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ON THE AUTOMORPHISMS OF THE GROUP RING OF A FINITELY GENERATED FREE ABELIAN GROUP

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Let R be an associative ring with 1 and G a finitely generated torsion-free abelian group. In this note, we classify all R-automorphisms of the group ring RG. The special case where G is infinite cyclic was previously settled in [**8**], and our interest in this problem was rekindled by the recent paper of Mehrvarz and Wallace [**7**], who carried out the classification in the case where R contains a nilpotent prime ideal.

It is interesting to compare this situation with the corresponding results for polynomial rings. Gilmer [**5**] and Coleman and Enochs [**3**] determined all R-automorphisms of the polynomial ring $R[x]$ in one indeterminate, and the study of the infinite cyclic case was originally motivated by their work. The problem of determining all R-automorphisms of $R[x, y]$, a polynomial ring in commuting indeterminants, has recently been settled but the case of three or more indeterminates is still open [**1**, **2**, **4**]. Nevertheless, we are able to answer the corresponding question for group rings.

An excellent general reference for results on group rings is [**10**], and we will follow the notation used in this text.

The referee of this paper kindly brought to our attention work by Lantz [**6**], in which a different description is given of the R-automorphisms of RG.

The statement of the main theorem involves the condition that certain elements be units in $z(R)G$ where $z(R)$ denotes the centre of R. Necessary and sufficient conditions for this to be the case are contained in the following proposition. We include a proof for completeness, although the result could be deduced from other equivalent characterizations given in either [**8**] or [**9**].

Proposition 1. *Let R be a commutative ring with 1 and let G be right-ordered. Then $\Sigma\alpha_g g$ is a unit in RG if and only if whenever P is a prime ideal of R, exactly one of the coefficients α_g does not belong to P.*

Proof. Let $\Sigma\alpha_g g$ be a unit in RG and P a prime ideal of R. Then $\Sigma\bar{\alpha}_g g$ is a unit in $(R/P)G \simeq RG/PG$. Since R/P is an integral domain and G is ordered, it is easy to see that this forces $\bar{\alpha}_g \neq 0$ for exactly one g [**8**].

Conversely, assume $\Sigma\alpha_g g$ satisfies the condition on prime ideals. Then $(\Sigma\alpha_g g)(\Sigma\alpha_g g^{-1}) = \Sigma(\alpha_g)^2 + N = (\Sigma\alpha_g)^2 + N'$ where N and N' are nilpotent elements of RG. The condition

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on α_g with respect to prime ideals also guarantees that $\Sigma\alpha_g$ does not belong to any maximal ideal of R, and hence must be a unit. It follows easily that $(\Sigma\alpha_g)^2 + N'$ is a unit in RG and the proof is complete.

We now state our main result.

Theorem 2. *Let $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_n \rangle$ be a finitely generated torsion-free abelian group. For each i, let $a^{(i)}_{j_1, j_2, \ldots, j_n}$ be nonzero ring elements for finitely many choices of $j_1, \ldots, j_n$. Then the mappings*

$$x_i \to \Theta(x_i) = \Sigma a^{(i)}_{j_1, j_2, \ldots, j_n} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$$

induce an R-automorphism Θ of RG if and only if the following condition is satisfied:

> *$\Theta(x_i)$ is a unit in $z(R)G$ for $1 \leqq i \leqq n$. Moreover, if P is any prime ideal of R and $\{a^{(i)}_{j_1(i), \ldots, j_n(i)}\}_{1 \leqq i \leqq n}$ are the coefficients which do not belong to P, then the corresponding group elements $\{x_1^{j_1(i)} x_2^{j_2(i)} \ldots x_n^{j_n(i)}\}_{1 \leqq i \leqq n}$ form a basis for G.* (*)

Remarks. (1) In condition (*), note that once we know $\Theta(x_i)$ is a unit in $z(R)G$, then Proposition 1 guarantees us that if P is any prime ideal of R and $1 \leqq i \leqq n$, exactly one of the coefficients $a^{(i)}_{j_1 \ldots j_n}$ does not belong to P.

(2) The final sentence in condition (*) is equivalent to saying that the $n \times n$ matrix whose (k, l)th entry is $j_l(k)$ has determinant ± 1.

Proof. First assume that the mappings $x_i \to \Theta(x_i)$ as described induce on R-automorphism of RG. Since x_i is a central unit, $\Theta(x_i)$ must also be a unit in $z(RG) \simeq z(R)G$. Let P be a prime ideal of R, and, for each i, let $a^{(i)}_{j_1(i), \ldots, j_n(i)}$ be the coefficient which is not in P (using Proposition 1).

Let $g \in G$. Since Θ is an automorphism of RG, we must have

$$g = \Sigma c_{t_1, t_2, \ldots, t_n} (\Theta(x_1))^{t_1} \ldots (\Theta(x_n))^{t_n}$$

for some finite set of nonzero elements $c_{t_1, t_2, \ldots, t_n}$ in R.

In $(R/P)G$, we have

$$\begin{aligned} g &= \Sigma \bar{c}_{t_1, t_2, \ldots, t_n} (\bar{\Theta}(x_1))^{t_1} \ldots (\bar{\Theta}(x_n))^{t_n} \\ &= \Sigma \bar{c}_{t_1, t_2, \ldots, t_n} (\bar{a}^{(1)}_{j_1(1) \ldots j_n(1)} x_1^{j_1(1)} \ldots x_n^{j_n(1)})^{t_1} \ldots (\bar{a}^{(n)}_{j_1(n) \ldots j_n(n)} x_1^{j_1(n)} \ldots x_n^{j_n(n)})^{t_n}. \end{aligned}$$

It follows that g is in the subgroup of G generated by the elements $\{x_1^{j_1(i)} x_2^{j_2(i)} \ldots x_n^{j_n(i)}\}_{1 \leqq i \leqq n}$, so these elements form a generating set for G. Because G is torsion-free abelian of rank n, these elements must form a basis for G.

Next we assume that we have elements $\Theta(x_i)$ as described which satisfy condition (*). We must show that the map $\Theta: RG \to RG$ induced by $x_i \to \Theta(x_i)$, $1 \leqq i \leqq n$, is injective and surjective.

Surjective. Let x_t be one of the given basis elements of G. We must see how to obtain x_t as an R-linear combination of the elements $\Theta(x_i)$, $1 \leqq i \leqq n$.

Let $\chi = \{a^{(i)}_{j_1(i),\dots,j_n(i)}\}_{1 \leqq i \leqq n}$ be any collection of n coefficients, one from each $\Theta(x_i)$, whose product is not nilpotent. It follows that there exists some prime ideal which does not contain any element of χ; select one of these and call it P_χ. Because of condition (*), we know that the corresponding group elements $\{x_1^{j_1(i)} x_2^{j_2(i)} \dots x_n^{j_n(i)}\}_{1 \leqq i \leqq n}$ form a basis for G. It follows that

$$x_t = \prod_{1 \leqq i \leqq n} (x_1^{j_1(i)} