrow \mathfrak{X}$ is a universal homeomorphism, so then is the morphism of schemes $\underline{\mathfrak{X}'} \rightarrow \underline{\mathfrak{X}}$. Thus, the induced morphism $\mathbf{Ét}_{\underline{\mathfrak{X}}} \rightarrow \mathbf{Ét}_{\underline{\mathfrak{X}'}}$ is an equivalence by [SP21, Tag 04DZ]. This clearly then reduces us to showing that, in general, the map $\mathbf{Ét}_{\mathfrak{X}} \rightarrow \mathbf{Ét}_{\underline{\mathfrak{X}}}$ is an equivalence. This assertion is clearly local on \mathfrak{X} and so we may assume without loss of generality that \mathfrak{X} is quasi-compact and quasi-separated and, hence, has an ideal sheaf of definition \mathcal{J} . As the map of schemes $\underline{\mathfrak{X}} \rightarrow \mathfrak{X}(\mathcal{J})$ induces an equivalence on étale sites by [SP21, Tag 04DZ] we are reduced to showing that the map $\mathbf{Ét}_{\mathfrak{X}} \rightarrow \mathbf{Ét}_{\mathfrak{X}(\mathcal{J})}$ is an equivalence. However, this follows by combining [SP21, Tag 04DZ] and [FK18, Chapter I, Proposition 1.4.2] because $\mathfrak{X} = \varinjlim \mathfrak{X}(\mathcal{J}^n)$ and each $\mathfrak{X}(\mathcal{J}^n) \rightarrow \mathfrak{X}(\mathcal{J}^{n+1})$ is a thickening of schemes. \square

3.4 Statement of the main result

Let \mathfrak{X} be a formal scheme locally of finite type over \mathcal{O}_K and let $X = \mathfrak{X}_\eta$ be its rigid generic fiber. We have the functors between the categories of étale objects

$$\mathbf{Ét}_X \xleftarrow{(-)_\eta} \mathbf{Ét}_{\mathfrak{X}} \xrightarrow{(-)_k} \mathbf{Ét}_{\mathfrak{X}_k}$$

where $(-)_\eta$ (respectively, $(-)_k$) is the functor which maps $\mathfrak{Y} \rightarrow \mathfrak{X}$ to $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta = X$ (respectively, $\mathfrak{Y}_k \rightarrow \mathfrak{X}_k$). The functor $(-)_k$ is an equivalence by Proposition 3.6. Let us denote by

$$u: \mathbf{Ét}_{\mathfrak{X}_k} \rightarrow \mathbf{Ét}_X$$

the composition $u = (-)_\eta \circ (-)_k^{-1}$. The main result of this section can then be stated as follows.

THEOREM 3.7. *Let \mathfrak{X} be a quasi-paracompact formal scheme locally of finite type over \mathcal{O}_K and let $X = \mathfrak{X}_\eta$. Then, the functor u maps $\mathbf{Cov}_{\mathfrak{X}_k}$ into $\mathbf{Cov}_X^{\text{oc}}$.*

Let us interpret this result in terms of fundamental groups. To this end, we first need to discuss compatible choices of base points for \mathfrak{X}_k and X . Let L be an algebraically closed non-archimedean field and let $\bar{\xi}: \text{Spf}(\mathcal{O}_L) \rightarrow \mathfrak{X}$ be a morphism of formal schemes, inducing geometric points $\bar{\xi}_\eta: \text{Spa}(L) \rightarrow X$ and $\bar{\xi}_k: \text{Spec}(\ell) \rightarrow \mathfrak{X}_k$, where $\ell = \mathcal{O}_L/\mathfrak{m}_L$ is the residue field of L . In this situation, if $\mathfrak{Y} \rightarrow \mathfrak{X}$ is an étale morphism, then the induced maps

$$\text{Hom}_X(\text{Spa}(L), \mathfrak{Y}_\eta) \leftarrow \text{Hom}_{\mathfrak{X}}(\text{Spf}(\mathcal{O}_L), \mathfrak{Y}) \rightarrow \text{Hom}_{\mathfrak{X}_k}(\text{Spec}(\ell), \mathfrak{Y}_k)$$

are bijective. To see this, we may replace \mathfrak{X} with $\text{Spf}(\mathcal{O}_L)$, and then it suffices to show that every étale morphism $\mathfrak{Y} \rightarrow \mathfrak{X}$ is a disjoint union of copies of \mathfrak{X} . This follows from Proposition 3.6 because $\underline{\mathfrak{X}} = \text{Spec}(\ell)$.

COROLLARY 3.8. *Let \mathfrak{X} be a quasi-paracompact formal scheme locally of finite type over \mathcal{O}_K whose special fiber \mathfrak{X}_k and rigid generic fiber $X = \mathfrak{X}_\eta$ are both connected. Then for a compatible choice of base points as previously discussed, the functor u induces a continuous homomorphism*

of Noohi groups

$$\pi_1^{\text{dJ}}(X, \bar{\xi}_\eta) \rightarrow \pi_1^{\text{proét}}(\mathfrak{X}_k, \bar{\xi}_0).$$

Its image is dense if \mathfrak{X} is η -normal (see Appendix A).

Proof. Given Theorem 3.7, the existence of the specialization map is immediate from [BS15, Theorem 7.2.5 2.], because by the previous discussion there is a natural identification of geometric fibers $F_{\bar{\xi}_\eta}^-(u(Y)) \simeq F_{\bar{\xi}_0}^-(Y)$.

The only part left to be justified is the claim of dense image. Note that by [Lar19, Proposition 2.37.(2)] and the proof of [ALY21a, Proposition 5.4.5] it suffices to show that if $Y \rightarrow \mathfrak{X}_k$ is a connected geometric covering and $\mathfrak{Y} \rightarrow \mathfrak{X}$ its unique étale deformation, then $\mathfrak{Y}_\eta = u(Y)$ is connected. However, because $\mathfrak{Y} \rightarrow \mathfrak{X}$ is étale, the formal scheme \mathfrak{Y} is η -normal by Proposition A.16, and we use Lemma A.3 to conclude. \square

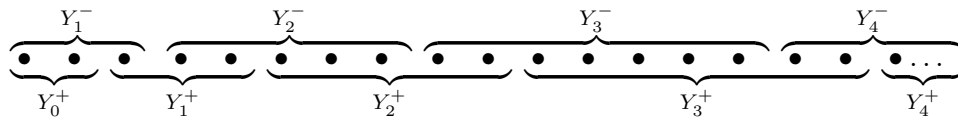
The natural converse to Theorem 3.7 is relatively straightforward given our setup, and we obtain the following (cf. [FK18, Chapter II, Theorem 7.5.17]).

COROLLARY 3.9. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be an étale map of admissible formal schemes over \mathcal{O}_K , with \mathfrak{X} quasi-paracompact. Then, the following are equivalent:*

- (a) $\mathfrak{Y}_k \rightarrow \mathfrak{X}_k$ is a geometric covering;
- (b) $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is a de Jong covering space;
- (c) $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is partially proper.

Proof. That part (a) implies part (b) is precisely the content of Theorem 3.7. To see that part (b) implies part (c) is simple because being partially proper is local on the target, and evidently a disjoint union of finite étale coverings is partially proper. That part (c) implies part (a) is then given by Lemma 2.11. \square

Remark 3.10. The following example shows that geometric coverings might not be the disjoint union of finite étale coverings étale locally on the target, illustrating the essential difficulty behind Theorem 3.7. Let k be an algebraically closed field, and let X be the nodal curve obtained by gluing two copies X^+, X^- of $\mathbf{G}_{m,k}$ at a closed point x . For $n \geq 0$, let $Y_n^\pm \rightarrow X^\pm$ be the connected cyclic covering of degree equal to the n th prime number. Let $Y^+ = \coprod_{n \geq 0} Y_n^+$ and $Y^- = \coprod_{n > 0} Y_n^-$, and let $Y \rightarrow X$ be the geometric covering of X with $Y|_{X^\pm} \simeq Y^\pm$ obtained by identifying the fibers of Y^\pm at x as in the following picture.



We sketch the proof that $Y \rightarrow X$ is not the disjoint union of finite étale coverings in any étale neighborhood of x . It suffices to observe that for a connected quasi-compact étale neighborhood $U^\pm \rightarrow X^\pm$ of x , the pull-backs $Y_n^\pm \times_{X^\pm} U^\pm \rightarrow U^\pm$ are connected for almost all n . However, U^\pm is an integral curve, and $Y_n^\pm \times_{X^\pm} U^\pm$ will be connected as long as the n th prime does not divide the degree of the field extension $k(U^\pm)/k(X^\pm)$.

3.5 The proof

The strategy of proof of Theorem 3.7 is to utilize the overconvergence criterion (Proposition 3.5). In particular, if all of the irreducible components of \mathfrak{X}_k are normal, then we are done by Lemma 3.2. Of course, this condition on the components of \mathfrak{X}_k will rarely be satisfied for a given formal scheme \mathfrak{X} .

To remedy this we perform a suitable admissible blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$ (Proposition 3.13). Namely, even though the normalization $n(\mathfrak{X}_k) \rightarrow \mathfrak{X}_k$ usually does not extend to a proper birational map to \mathfrak{X} , in most situations it is a blowup, and in any case it is dominated by some blowup $W \rightarrow \mathfrak{X}_k$. Then, blowing up the same center on \mathfrak{X} gives an admissible blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$ such that the strict transform of \mathfrak{X}_k factors through $n(\mathfrak{X}_k)$. The restriction of a given geometric covering of \mathfrak{X}_k to $W \subseteq \mathfrak{X}'_k$ is thus the disjoint union of finite étale maps. However, the irreducible components of \mathfrak{X}'_k belong to two groups: the strict transforms of the irreducible components of \mathfrak{X}_k , and those whose image in \mathfrak{X}_k has positive codimension. We handle the latter in a similar way, and the proof proceeds by induction.

LEMMA 3.11. *Let \mathfrak{X} be a quasi-compact admissible formal scheme over \mathcal{O}_K and let Z_1, \dots, Z_r be closed integral subschemes of \mathfrak{X}_k such that $Z_i \not\subseteq Z_j$ for $i \neq j$. Then there exists an admissible blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$ whose center does not contain any Z_i and such that the strict transforms $Z'_i \rightarrow Z_i$ in \mathfrak{X}' are birational and factor through the normalization of Z_i for each $i = 1, \dots, r$.*

Birationality of $Z'_i \rightarrow Z_i$ means that the blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$ is an isomorphism in a neighborhood of the generic point of Z_i .

Proof of Lemma 3.11. For every i , there is a non-zero quasi-coherent ideal sheaf $J_i \subseteq \mathcal{O}_{Z_i}$ whose blowup $Z''_i \rightarrow Z_i$ factors through the normalization of Z_i . Indeed, if $U \subseteq Z_i$ is the normal locus [Gro65, Scholie 7.8.3(iv)], then the normalization is a proper map which is an isomorphism over U and, hence, is dominated by a U -admissible blowup (see [SP21, Tag 080K]) by [RG71, Première partie, Corollaire 5.7.12] (see also [SP21, Tag 081T]). We also denote by $J_i \subseteq \mathcal{O}_{\mathfrak{X}_k}$ the induced quasi-coherent ideal sheaf. By Lemma 3.12 there exist admissible ideal sheaves $\mathcal{J}_i \subseteq \mathcal{O}_{\mathfrak{X}}$ with $\mathcal{J}_i \cdot \mathcal{O}_{\mathfrak{X}_k} = J_i$. If $\mathfrak{X}'_i \rightarrow \mathfrak{X}$ is the admissible blowup of \mathcal{J}_i , then Z''_i is the strict transform of Z_i in \mathfrak{X}'_i .

Let $\mathcal{J} = \prod_{i=1}^r \mathcal{J}_i$ which is again an admissible ideal sheaf by [FK18, Chapter I, Proposition 3.7.9], and let $\mathfrak{X}' \rightarrow \mathfrak{X}$ be the corresponding admissible blowup. As the generic point of Z_i does not belong to any other Z_j , the map $\mathfrak{X}' \rightarrow \mathfrak{X}$ is an isomorphism generically on Z_i and, in particular, the blowup center $V(\mathcal{J})$ does not contain any Z_i . If Z'_i is the strict transform of Z_i in \mathfrak{X}' , then $Z'_i \rightarrow Z_i$ factors through $Z''_i \rightarrow Z_i$ and, hence, through the normalization of Z_i . \square

LEMMA 3.12. *Let \mathfrak{X} be a formal scheme of finite type over \mathcal{O}_K and let $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}_k}$ be a coherent ideal sheaf. Then there exists an admissible ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{X}}$ such that $\mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}_k} = \mathcal{J}$.*

Proof. We start the proof with an observation. Suppose that $\mathfrak{X} = \text{Spf}(A)$ and let $J = (a_1, \dots, a_r) \subseteq A_k$ be the ideal corresponding to \mathcal{J} . If $a'_i \in A$ are liftings of the a_i and $I' = (a'_1, \dots, a'_r, \varpi) \subseteq A$, then I' is an admissible ideal and $I'A_k = J$, so in particular the result holds if \mathfrak{X} is affine. Moreover, if $I' = (a'_1, \dots, a'_r, \varpi)$ and $I'' = (b'_1, \dots, b'_s, \varpi)$ are two admissible ideals of A with $I'A_k = J = I''A_k$, then we can write $b'_i = \sum_j \alpha_{ij} a'_j + \varpi_i \gamma_i$ with $\alpha_{ij}, \gamma_i \in A$ and $\varpi_i \in \mathfrak{m}_K$. It follows that if $(\varpi, \varpi_1, \dots, \varpi_s)\mathcal{O}_K = (\varpi')$, then $I'' \subseteq I' + (\varpi')$. Swapping the roles of I' and I'' , we obtain $I' \subseteq I'' + (\varpi'')$ for some pseudo-uniformizer ϖ'' . Taking $(\omega) = (\varpi', \varpi'')$, we see that $I' + (\omega) = I'' + (\omega)$ for a suitably chosen pseudo-uniformizer ω .

For the general case, cover \mathfrak{X} by finitely many affine opens $\mathfrak{U}_i = \text{Spf}(A_i)$ and cover the pairwise intersections $\mathfrak{U}_i \cap \mathfrak{U}_{i'}$ by finitely many affine opens $\mathfrak{U}_{ii'j} = \text{Spf}(A_{ii'j})$. By the previous paragraph, we can find admissible ideals $I'_i \subseteq A_i$ such that $I'_i(A_i)_k$ is the ideal corresponding to $\mathcal{J}|_{(\mathfrak{U}_i)_k}$. Moreover, for each triple (i, i', j) there exists a pseudo-uniformizer $\omega_{ii'j}$ such that $I'_i + (\omega_{ii'j})$ and $I'_{i'} + (\omega_{ii'j})$ generate the same ideal in $A_{ii'j}$. If $(\omega) = (\omega_{ii'j})_{(i,i',j)}$, then the admissible ideals $I_i = I'_i + (\omega) \subseteq A_i$ glue to give the desired admissible ideal sheaf \mathcal{J} . \square

Next, we show that we may iteratively apply Lemma 3.11 to produce an admissible blowup \mathfrak{X}' of \mathfrak{X} whose irreducible components factor through the normalizations of their images in \mathfrak{X} .

PROPOSITION 3.13. *Let \mathfrak{X} be a quasi-compact admissible formal scheme over \mathcal{O}_K . Then there exists an admissible blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$ such that for every irreducible component Z' of \mathfrak{X}'_k with image $Z \subseteq \mathfrak{X}_k$, the map $Z' \rightarrow Z$ factors through the normalization of Z .*

Proof. We prove by induction on $n \geq 0$ the following statement: there exists an admissible blowup $\mathfrak{X}'_n \rightarrow \mathfrak{X}$ such that the assertion of the proposition holds for all Z' such that $\text{codim } Z \leq n$. Here codimension means the Krull dimension of the local ring $\mathcal{O}_{\mathfrak{X}_k, \eta_Z}$ where η_Z is the generic point of Z . This claim for $n = \dim \mathfrak{X}_k$ will then imply the required assertion.

Starting with $n = 0$, let Z_1, \dots, Z_r be the irreducible components of \mathfrak{X}_k . Clearly Z_i is not contained in Z_j for $i \neq j$. Let $\mathfrak{X}'_0 \rightarrow \mathfrak{X}$ be the admissible blowup given by Lemma 3.11. This blowup is birational at the generic points of the Z_i , and hence every irreducible component of $(\mathfrak{X}'_0)_k$ which is not the strict transform of some Z_i has image of codimension at least one. The claim is thus proved for $n = 0$.

For the induction step from n to $n + 1$, suppose that we have constructed $\mathfrak{X}'_n \rightarrow \mathfrak{X}$ as above. Let Z'_1, \dots, Z'_r be the irreducible components of $(\mathfrak{X}'_n)_k$ whose images Z_1, \dots, Z_r in \mathfrak{X}_k have codimension exactly $n + 1$. Again, Z_i does not properly contain another Z_j (we could have $Z_i = Z_j$ for some $i \neq j$). Apply Lemma 3.11 to obtain an admissible blowup $\mathfrak{X}''_n \rightarrow \mathfrak{X}$ relative to an admissible ideal sheaf \mathcal{J} with $\text{codim } V(\mathcal{J}) > n + 1$ where $J = \mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}_k}$. If \mathfrak{X}'_n is the blowup relative to an admissible ideal sheaf \mathcal{J} and \mathfrak{X}''_n , let us take $\mathfrak{X}'_{n+1} \rightarrow \mathfrak{X}$ to be the blowup relative to $\mathcal{J} \cdot \mathcal{J}$, which is also the blowup of \mathfrak{X}'_n relative to $\mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}'_n}$.

We check that $\mathfrak{X}'_{n+1} \rightarrow \mathfrak{X}$ satisfies the assertion for codimension $n + 1$. Let Z'' be an irreducible component of $(\mathfrak{X}'_{n+1})_k$, and let Z' and Z be its images in $(\mathfrak{X}'_n)_k$ and \mathfrak{X}_k , respectively. There are four cases to consider:

- (1) Z' is an irreducible component of $(\mathfrak{X}'_n)_k$ and
 - (a) $\text{codim } Z \leq n$,
 - (b) $\text{codim } Z = n + 1$,
 - (c) $\text{codim } Z > n + 1$; or
- (2) Z' is not an irreducible component of $(\mathfrak{X}'_n)_k$.

In case (1a) we are done by the assumed property of \mathfrak{X}'_n , and in case (1c) there is nothing to show. In case (1b), we have $Z' = Z'_i$ for some i . As the generic point of $Z = Z_i$ is not in $V(\mathcal{J})$, the generic point of Z' is not in $V(\mathcal{J} \cdot \mathcal{O}_{(\mathfrak{X}'_n)_k})$. As $\mathfrak{X}'_{n+1} \rightarrow \mathfrak{X}'_n$ is the blowup at $\mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}'_n}$, it is an isomorphism in a neighborhood of the generic point of Z' , and we conclude that Z'' is the strict transform of Z' in \mathfrak{X}'_{n+1} . That is, $Z'' = \text{Bl}_{\mathcal{J} \cdot \mathcal{O}_{Z'}}(Z')$, and so by functoriality it maps to the strict transform $\text{Bl}_{\mathcal{J} \cdot \mathcal{O}_Z}(Z)$ of Z in \mathfrak{X}''_n . By construction, the latter factors through the normalization of Z , and we are done. See the following diagram.

$$\begin{array}{ccc}
 \mathfrak{X}'_{n+1} & \xrightarrow{\mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}'_n}} & \mathfrak{X}'_n \\
 \mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}''_n} \downarrow & & \downarrow \mathcal{J} \\
 \mathfrak{X}''_n & \xrightarrow{\mathcal{J}} & \mathfrak{X}
 \end{array}
 \qquad
 \begin{array}{ccc}
 Z'' = \text{Bl}_{\mathcal{J} \cdot \mathcal{O}_{Z'}}(Z') & \longrightarrow & Z' = Z'_i \\
 \downarrow & & \downarrow \\
 \text{Bl}_{\mathcal{J} \cdot \mathcal{O}_Z}(Z) & \longrightarrow & n(Z) \longrightarrow Z = Z_i
 \end{array}$$

Finally, in case (2), Z'' is contained in the exceptional locus of $\mathfrak{X}'_{n+1} \rightarrow \mathfrak{X}'_n$, which is contained in the preimage of $V(\mathcal{J})$ in \mathfrak{X}'_n . Thus $Z \subseteq V(\mathcal{J})$ and, hence, $\text{codim } Z \geq \text{codim } V(\mathcal{J}) > n + 1$. □

We are now ready to put the pieces together and present the proof of Theorem 3.7.

Proof of Theorem 3.7. We first reduce to the case \mathfrak{X} is of finite type. As \mathfrak{X} is quasi-paracompact, its rigid generic fiber $X = \mathfrak{X}_\eta$ is quasi-paracompact and quasi-separated and, hence, taut by [FK18, Chapter 0, Proposition 2.5.15]. Therefore, for each point $x \in X$ there exists a quasi-compact open $U_x \in X$ which is an overconvergent neighborhood of x , i.e. it contains an overconvergent open containing x (see [ALY21a, Proposition 2.3.5]). It then suffices to show that for any geometric covering $Y \rightarrow \mathfrak{X}_k$ that $u(Y)$ is a de Jong covering space when restricted to U_x . As U_x is quasi-compact, by (the proof of) [Bos14, Lemma 8.4/5] there exists an admissible blowup $\mathfrak{X}' \rightarrow \mathfrak{X}$ such that $U_x = \mathfrak{U}'_\eta$ for an open formal subscheme $\mathfrak{U}' \subseteq \mathfrak{X}'$. If the result holds for \mathfrak{U}' (which is of finite type) and the base change $Y' = Y \times_{\mathfrak{X}_k} \mathfrak{U}'_k$ of the given geometric covering $Y \rightarrow \mathfrak{X}_k$, we deduce that $\mathfrak{Y}'_\eta \rightarrow \mathfrak{U}'_\eta = U_x$ is a de Jong covering, where $\mathfrak{Y}' \rightarrow \mathfrak{U}'$ is the unique étale lifting of $Y' \rightarrow \mathfrak{U}'_k$. However, $\mathfrak{Y}'_\eta = u(Y) \times_X U_x$, and we conclude.

Suppose now that \mathfrak{X} is of finite type. Blowing up an ideal sheaf of definition, we may assume that \mathfrak{X} is admissible. Proposition 3.13 yields an admissible blowup $f: \mathfrak{X}' \rightarrow \mathfrak{X}$ such that for every irreducible component Z' of \mathfrak{X}'_k the induced map $Z' \rightarrow \mathfrak{X}'_k \rightarrow \mathfrak{X}_k$ factors through a normal scheme Z of finite type over k . By Lemma 3.1 we know that $Y_{Z'}$ is the disjoint union of finite étale coverings of Z' . As f_η is an isomorphism, we may identify $X' = \mathfrak{X}'_\eta$ with X and we have the following commutative diagram.

$$\begin{array}{ccc} \mathbf{Et}_{X'} & \xleftarrow{u'} & \mathbf{Et}_{\mathfrak{X}'_k} \\ \parallel & & \uparrow f^* \\ \mathbf{Et}_X & \xleftarrow{u} & \mathbf{Et}_{\mathfrak{X}_k} \end{array}$$

By Proposition 3.5, the functor u' maps the image of $f^*: \mathbf{Cov}_{\mathfrak{X}_k} \rightarrow \mathbf{Cov}_{\mathfrak{X}'_k}$ into $\mathbf{Cov}_{X'}^{\text{oc}} = \mathbf{Cov}_X^{\text{oc}}$ and, hence, u sends $\mathbf{Cov}_{\mathfrak{X}_k}$ into $\mathbf{Cov}_X^{\text{oc}}$. □

Remark 3.14. The authors suspect that the above results hold in a greater generality. Namely, let \mathfrak{X} be a universally rigid-Noetherian formal scheme (in the sense of [FK18, Chapter I, Definition 2.1.7]) such that $\underline{\mathfrak{X}}$ is topologically Noetherian and universally Japanese (see [SP21, Tag 032R]). There is then still a functor $\mathbf{Ét}_{\underline{\mathfrak{X}}}$ to $\mathbf{Ét}_{\mathfrak{X}_{\text{rig}}}$ (where $\mathfrak{X}_{\text{rig}}$ is as in [FK18, Chapter II]), and the same methods should show that $\mathbf{Cov}_{\underline{\mathfrak{X}}}$ is mapped into $\mathbf{Cov}_{\mathfrak{X}_{\text{rig}}}^{\text{oc}}$. However, an interpretation of such a result in terms of fundamental groups is clearly missing, because we do not know if for $\mathfrak{X}_{\text{rig}}$ connected the pair $(\mathbf{Cov}_{\mathfrak{X}_{\text{rig}}}^{\text{oc}}, F_{\overline{\mathfrak{X}}})$ is a tame infinite Galois category in this generality.

4. Tame coverings

In this section, we assume that the base non-archimedean field K is discretely valued and we fix a uniformizer ϖ . We aim to show that, under certain smoothness hypotheses, the notion of tame (see Definition 4.11) ét-covering spaces, adm-covering spaces, and de Jong coverings of a rigid K -space all coincide (Theorem 4.14). The method of proof mirrors that of Theorem 3.7. Namely we formulate a simple topological lemma (see Proposition 4.5) which will allow us to apply the overconvergence criterion of Proposition 3.5. We then employ the link between tameness and logarithmic geometry (the logarithmic Abhyankar’s lemma, Theorem 4.8) to verify that our topological lemma applies to suitable models of tame covering spaces.

4.1 Preliminaries about finite maps and formal models

The following lemma shows that, as in algebraic geometry (see [SP21, Tag 0BAK]), given a formal scheme \mathfrak{X} and a finite morphism of rigid K -spaces $Y \rightarrow \mathfrak{X}_\eta$, one can ‘normalize \mathfrak{X} in Y ’,

producing a formal model $\mathfrak{Y} \rightarrow \mathfrak{X}$ which is again a finite morphism. There is a unique such map with the property that \mathfrak{Y} is η -normal (see Appendix A for the definition and basic properties of η -normality).

LEMMA 4.1. *Let \mathfrak{X} be an admissible formal scheme over \mathcal{O}_K , let $X = \mathfrak{X}_\eta$, and let $f: Y \rightarrow X$ be a finite morphism with Y reduced. Then there exists a unique up to unique isomorphism finite morphism of formal schemes $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ with $f = f_\eta$ and \mathfrak{Y} η -normal.*

Proof. Suppose that $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ is a finite map with $f_\eta = f$ and \mathfrak{Y} η -normal. On the one hand, $\mathrm{sp}_{\mathfrak{Y},*} \mathcal{O}_Y^+ = \mathcal{O}_{\mathfrak{Y}}$ because \mathfrak{Y} is η -normal. On the other hand, \mathfrak{Y} is the relative formal spectrum [FK18, Chapter I, Proposition 4.2.6] of $f_* \mathcal{O}_{\mathfrak{Y}} = f_*(\mathrm{sp}_{\mathfrak{Y},*} \mathcal{O}_Y^+) = \mathrm{sp}_{\mathfrak{X},*}(f_* \mathcal{O}_Y^+)$. This implies the uniqueness, and that to show existence we need to check that $\mathrm{sp}_{\mathfrak{X},*}(f_* \mathcal{O}_Y^+)$ is an adically quasi-coherent and finitely generated $\mathcal{O}_{\mathfrak{X}}$ -algebra. For an affinoid open $\mathrm{Spf}(A)$ in X , write $Y \times_X \mathrm{Spa}(A_K) = \mathrm{Spa}(B)$ for a finite and reduced A_K -algebra B . The value of the sheaf in question on $\mathrm{Spf}(A)$ is B° . Choose a surjection $\mathcal{O}_K \langle x_1, \dots, x_n \rangle \rightarrow A$. As K is discretely valued, by [BGR84, Corollary 6.4.1/6] applied to $K \langle x_1, \dots, x_n \rangle \rightarrow A_K$, the algebra A_K° is finite over A , and by the same result applied to $A_K \rightarrow B$, the algebra B° is finite over A_K° and, hence, also over A . Moreover, the sheaf $\mathrm{sp}_{\mathfrak{X},*}(f_* \mathcal{O}_Y^+)$ is adically quasi-coherent because for a in A the previous discussion, and the sheaf property for \mathcal{O}_Y^+ , shows that $B^\circ \otimes_A A \langle a^{-1} \rangle$ is precisely $B^\circ \langle a^{-1} \rangle^\circ$. \square

The following corollary extends the above result to the case of maps which are locally on the target the disjoint union of finite maps.

COROLLARY 4.2. *Let \mathfrak{X} be an admissible formal scheme over \mathcal{O}_K , let $X = \mathfrak{X}_\eta$, and let $f: Y \rightarrow X$ be a morphism with Y reduced for which there exists an open cover $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{U}_i$ such that $Y \times_X \mathfrak{U}_{i,\eta}$ is a disjoint union of finite spaces over $\mathfrak{U}_{i,\eta}$. Then there exists a unique up to unique isomorphism morphism of formal schemes $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ with $f = f_\eta$ and \mathfrak{Y} η -normal which locally on \mathfrak{X} is the disjoint union of finite formal schemes over \mathfrak{X} .*

Proof. Suppose first that $Y = \coprod_{j \in J} Y_j$ with each Y_j finite over X . In this case, we can apply Lemma 4.1 to each Y_j and take $\mathfrak{Y} = \coprod_{j \in J} \mathfrak{Y}_j$, which clearly satisfies the required property. In particular, this construction does not depend on the way Y is presented as a disjoint union of finite X -spaces, and is compatible with passing to an open formal subscheme of \mathfrak{X} . This implies that in the general case, the formal schemes $\mathfrak{Y}_i \rightarrow \mathfrak{U}_i$ constructed in the first step will glue to provide the desired map $\mathfrak{Y} \rightarrow \mathfrak{X}$. \square

We call the model $\mathfrak{Y} \rightarrow \mathfrak{X}$ constructed in Corollary 4.2 the *normalization* of \mathfrak{X} in Y .

Remark 4.3. The assertions of Lemma 4.1 and Corollary 4.2 remain valid if K is stable [BGR84, Definition 3.6.1/1] and with divisible value group (e.g. if K is algebraically closed). Indeed, under these assumptions the conclusions of [BGR84, Corollary 6.4.1/6] hold (see [BGR84, Corollary 6.4.1/5]).

The following lemma will help us construct formal models in §4.4.

LEMMA 4.4. *Let X be a quasi-paracompact and quasi-separated rigid space over K , and let $X = \bigcup_{i \in I} U_i$ be an open cover. Then there exists an admissible formal model \mathfrak{X} of X and an open cover $\mathfrak{X} = \bigcup_{j \in J} \mathfrak{Y}_j$ such that $\{\mathfrak{Y}_{j,\eta}\}_{j \in J}$ refines $\{U_i\}_{i \in I}$.*

Proof. Refining the given cover, we may assume that the U_i are quasi-compact. By assumption, there is an open cover $X = \bigcup_{s \in S} X_s$ of finite type (i.e. each member intersects only finitely many others) by quasi-compact opens $X_s \subseteq X$. As X_s are quasi-compact, for each s there exists a finite

subset $F_s \subset I$ such that $X_s \subset \bigcup_{i \in F_s} U_i$. Now, the open cover $\bigcup_{s \in S} \bigcup_{i \in F_s} (U_i \cap X_s)$ is a cover of finite type. Moreover, its members are quasi-compact, as X was quasi-separated. By [Bos14, 8.4/5], there exists a formal model \mathfrak{X} of X and an open cover $\mathfrak{X} = \bigcup_{s \in S} \bigcup_{i \in F_s} \mathfrak{Y}_{i,s}$ such that $(\mathfrak{Y}_{i,s})_\eta = U_i \cap X_s$. \square

4.2 A special case of the overconvergence criterion

In order to apply our overconvergence criterion (Proposition 3.5) to tame covering spaces, we show the following result.

PROPOSITION 4.5. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of formal schemes locally of finite type over \mathcal{O}_K which locally on \mathfrak{X} is the disjoint union of finite morphisms. Suppose that for every irreducible component Z of \mathfrak{X}_k , the base change $\mathfrak{Y} \times_{\mathfrak{X}} Z$ is locally irreducible⁶ (e.g. $(\mathfrak{Y} \times_{\mathfrak{X}} Z)_{\text{red}}$ is normal). Then for all such Z , the scheme $\mathfrak{Y} \times_{\mathfrak{X}} Z$ is a disjoint union of finite Z -spaces.*

Proof. Let T be a connected component of $\mathfrak{Y} \times_{\mathfrak{X}} Z$. By assumption, T is locally irreducible, and hence irreducible. We show that $T \rightarrow Z$ is a finite morphism, which implies the assertion. If $Z = \bigcup U_i$ is an open cover such that $\mathfrak{Y} \times_{\mathfrak{X}} U_i$ is the disjoint union of finite maps $V_{ij} \rightarrow U_i$ with V_{ij} connected, then $T \times_Z U_i$ is the disjoint union of some of the V_{ij} . However, because T is irreducible, so is the open subset $T \times_Z U_i \subseteq T$. Therefore, we must have $T \times_Z U_i = V_{ij}$ for some index j . Thus, $T \times_Z U_i \rightarrow U_i$ is finite for all i and, hence, $T \rightarrow Z$ is finite. \square

Using Proposition 3.5 we obtain the following corollary.

COROLLARY 4.6. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of formal schemes satisfying the assumptions in Proposition 4.5 and such that $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is étale. Then, $\mathfrak{Y}_\eta \rightarrow \mathfrak{X}_\eta$ is a de Jong covering space.*

Example 4.7. In [ALY21b, § 2], we constructed over K of equal characteristic $p > 0$ an example of an adm-covering space $Y \rightarrow X$ which is not a de Jong covering space, and it is instructive to see why the above criterion does not apply to it. We recall the setup: X is the annulus $\{|\varpi| \leq |x| \leq |\varpi|^{-1}\}$ covered by the two annuli

$$U^- = \{|\varpi| \leq |x| \leq 1\} \quad \text{and} \quad U^+ = \{1 \leq |x| \leq |\varpi|^{-1}\},$$

intersecting along the unit circle $U^+ \cap U^- = C = \{|x| = 1\}$. The restriction of $Y \rightarrow X$ to U^\pm is a disjoint union of well-chosen Artin–Schreier coverings Y_n^\pm ($n \in \mathbf{Z}$) which are split over C , and $\coprod_n Y_n^-$ and $\coprod_n Y_n^+$ are identified suitably over C in order to make Y connected.

Construct a formal model \mathfrak{X} of X by gluing together the formal models

$$\mathfrak{U}_+ = \text{Spf}(\mathcal{O}_K\langle x^+, y^+ \rangle / (x^+ y^+ - \varpi)) \quad \text{and} \quad \mathfrak{U}_- = \text{Spf}(\mathcal{O}_K\langle x^-, y^- \rangle / (x^- y^- - \varpi))$$

of U^+ and U^- along $\mathfrak{C} = \text{Spf}(\mathcal{O}_K\langle x^+, x^- \rangle / (x^+ x^- - 1))$ identified with $D(x^\pm) \subseteq \mathfrak{U}^\pm$. (In terms of the coordinate x , we have $x^+ = 1/x$, $x^- = x$, $y^+ = \varpi x$, $y^- = \varpi/x$.) The irreducible component Z of \mathfrak{X}_k containing $\mathfrak{C}_k \simeq \mathbf{G}_{m,k}$ is isomorphic to \mathbf{P}_k^1 . The restriction of the normalization \mathfrak{Y} of \mathfrak{X} in $Y \rightarrow X$ to Z will look like Figure 1: it will consist of infinitely many copies of Z glued together in a complicated way over $0, \infty \in Z$, and will not be locally irreducible.

4.3 Logarithmic Abhyankar’s lemma

The following result describes the local structure of a finite covering of a regular scheme tamely ramified along a strict normal crossings divisor [SP21, Tag 0CBN]. It is a basic consequence of the logarithmic Abhyankar’s lemma.

⁶ A topological space is locally irreducible if every point has an irreducible open neighborhood (equivalently, a basis of irreducible open neighborhoods) or, equivalently, if it is the disjoint union of irreducible spaces.

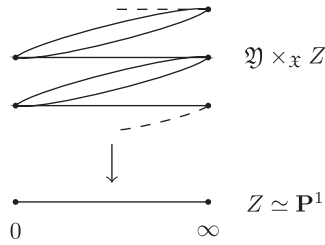


FIGURE 1. Example 4.7 ($p = 3$), compare with [ALY21b, Figure 1].

In order to state the result, we first recall some basic terminology from log geometry. For a commutative monoid P (written additively), we write $P \rightarrow P^{\text{gp}}$ for the initial homomorphism into a group. The monoid P is called *integral* if $P \rightarrow P^{\text{gp}}$ is injective, *fine* if it is finitely generated and integral, and *saturated* if it is integral and for $x \in P^{\text{gp}}$ such that $nx \in P$ for some $n \geq 1$, we have $x \in P$. Finally, a morphism of fine and saturated monoids $\alpha: P \rightarrow Q$ is called *Kummer* (see [III02, § 1.6]) if it is injective and for every $x \in Q$ there exists an $n \geq 1$ such that $nx \in \alpha(P)$. Equivalently, $P = Q \cap \Lambda$ for a subgroup $\Lambda \subseteq Q^{\text{gp}}$ of finite index.

Recall that if D is a prime divisor on a scheme X with generic point ξ such that $\mathcal{O}_{X,\xi}$ is a discrete valuation ring, we say that an étale morphism $V \rightarrow X$ is *tamely ramified along D* if $V \otimes \text{Frac}(\mathcal{O}_{X,\xi}^h)$ is the disjoint union of spectra of finite extensions of $\text{Frac}(\mathcal{O}_{X,\xi}^h)$ which are tame with respect to the given discrete valuation.

THEOREM 4.8 (See [Moc99, Theorem 3.3] and [GR04, Theorem 13.3.43]). *Let R be a strictly Henselian regular local ring, and let $x_1, \dots, x_r \in \mathfrak{m}_R$ be elements whose images in $\mathfrak{m}_R/\mathfrak{m}_R^2$ are linearly independent. Let $X = \text{Spec}(R)$, $D_i = V(x_i) \subseteq X$ ($i = 1, \dots, r$), and $U = D(x_1 \cdots x_r) = X \setminus (D_1 \cup \dots \cup D_r)$. Let $V \rightarrow U$ be a connected finite étale covering which is tamely ramified along D_1, \dots, D_r , and let $f: Y \rightarrow X$ be the normalization of X in V . Then there exists a Kummer homomorphism $\alpha: \mathbf{N}^r \rightarrow Q$ with $Q^{\text{gp}}/\mathbf{Z}^r$ of order invertible in R and an isomorphism of X -schemes*

$$Y \simeq \text{Spec}(R \otimes_{\mathbf{Z}[\mathbf{N}^r]} \mathbf{Z}[Q]),$$

where $\mathbf{Z}[\mathbf{N}^r] \rightarrow R$ sends e_i to x_i .

We need the following lemma, which follows from basic facts about log regularity [Kat94, Theorems 4.1 and 8.2].

LEMMA 4.9. *Let R be a regular local ring, and let $x_1, \dots, x_r \in \mathfrak{m}_R$ be elements whose images in $\mathfrak{m}_R/\mathfrak{m}_R^2$ are linearly independent. Let $\alpha: \mathbf{N}^r \rightarrow Q$ be a Kummer homomorphism of fine and saturated monoids with $Q^{\text{gp}}/\mathbf{Z}^r$ of order invertible in R and let $\mathbf{N}^r \rightarrow R$ send e_i to x_i . Then the ring $R \otimes_{\mathbf{Z}[\mathbf{N}^r]} \mathbf{Z}[Q]$ is normal.*

The following corollary is the crux of the argument in this section.

COROLLARY 4.10. *Let X be a regular scheme and let $D = D_1 \cup \dots \cup D_r$ be a divisor with strict normal crossings. Let $U = X \setminus D$ and let $V \rightarrow U$ be a finite étale morphism which is tamely ramified along D_1, \dots, D_r . Let $Y \rightarrow X$ be the normalization of X in V . Then the schemes $(Y \times_X D_i)_{\text{red}}$ are normal for $i = 1, \dots, r$.*

Proof. Using standard arguments one quickly reduces to the case when X is strictly local. Write $X = \text{Spec}(R)$ and $D_i = V(x_i)$ for some $x_i \in R$. Let $Y \simeq \text{Spec}(R \otimes_{\mathbf{Z}[\mathbf{N}^r]} \mathbf{Z}[Q])$ be

as in Theorem 4.8. Our goal is to show that the ring

$$((R \otimes_{\mathbf{Z}[\mathbf{N}^r]} \mathbf{Z}[Q]) \otimes_R R/(x_i))_{\text{red}}$$

is normal for $i = 1, \dots, r$. For notational convenience, we prove this for $i = r$. The ring $R' = R/(x_r)$ is again a regular local ring, and x_1, \dots, x_{r-1} are elements of the maximal ideal which are linearly independent in $\mathfrak{m}_{R'}/\mathfrak{m}_{R'}^2$. We claim that there exists a Kummer homomorphism of fine and saturated monoids $\alpha': \mathbf{N}^{r-1} \rightarrow Q'$ with $(Q')^{\text{gp}}/\mathbf{Z}^{r-1}$ of order invertible in R' such that

$$((R \otimes_{\mathbf{Z}[\mathbf{N}^r]} \mathbf{Z}[Q]) \otimes_R R')_{\text{red}} \simeq R' \otimes_{\mathbf{Z}[\mathbf{N}^{r-1}]} \mathbf{Z}[Q'].$$

The right-hand side of this isomorphism is normal by Lemma 4.9, so the claim implies the required assertion. Note that the left-hand side can also be written as $(R \otimes_{\mathbf{Z}[\mathbf{N}^r]/(\alpha(e_r))} \mathbf{Z}[Q]/(\alpha(e_r)))_{\text{red}}$.

Let us first compute $(\text{Spec}(\mathbf{Z}[Q]/(\alpha(e_r))))_{\text{red}}$. To this end, let $\mathbf{N}^{r-1} \subseteq \mathbf{N}^r$ be the inclusion omitting the r th factor, let $Q' \subseteq Q$ consist of elements whose positive multiple lies in $\alpha(\mathbf{N}^{r-1})$, and let $\alpha': \mathbf{N}^{r-1} \rightarrow Q'$ be the induced map. Every element of $Q \setminus Q'$ has a multiple in the ideal of $\mathbf{Z}[Q]$ generated by $\alpha(e_r)$. Consequently, the ideal J generated by $Q \setminus Q'$ in $\mathbf{Z}[Q]/(\alpha(e_r))$ is nilpotent. As $\mathbf{Z}[Q]/(\alpha(e_r), J) = \mathbf{Z}[Q]/(J) \simeq \mathbf{Z}[Q']$ is reduced, we have $(\text{Spec}(\mathbf{Z}[Q]/(\alpha(e_r))))_{\text{red}} \simeq \mathbf{Z}[Q']$. Note that $\alpha': \mathbf{N}^{r-1} \rightarrow Q'$ is a Kummer homomorphism of fine and saturated monoids, and the order of $(Q')^{\text{gp}}/\mathbf{Z}^{r-1} \subseteq Q^{\text{gp}}/\mathbf{Z}^r$ is invertible in R .

Applying $R' \otimes_{\mathbf{Z}[\mathbf{N}^{r-1}]} (-)$ to the surjection with nilpotent kernel $\mathbf{Z}[Q]/(\alpha(e_r)) \rightarrow \mathbf{Z}[Q']$ we get a surjection with nilpotent kernel

$$R' \otimes_{\mathbf{Z}[\mathbf{N}^{r-1}]} \mathbf{Z}[Q]/(\alpha(e_r)) \rightarrow R' \otimes_{\mathbf{Z}[\mathbf{N}^{r-1}]} \mathbf{Z}[Q'].$$

The target is reduced (as it is normal) and, hence, it is the reduction of the source. □

4.4 Tame coverings

We now begin the setup to the main theorem of this section, which shows that under the additional hypotheses including tameness, every ét-covering space is a de Jong covering space. The general definition we use is due to Hübner [Hüb21].⁷

DEFINITION 4.11. An étale morphism of rigid K -spaces $f: Y \rightarrow X$ is *tame* if for every $y \in Y$, the corresponding finite extension of valued fields $k(f(x)) \subseteq k(y)$ is tame.

The second assumption we shall put on the base space is that it has sufficiently many semistable models, as defined in the following. Let us call a formal scheme \mathfrak{X} locally of finite type over \mathcal{O}_K *gss* (generalized strictly semistable) if locally \mathfrak{X} admits an étale map to the formal scheme $\text{Spf}(\mathcal{O}_K\langle x_1, \dots, x_n \rangle / (x_1^{m_1} \cdots x_n^{m_n} - \varpi))$ for some $n \geq 0$ and $m_1, \dots, m_n \geq 0$ not all zero.

DEFINITION 4.12. Let X be a smooth quasi-paracompact and quasi-separated rigid K -space. We say that *gss formal models of X are cofinal* if for every admissible formal model \mathfrak{X} of X there exists a finite extension K' of K with valuation ring $\mathcal{O}_{K'}$, a gss formal model \mathfrak{X}' of $X_{K'}$ over $\mathcal{O}_{K'}$, and a morphism

$$f: \mathfrak{X}' \rightarrow \mathfrak{X}_{\mathcal{O}_{K'}}$$

such that f_η is an isomorphism.

The cofinality of gss formal models for a smooth quasi-paracompact and quasi-separated rigid K -space condition is satisfied (with $K' = K$) if K is of equal characteristic zero (see [Tem12]),

⁷ A more restrictive notion of tame coverings of non-archimedean curves has been studied by Berkovich [Ber93, § 6.3], who showed, for example, that finite étale coverings of the unit disc which are tame in his sense are trivial. This is not the case for the notion we use here.

or if X is of dimension one. Moreover, it would follow from functorial resolution of singularities for quasi-excellent schemes of dimension $\dim(X) + 1$ (again with $K' = K$).

We note that for a gss formal scheme \mathfrak{X} , the complete local rings $\widehat{\mathcal{O}}_{\mathfrak{X},x}$ are regular, and the preimages of the irreducible components of \mathfrak{X}_k in $\text{Spec}(\widehat{\mathcal{O}}_{\mathfrak{X},x})$ are of the form $V(x_i)$ for elements $x_1, \dots, x_r \in \mathfrak{m}_x$ which are linearly independent modulo \mathfrak{m}_x^2 . In particular, \mathfrak{X} is formally normal and, hence, η -normal by Lemma A.2.

LEMMA 4.13. *Let $\mathfrak{X} = \text{Spf}(A)$ be a gss formal scheme over \mathcal{O}_K and let $f: Y \rightarrow \mathfrak{X}_\eta = X$ be a finite étale and tame map of rigid K -spaces. Then, writing $Y = \text{Spa}(B_K)$, the morphism of schemes $\text{Spec}(B_K) \rightarrow \text{Spec}(A) = \mathfrak{X}$ is tamely ramified along the irreducible components of $\text{Spec}(A_k)$.*

Proof. Let ξ_1, \dots, ξ_r be the generic points of the irreducible components of \mathfrak{X}_k . The complete local rings $V_i = \widehat{\mathcal{O}}_{\mathfrak{X},\xi_i}$ are discrete valuation rings and ϖ is in the maximal ideal of V_i . We let $K_i = V_i[1/\varpi] = \text{Frac}(V_i)$. We have maps of Huber pairs $(A_K, A) \rightarrow (K_i, V_i)$ and pull-back squares as follows.

$$\begin{CD} \text{Spa}(B_K \otimes_A K_i) @>>> Y \\ @VVV @VVV \\ \text{Spa}(K_i) @>>> X \end{CD}$$

Thus, the left map is a tame étale map of adic spaces [Hüb21, §3] and so the map $\text{Spec}(B_K \otimes_A K_i) \rightarrow \text{Spec}(V_i)$ is tamely ramified along the closed point of $\text{Spec}(V_i)$. Then the same holds if we replace V_i with the Henselization $\mathcal{O}_{\mathfrak{X},\xi_i}^h$ (because $\mathcal{O}_{\mathfrak{X},\xi_i}^h$ and $V_i = \widehat{\mathcal{O}}_{\mathfrak{X},\xi_i}$ have the same inertia group) and, hence, $\text{Spec}(B_K) \rightarrow \text{Spec}(A)$ is tamely ramified at ξ_i . \square

We now come to our main theorem.

THEOREM 4.14. *Let X be a smooth quasi-paracompact and quasi-separated rigid-analytic space and suppose that gss formal models of X are cofinal. Let $Y \rightarrow X$ be an adm-covering space. If $Y \rightarrow X$ is tame, then it is a de Jong covering space.*

Proof. To show the assertion, it suffices to prove that $Y_{K'} \rightarrow X_{K'}$ is a de Jong covering space for some finite extension K' of K . Indeed, in the case of a separable extension, this follows from [dJ95b, Lemma 2.3], and in the purely inseparable case this is simple because $\text{Spa}(K') \rightarrow \text{Spa}(K)$ is a universal homeomorphism. By assumption, there exists an open cover $X = \bigcup_{i \in I} U_i$ such that $Y|_{U_i}$ is the disjoint union of finite étale covers of U_i . By Lemma 4.4, we may assume that $U_i = \mathfrak{U}_{i,\eta}$ for an open cover $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{U}_i$ of some admissible formal model \mathfrak{X} of X . Replacing K with a finite extension and performing an admissible blowup, we may assume that \mathfrak{X} is gss and, in particular, η -normal.

Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be the unique model of $Y \rightarrow X$ with \mathfrak{Y} η -normal and which is locally on \mathfrak{X} the disjoint union of finite morphisms, provided by Corollary 4.2. Let D_1, \dots, D_r be the irreducible components of $(\mathfrak{X}_k)_{\text{red}}$. Thanks to Corollary 4.6, it suffices to show that the schemes $(\mathfrak{Y} \times_{\mathfrak{X}} D_i)_{\text{red}}$ are normal.

This assertion being local, we may assume that $\mathfrak{X} = \text{Spf}(A)$ is affine and that Y is the disjoint union of finite étale covers of $X = \text{Spa}(A_K)$. Working on one connected component of Y at a time, we may assume that $Y \rightarrow X$ is finite étale and, therefore, $\mathfrak{Y} = \text{Spf}(B)$ where $B = B_K^\circ$, $A \rightarrow B$ is finite, and $A_K \rightarrow B_K$ is finite étale.

As $\text{Spf}(A)$ and $\text{Spec}(A)$ have the same completed local rings at closed points, and the same special fiber, the scheme $\text{Spec}(A)$ is regular and the divisor $V(\varpi)_{\text{red}} = D_1 \cup \dots \cup D_r$ has strict normal crossings. By Lemma 4.13, the map of schemes $\text{Spec}(B_K) \rightarrow \text{Spec}(A)$ is tamely ramified

along the D_i . Therefore, by Corollary 4.10, the schemes $T_i = (\text{Spec}(B) \times_{\text{Spec}(A)} D_i)_{\text{red}}$ are normal. As the D_i reside in the special fiber, we have $T_i = \mathfrak{Y} \times_{\mathfrak{X}} D_i$, and the theorem is proved. \square

Remark 4.15. In the language of logarithmic geometry, the proof of Theorem 4.14 gives the following result. If \mathfrak{X} is a gss formal scheme over \mathcal{O}_K with rigid generic fiber $X = \mathfrak{X}_\eta$ and if $Y \rightarrow X$ is a tame finite étale covering, then Y extends uniquely to a Kummer étale morphism $\mathfrak{Y} \rightarrow \mathfrak{X}$ where \mathfrak{Y} and \mathfrak{X} are endowed with the natural log structures induced by \mathfrak{Y}_k and \mathfrak{X}_k . Thus, the category $\mathbf{F\acute{E}t}_X$ is equivalent to the category of finite and Kummer étale coverings of \mathfrak{X} treated as a log formal scheme.

We finally extend our result to the full generality by showing that every tame ét-covering space is a de Jong covering space. We essentially use a bootstrap argument to reduce to Theorem 4.14.

COROLLARY 4.16. *Let X be a smooth quasi-paracompact and quasi-separated rigid K -space and suppose that the following conditions hold:*

- (i) *gss formal models of X are cofinal;*
- (ii) *for every finite étale covering Y of an affinoid open of X , gss formal models of Y are cofinal.*

Then every tame ét-covering space of X is a de Jong covering space.

Proof. Let $Y \rightarrow X$ be a tame ét-covering space, and let $U \rightarrow X$ be an étale cover such that Y_U is the disjoint union of finite étale coverings. By [ALY21a, Proposition 3.7.2], the cover $\{U \rightarrow X\}$ admits as a refinement a cover of the form $\{V_{ij} \rightarrow X\}_{i \in I, j \in J_i}$ such that for all $i \in I$ and $j \in J_i$ one has a factorization

$$V_{ij} \rightarrow W_i \rightarrow U_i \rightarrow X,$$

where $\{U_i \rightarrow X\}_{i \in I}$ forms an open cover, the maps $W_i \rightarrow U_i$ are finite étale, and for every $i \in I$, the family $\{V_{ij} \rightarrow W_i\}_{j \in J_i}$ is a finite affinoid open cover. Moreover, refining further we may assume that $\{U_i \rightarrow X\}_{i \in I}$ is an affinoid cover of finite type. Now, each $Y_{W_i} \rightarrow W_i$ is a tame adm-covering space, and by assumption (ii) and Theorem 4.14, we see that it is a de Jong covering space. Therefore, by [dJ95b, Lemma 2.3] also each $Y_{U_i} \rightarrow U_i$ is a de Jong covering space. Thus, $Y \rightarrow X$ is an adm-covering space, and we conclude using assumption (i) and Theorem 4.14. \square

As remarked before all smooth quasi-paracompact and quasi-separated rigid K -varieties, where K has equal characteristic zero, have cofinal gss formal models. Moreover, because tameness is automatic when the residue field k is of characteristic zero, we deduce the following result.

COROLLARY 4.17. *Let K be a complete discrete valuation field of residue characteristic zero, and let X be a smooth quasi-paracompact and quasi-separated rigid-analytic space over K . Then every ét-covering space of X is a de Jong covering space.*

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Appendix A. η -normality

The rigid generic fiber of a connected admissible formal scheme over \mathcal{O}_K may be disconnected. In order to control its connectedness, we introduce in this appendix the condition of η -normality, i.e. being ‘integrally closed in the rigid generic fiber’. To be able to apply it to questions concerning covering spaces, we need to show that if $\mathfrak{X}' \rightarrow \mathfrak{X}$ is étale and \mathfrak{X} is η -normal, then so is \mathfrak{X}' (Proposition A.16). To this end, we provide a version of Serre’s criterion ‘normal = $(S_2) + (R_1)$ ’ (Proposition A.15). The reader should be aware that many of the proofs in the following are considerably easier in the situation when K is discretely valued.

A.1 The definition of η -normality

We start with a definition of η -normality (cf. [dJ95a, Remark 7.4.2]).

DEFINITION A.1 (Cf. [Tem10, p. 619]). An \mathcal{O}_K -algebra A topologically of finite type over \mathcal{O}_K is called η -normal if it is \mathcal{O}_K -torsion free and integrally closed in A_K . A formal scheme \mathfrak{X} locally of finite type over \mathcal{O}_K is called η -normal if for every affine open $\mathrm{Spf}(A) \subseteq \mathfrak{X}$, the algebra A is η -normal.

In other words, A is η -normal if $A = A_K^\circ$. In particular, A is reduced because if f is nilpotent, then $K \cdot f \subseteq A_K^\circ$ and A_K° is not bounded which contradicts that it is equal to A . One can rephrase the condition that \mathfrak{X} is η -normal by saying that the map of sheaves $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathrm{sp}_{\mathfrak{X},*} \mathcal{O}_{\mathfrak{X},\eta}^+$ is an isomorphism. In particular, the property of being η -normal can be checked on a single affine open cover of \mathfrak{X} .

The following lemma gives common conditions which guarantee η -normality.

LEMMA A.2. *Suppose that one of the following conditions holds:*

- (a) \mathfrak{X}_k is reduced;
- (b) \mathfrak{X} is formally normal (i.e. $\mathcal{O}_{\mathfrak{X},x}$ is normal for all x in \mathfrak{X}).

Then \mathfrak{X} is η -normal.

Proof. For condition (a) we may assume that \mathfrak{X} is affine, in which case the claim follows from [BLR95, Proposition 1.1]. For condition (b) it suffices to show that if $\mathrm{Spf}(A)$ is formally normal and connected, then A is a normal domain. As $\mathrm{Spec}(A)$ is connected and topologically Noetherian, we know by [SP21, Tag 0357] that it suffices to show that $\mathrm{Spec}(A)$ is a normal scheme. However, by [SP21, Tag 030B] it suffices to show that for every maximal ideal \mathfrak{m} of A that $A_{\mathfrak{m}}$ is a normal domain. Note though that since A is ϖ -adically complete, that ϖ is contained in the Jacobson radical of A (see [FK18, Chapter 0, Lemma 7.2.13]), and so $\mathfrak{m} \supseteq \sqrt{(\varpi)}A = \mathfrak{m}_K A$. Thus, we see that \mathfrak{m} corresponds to a point x of $\mathrm{Spf}(A)$. Moreover, we have a natural map $A_{\mathfrak{m}} \rightarrow \mathcal{O}_{\mathfrak{X},x}$ obtained as the colimit of the natural maps $A[f^{-1}] \rightarrow A\langle f^{-1} \rangle$. By Gabber’s theorem (see [Bos14, Lemma 8.2/2]) each map $A[f^{-1}] \rightarrow A\langle f^{-1} \rangle$ is flat and, thus, is the map $A_{\mathfrak{m}} \rightarrow \mathcal{O}_{\mathfrak{X},x}$. This map is local (cf. [Bos14, Remark 7.2/1]) and faithfully flat by [SP21, Tag 00HQ]. Thus, $A_{\mathfrak{m}}$ is normal by [SP21, Tag 033G]. □

For us, the main use for η -normality is the fact that connectedness is preserved when passing to the rigid generic fiber.

LEMMA A.3. *Let \mathfrak{X} be formal scheme locally of finite type over \mathcal{O}_K which is η -normal. If \mathfrak{X} is connected, then so is \mathfrak{X}_{η} .*

Proof. By assumption, the map $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathrm{sp}_{\mathfrak{X},*} \mathcal{O}_{\mathfrak{X}_\eta}^+$ is an isomorphism. As every idempotent $\Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta})$ belongs to $\Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta}^+) = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, we deduce that $\Gamma(\mathfrak{X}_\eta, \mathcal{O}_{\mathfrak{X}_\eta})$ has no non-trivial idempotents, so \mathfrak{X}_η is connected. \square

A.2 Conditions (S'_1) and (S'_2)

In the absence of a robust theory of depth and embedded components in the non-Noetherian setting, we use the following theory condition as a substitute for the condition (S_1) .

DEFINITION A.4. Let X be a locally topologically Noetherian scheme. We say that X satisfies *condition (S'_1)* if every dense open subscheme $U \subseteq X$ is schematically dense (i.e. $\mathcal{O}_X \rightarrow j_*\mathcal{O}_U$ is injective, where $j: U \rightarrow X$ is the inclusion).

LEMMA A.5. *Let X be a locally topologically Noetherian scheme.*

- (a) *If $X = \mathrm{Spec}(A)$ is affine, then X satisfies (S'_1) if and only if every zero divisor of A belongs to at least one minimal prime of A .*
- (b) *The scheme X satisfies condition (S'_1) if and only if for one (equivalently, for all) open covers $\{U_i\}$ of X each U_i satisfies condition (S'_1) .*
- (c) *If X is locally Noetherian, then X satisfies (S'_1) if and only if it has no embedded points or, equivalently, if it satisfies condition (S_1) .*

Proof. (a) Let us first note that an open of the form $D(g)$, $g \in A$ is dense in $X = \mathrm{Spec}(A)$ if and only if g belongs to no minimal prime of A . To see that (S'_1) implies the given condition, take $U = D(g)$ and note that the equation $gf = 0$ implies that $f|_U = 0$ and, hence, $f = 0$.

Conversely, assume the condition holds and let $U \subseteq X$ be a dense open. We first claim that U contains a dense open of the form $D(g)$ for some $g \in A$. Indeed, writing $X \setminus U = V(I)$ for some ideal $I \subseteq A$, and denoting by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the minimal primes of A , the condition that U is dense means that $I \not\subseteq \mathfrak{p}_i$ for all i . By prime avoidance [SP21, Tag 00DS] there exists a $g \in I$ with $g \notin \mathfrak{p}_i$ for all i . Then $D(g) \subseteq U$ and $D(g)$ is dense. Thus, we may assume that $U = D(g)$.

Now, the inclusion $j: U \rightarrow X$ is affine and, hence, $\mathcal{O}_X \rightarrow j_*\mathcal{O}_U$ is injective if and only if $A \rightarrow A[1/g]$ is injective. Its kernel consists of elements $f \in A$ such that $g^n f = 0$ for some $n \geq 1$. As the powers g^n belong to no minimal prime of A , the given condition implies that $f = 0$.

(b) If $X = \bigcup U_i$ and the U_i satisfy (S'_1) , then so does X . For the converse, it suffices to show that if X satisfies (S'_1) , then every open $X' \subseteq X$ satisfies (S'_1) . If $U' \subseteq X'$ is a dense open, then let U be the union of all opens of X whose intersection with X' equals U' . Then U is a dense open of X , so $\mathcal{O}_X \rightarrow j_*\mathcal{O}_U$ is injective, and so is its restriction to X' .

(c) Combine [SP21, Tag 083P] and [SP21, Tag 0346]. \square

LEMMA A.6. *Let $Y \rightarrow X$ be an étale morphism of locally topologically Noetherian schemes.⁸ If X satisfies (S'_1) , then so does Y .*

Proof. As the condition is local on Y , we may assume that X and Y are affine and f is standard étale, i.e.

$$X = \mathrm{Spec}(A), \quad Y = \mathrm{Spec}(B), \quad B = (A[T]/(f))_g,$$

where $f \in A[T]$ is a monic polynomial whose derivative f' is invertible in B (see [SP21, Tag 02GT]). Let $C = A[T]/(f)$ and let $Z = \mathrm{Spec}(C)$, so that Y is an open subset of Z .

⁸ Note that if $Y \rightarrow X$ is étale and X is locally topologically Noetherian, then so is Y , by [BS15, Lemma 6.6.10(3)].

As f is monic, C is a free A -module of rank $\text{deg}(f)$ and, hence, the morphism $\varphi: Z \rightarrow X$ is finite and locally free and therefore locally of finite presentation (see [SP21, Tag 02KB]).

As Y is an open in Z , it suffices to show that Z satisfies (S'_1) . To this end, suppose that $W \subseteq Z$ is a dense open and $x \in C$ is an element whose restriction to W is zero. We have to show that $x = 0$. Let $U = X \setminus \varphi(Z \setminus W)$; this is an open subset because φ is a closed map, and $\varphi^{-1}(U)$ is contained in W . We claim that $\varphi^{-1}(U)$ is also dense in Z ; otherwise there exists a non-empty open $V \subseteq Z$ with $\varphi(V)$ contained in $\varphi(Z \setminus W)$. But $\varphi(V)$ is open (because φ is flat and finitely presented, see [SP21, Tag 01UA]) and so contains a generic point ξ . That said, the fiber $\varphi^{-1}(\xi)$ consists entirely of generic points (cf. [SP21, Tag 0ECG]) which would imply that $Z \setminus W$ contains a generic point, which is impossible.

Writing $x = \sum_{i < \text{deg}(f)} a_i T^i$ with $a_i \in A$, if x vanishes on W , it vanishes on $\varphi^{-1}(U)$, and then the coefficients a_i vanish on the dense open $U \subseteq X$. As X satisfies (S'_1) , we must have $a_i = 0$ for all i , and hence $x = 0$, as desired. \square

PROPOSITION A.7. *Let A be an admissible \mathcal{O}_K -algebra. Then the following conditions are equivalent:*

- (a) *for some pseudo-uniformizer ϖ , the scheme $\text{Spec}(A/\varpi A)$ satisfies (S'_1) ;*
- (b) *for every pseudo-uniformizer ϖ , the scheme $\text{Spec}(A/\varpi A)$ satisfies (S'_1) .*

Proof. It suffices to show the following two claims.

- (i) If $\text{Spec}(A/\varpi' A)$ and $\text{Spec}(A/\varpi'' A)$ both satisfy (S'_1) , then so does $\text{Spec}(A/\varpi A)$, where $\varpi = \varpi' \varpi''$.
- (ii) If $\text{Spec}(A/\varpi A)$ satisfies (S'_1) and ϖ' divides ϖ , then $\text{Spec}(A/\varpi' A)$ satisfies (S'_1) .

For the first claim, suppose that $g \in A/\varpi A$ belongs to no minimal prime and $f \in A/\varpi A$ is such that $gf = 0$. Reducing modulo ϖ' and using the (S'_1) property of $\text{Spec}(A/\varpi' A)$, we see that $f \in \varpi' A/\varpi A$, say $f = \varpi' f'$. Then $\varpi' f' g = 0$ implies (because A is flat over \mathcal{O}_K) that $f' g = \varpi'' h$ for some h .⁹ Reducing modulo ϖ'' we see that $f' g = 0$ in $A/\varpi'' A$ and so using the (S'_1) property of $\text{Spec}(A/\varpi'' A)$, we see that $f' = 0$ in $A/\varpi' A$ and so $f = \varpi' f' = 0$ in $A/\varpi A$, so $f = \varpi' f' = \varpi' \varpi'' f'' = 0$ as desired.

For the second claim, suppose that $g' \in A/\varpi' A$ belongs to no minimal prime and $f' \in A/\varpi' A$ is such that $g' f' = 0$. Let f and g be lifts to $A/\varpi A$, so that $gf = \varpi' h$ for some h . Write $\varpi = \varpi' \varpi''$, so $f'' := \varpi'' f$ has the property that $g f'' = \varpi'' g f = \varpi'' \varpi' h = 0$. By the (S'_1) property of $\text{Spec}(A/\varpi A)$, we thus have $f'' = \varpi'' f = 0$. As A is flat over \mathcal{O}_K , we must have $f \in \varpi' A$, so $f' = 0$. \square

Remark A.8. We do not know if the conditions in Proposition A.7 are further equivalent to the condition that $\text{Spec}(A_k)$ satisfies (S'_1) (or, equivalently, (S_1)) in the case when K is not discretely valued.

DEFINITION A.9. Let \mathfrak{X} be a formal scheme locally of finite type over \mathcal{O}_K . We say that \mathfrak{X} satisfies condition (S'_2) if it is flat over \mathcal{O}_K and for every (equivalently, any) pseudo-uniformizer ϖ , the scheme $\mathfrak{X} \otimes (\mathcal{O}_K/\varpi)$ satisfies condition (S'_1) .

The following result follows from Lemma A.6 and [FK18, Chapter I, Proposition 5.3.11].

LEMMA A.10. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be an étale morphism of formal schemes locally of finite type over \mathcal{O}_K . If \mathfrak{X} satisfies (S'_2) , then so does \mathfrak{Y} .*

⁹ Indeed, $\varpi' f' g = 0$ means that $\varpi' f' g = \varpi h$ in A . This implies that $\varpi'(f' g - \varpi'' h) = 0$ in A which, because A has no \mathcal{O}_K -torsion, implies that $f' g - \varpi'' h = 0$ and so $f' g = \varpi'' h$ as desired.

A.3 Condition (R'_1)

In our relative situation over \mathcal{O}_K the replacement for the usual condition (R_1) for a formal scheme \mathfrak{X} is a statement about the local rings of \mathfrak{X} at the generic points ζ of \mathfrak{X}_k . As ζ is codimension zero in \mathfrak{X}_k it is intuitively of codimension one in \mathfrak{X} and so we would then like to claim that $\mathcal{O}_{\mathfrak{X},\zeta}$ is a discrete valuation ring. As \mathfrak{X} is not necessarily Noetherian, we instead require that $\mathcal{O}_{\mathfrak{X},\zeta}$ is a valuation ring.

To this end, for a formal scheme \mathfrak{X} locally of finite type over \mathcal{O}_K let us denote by $I(\mathfrak{X})$ the set of generic points of the scheme \mathfrak{X}_k or, equivalently, the set of generic points of the locally spectral space $|\mathfrak{X}|$.

DEFINITION A.11. Let \mathfrak{X} be an admissible formal scheme over \mathcal{O}_K . We say that \mathfrak{X} satisfies condition (R'_1) if for every ζ in $I(\mathfrak{X})$, the local ring $\mathcal{O}_{\mathfrak{X},\zeta}$ is a valuation ring.

Assume that \mathfrak{X} is quasi-compact and set $X = \mathfrak{X}_\eta$. It will be useful in the proof of Lemma A.14 and Proposition A.15 to relate the ring $\mathcal{O}_{\mathfrak{X},\zeta}$ for ζ in $I(\mathfrak{X})$ to local rings of \mathcal{O}_X^+ . To this end, following [FK18, Chapter II, Definition 11.1.2] let us denote by $D(\mathfrak{X})$ the set of divisorial points of X which are residually finite over \mathfrak{X} . By [FK18, Chapter II, Proposition 11.1.3] and [FK18, Chapter II, Theorem 11.2.1] one has that the set $D(\mathfrak{X})$ is a finite set of maximal points, and that the equality

$$D(\mathfrak{X}) = \bigcup_{\zeta \in I(\mathfrak{X})} \text{sp}_{\mathfrak{X}}^{-1}(\zeta)$$

holds. Moreover, one has by [FK18, Chapter II, Theorem 11.2.1] and [BGR84, Proposition 6.3/3] that

$$\|f\|_{\text{sup}} = \sup_{x \in D(\mathfrak{X})} |f(x)|, \tag{A.3.1}$$

where $\|\cdot\|_{\text{sup}}$ is the supremum semi-norm as in [BGR84, §6.2.1].

Now, for x in $D(\mathfrak{X})$ with specialization $\zeta = \text{sp}_{\mathfrak{X}}(x)$ one has a natural map of topological rings $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{X,x}^+$ (this map exists even if \mathfrak{X} is only admissible over \mathcal{O}_K). Indeed, because (X, \mathcal{O}_X^+) is isomorphic to the inverse limit $\varprojlim(\mathfrak{X}', \mathcal{O}_{\mathfrak{X}'})$ where \mathfrak{X}' travels over the admissible blowups of \mathfrak{X} one sees that there is a canonical identification

$$\mathcal{O}_{X,x}^+ = \varprojlim \mathcal{O}_{\mathfrak{X}', \text{sp}_{\mathfrak{X}'}(x)} \tag{A.3.2}$$

(see [FK18, Chapter 0, Proposition 4.1.10]) from where the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{X,x}^+$ is clear. Thus, if $k(x)^+$ denotes the (uncompleted) residue valuation ring, one obtains a map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow k(x)^+$.

LEMMA A.12. *Let \mathfrak{X} be an admissible formal scheme over \mathcal{O}_K , let $x \in D(\mathfrak{X})$ with image $\zeta \in I(\mathfrak{X})$. Then, the following are equivalent:*

- (a) *the ring $\mathcal{O}_{\mathfrak{X},\zeta}$ is a valuation ring;*
- (b) *the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow k(x)^+$ is an isomorphism.*

In particular, under these conditions, the ring $\mathcal{O}_{\mathfrak{X},\zeta}$ is a height one valuation ring which is ϖ -adically separated and Henselian.

Proof. Clearly condition (b) implies condition (a). Moreover, the last statement follows then immediately from the fact that x is a maximal point and [FK18, Chapter II, Corollary 3.2.8]. To see that condition (a) implies condition (b), let us begin by observing that for any admissible blowup \mathfrak{X}' , the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{\mathfrak{X}',\zeta'}$ is an isomorphism where $\zeta' = \text{sp}_{\mathfrak{X}'}(x)$. Indeed, suppose that $\mathfrak{X}' \rightarrow \mathfrak{X}$ is the blowup at an admissible ideal sheaf \mathcal{J} . As \mathcal{J} is finitely generated we know that

$\mathcal{J}_\zeta \subseteq \mathcal{O}_{\mathfrak{X},\zeta}$ is a finitely generated ideal which, because $\mathcal{O}_{\mathfrak{X},\zeta}$ is a valuation ring, must be invertible. We may then find an open \mathfrak{U} of \mathfrak{X} containing ζ such that $\mathcal{J}|_{\mathfrak{U}}$ is invertible. However, $\mathfrak{X}'_{\mathfrak{U}} \rightarrow \mathfrak{U}$ coincides with the blowup of \mathfrak{U} along $\mathcal{J}|_{\mathfrak{U}}$ (see [FK18, Chapter II, Proposition 1.1.8]) but because $\mathcal{J}|_{\mathfrak{U}}$ is invertible this implies that $\mathfrak{X}'_{\mathfrak{U}} \rightarrow \mathfrak{U}$ is an isomorphism and so evidently $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{\mathfrak{X}',\zeta'}$ is an isomorphism as desired.

Combining this with (A.3.2) we deduce that the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{X,x}^+$ is an isomorphism. To show then that the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow k(x)^+$ is an isomorphism note that for any admissible blowup \mathfrak{X}' of \mathfrak{X} the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{\mathfrak{X}',\zeta'}$ is an isomorphism and commutes with the natural maps to $k(x)^+$. Thus, we may replace $\mathcal{O}_{\mathfrak{X},\zeta}$ with any $\mathcal{O}_{\mathfrak{X}',\zeta'}$. However, we in particular may take \mathfrak{X}' as in [FK18, Chapter II, Proposition 11.2.8]. As the kernel of $\mathcal{O}_{\mathfrak{X},\zeta} \simeq \mathcal{O}_{X,x}^+ \rightarrow k(x)^+$ is precisely those f with $|f(x)| = 0$ we are done if we can show that for f in $\mathcal{O}_{\mathfrak{X},\zeta}$ one has that $|f(x)| = 0$ implies that $f = 0$. However, we may as well assume that f is in $\mathcal{O}(\mathfrak{U}_\eta)$ for some affine irreducible neighborhood \mathfrak{U} of ζ . However, we then see that by our choice of model $D(\mathfrak{U}) = \text{sp}_{\mathfrak{X}}^{-1}(\zeta) = \{x\}$. Thus, by (A.3.1) we deduce that $\|f\|_{\text{sup}} = 0$ (where this supremum is taken relative to \mathfrak{U}_η) and, thus, by [Bos14, Proposition 3.1/10] we deduce that f is a nilpotent element, and because $\mathcal{O}_{\mathfrak{X},\zeta}$ is a domain, has trivial image in $\mathcal{O}_{\mathfrak{X},\zeta}$. \square

We now wish to establish that the (R'_1) condition ascends through étale maps of formal schemes. To do this, we first need the following lemma.

LEMMA A.13. *Let $f: \text{Spf}(B) \rightarrow \text{Spf}(A)$ be a finite étale morphism of admissible formal schemes over \mathcal{O}_K . Then, the map $A \rightarrow B$ is finite étale.*

Proof. We claim that the ring map $A \rightarrow B$ is weakly étale (see [BS15, Definition 1.2]). Indeed, we must show that $A \rightarrow B$ is flat, and that $B \rightarrow B \otimes_A B$ are flat. However, note that because $A \rightarrow B$ is finite, $B \otimes_A B$ is finite over B , and so complete. From this and [FK18, Chapter I, Proposition 4.8.1] we see that it suffices to check that $A/\varpi^m A \rightarrow B/\varpi^m B$ is weakly étale for all $m \geq 0$, but this is clear because this map is étale. Note though that because $A \rightarrow B$ is finite, B is actually finitely presented as an A -module (see [Bos14, Theorem 7.2/4]) and so $A \rightarrow B$ is finitely presented (see [SP21, Tag 0D46]). As $A \rightarrow B$ is both finitely presented and weakly étale, it is étale (see [BS15, Theorem 1.3]). \square

LEMMA A.14. *Let $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ be an étale morphism of admissible formal schemes over \mathcal{O}_K . If \mathfrak{X} satisfies (R'_1) , then so does \mathfrak{Y} .*

Proof. Let ξ be a generic point of \mathfrak{Y} and let $\zeta = f(\xi)$. We first show that the map $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{\mathfrak{Y},\xi}$ is finite étale. Note that ζ is a generic point of \mathfrak{X} . Indeed, it suffices to show that $f_1(\xi)$ is a generic point where f_1 is the reduction of f modulo (ϖ) , but this is clear, as f_1 is an open morphism of locally topologically Noetherian schemes. Note then, that because f_1 is an étale morphism of schemes, there exists open neighborhoods U (respectively, V) of ζ (respectively, ξ) such that $f(V) \subseteq U$, $f_1|_V: V \rightarrow U$ is finite étale, and $f_1^{-1}(\zeta) \cap V = \{\xi\}$. Indeed, this follows from the fact that the local ring at ζ has reduced subscheme which is a field, and so (by [SP21, Tag 04DZ]) all étale algebras are finite product of finite étale algebras. From this and [FK18, Chapter I, Proposition 4.2.3] we see that we may find affine neighborhoods $\text{Spf}(A)$ (respectively, $\text{Spf}(B)$) of ζ (respectively, ξ) such that $f(\text{Spf}(B)) \subseteq \text{Spf}(A)$ and $f|_{\text{Spf}(B)}: \text{Spf}(B) \rightarrow \text{Spf}(A)$ is finite étale and $f^{-1}(\zeta) \cap \text{Spf}(B) = \{\xi\}$. Replace $f: \mathfrak{Y} \rightarrow \mathfrak{X}$ by $f|_{\text{Spf}(B)}$. By Lemma A.13 we then know that $A \rightarrow B$ is finite étale. Now, because f is finite and $f^{-1}(\zeta) = \{\xi\}$ the neighborhoods of the form $f^{-1}(D(f))$ for $D(f)$ a neighborhood of ζ are cofinal in the neighborhoods of ξ . Thus, $\mathcal{O}_{\mathfrak{Y},\xi} = \mathcal{O}_{\mathfrak{X},\zeta} \otimes_A B$, and because $A \rightarrow B$ is finite étale

we deduce that $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{\mathfrak{Y},\xi}$ is finite étale as desired. In particular, because $\mathcal{O}_{\mathfrak{X},\zeta}$ is a normal domain, so is $\mathcal{O}_{\mathfrak{Y},\xi}$ (see [SP21, Tag 025P]).

Using [Bou64, Chapitre VI, §6, Proposition 6] applied to the map of local rings $\mathcal{O}_{\mathfrak{X},\zeta} \rightarrow \mathcal{O}_{\mathfrak{Y},\xi}$, we deduce that $\mathcal{O}_{\mathfrak{Y},\xi}$ is a valuation ring if $\mathcal{O}_{\mathfrak{X},\zeta}$ is, as desired. \square

A.4 Serre’s criterion for η -normality

Serre’s criterion states that a locally Noetherian scheme is normal if and only if it is both (S_2) and (R_1) . In our situation of admissible formal schemes over \mathcal{O}_K , we have the following analog (compare with condition (N) in [Ray70, §6.1]).

PROPOSITION A.15 (Serre’s criterion for η -normality). *Let \mathfrak{X} be an admissible formal scheme over \mathcal{O}_K . The following are equivalent:*

- (a) \mathfrak{X} is η -normal;
- (b) \mathfrak{X} satisfies (S'_2) and (R'_1) .

Proof. As being η -normal and condition (b) are Zariski local on \mathfrak{X} , we may assume that $\mathfrak{X} = \text{Spf}(A)$ where A is an admissible \mathcal{O}_K -algebra. Set X to be $\mathfrak{X}_\eta = \text{Spa}(A_K)$.

Suppose first that condition (a) holds. To show condition (S'_2) it suffices to show that if a is an element of A such $a \in \varpi\mathcal{O}_{\mathfrak{X},\zeta}$ for all ζ in $I(\mathfrak{X})$, then $a \in \varpi A$. Consider $\varpi^{-1}a$ as an element of A_K . Note then that, by assumption, we have that $\varpi^{-1}a \in \mathcal{O}_{\mathfrak{X},\zeta} \subseteq k(x)^+$ for all x in $D(\mathfrak{X})$. Thus, we see from (A.3.1) and [BGR84, Proposition 6.2.3/1] that $\varpi^{-1}a$ is in $A_K^\circ = A$ as desired.

To see that (R'_1) holds, passing to an affine open neighborhood of a given $\zeta \in I(\mathfrak{X})$ we may assume that $\mathfrak{X} = \text{Spf}(A)$ and $|\mathfrak{X}|$ is irreducible. Moreover, because the equality $A_K^\circ = A$ shows that $\text{Spec}(\tilde{A}_K) = \text{Spec}(A_k)_{\text{red}}$ (where $\tilde{A} = A_K^\circ/A_K^\circ$ as in [BGR84, §1.2.5]) we deduce that \tilde{A} is a domain, and because A is moreover reduced, the supremum semi-norm $\|\cdot\|_{\text{sup}}$ is a multiplicative norm by [BGR84, Proposition 6.2.3/5]. This implies that A is a domain; by the same argument, for every $f \in A$ with $f(\zeta) \neq 0$, the complete localization $A\langle f^{-1} \rangle$ is a domain, and passing to the limit also $\mathcal{O}_{\mathfrak{X},\zeta}$ is a domain. To check that $\mathcal{O}_{\mathfrak{X},\zeta}$ is a valuation ring, take a non-zero fraction $f/g \in \text{Frac}(\mathcal{O}_{\mathfrak{X},\zeta})$; shrinking \mathfrak{X} further, we may assume $f, g \in A$ and $f^{-1}, g^{-1} \in A_K$. For the latter claim, because $A = A_K^\circ$, we can by [BGR84, Proposition 6.2.1/4 (ii)] write $f^n = \alpha u$, $g^m = \beta v$ where $\alpha, \beta \in \mathcal{O}_K$ and u, v have supremum norm one and, hence, their images in \tilde{A} are non-zero; passing to a smaller affine neighborhood of ζ , we may assume that u and v are invertible in \tilde{A} and, hence, in A (because A and \tilde{A} have the same maximal ideals), so that f and g become invertible in A_K . Then if say $\|f\|_{\text{sup}} \leq \|g\|_{\text{sup}}$, then $f/g \in A_K$ and $\|f/g\|_{\text{sup}} \leq 1$, so $f/g \in A_K^\circ = A$ and so $f/g \in \mathcal{O}_{\mathfrak{X},\zeta}$. Otherwise, $g/f \in \mathcal{O}_{\mathfrak{X},\zeta}$ by the same argument.

Conversely, suppose that condition (b) holds. On the one hand, again by (A.3.1) we have

$$A_K^\circ = \{a \in A_K : a \in k(x)^+ \text{ for each } x \in D(\mathfrak{X})\},$$

where we are implicitly identifying $a \in A_K$ with its image in $k(x)$. On the other hand, we claim that

$$A = \{a \in A_K : a \in \mathcal{O}_{\mathfrak{X},\zeta} \text{ for all } \zeta \in I(\mathfrak{X})\},$$

where we identify $a \in A_K$ with its image in $\mathcal{O}_{\mathfrak{X},\zeta}[1/\varpi]$. This would imply the required assertion by Lemma A.12 and the (R'_1) property.

To show the claim, we note that evidently A maps into each $\mathcal{O}_{\mathfrak{X},\zeta}$. Conversely, let $a \in A_K$ be an element of the right-hand side, and write $a = \varpi^{-n}a'$ with $a' \in A$ and $n \geq 0$. By the (S'_2) property of \mathfrak{X} or, equivalently, the (S'_1) property of $\text{Spec}(A/\varpi^n)$, our assumption that a' lies in $\varpi^n\mathcal{O}_{\mathfrak{X},\zeta}$ for all ζ implies that $a' \in \varpi^n A$ and, hence, $a \in A$ as desired. \square

Using the criterion of Proposition A.15, Lemmas A.10 and A.14 imply the following result.

COROLLARY A.16. *Let $\mathfrak{Y} \rightarrow \mathfrak{X}$ be an étale morphism of admissible formal schemes over \mathcal{O}_K . If \mathfrak{X} is η -normal, then so is \mathfrak{Y} .*

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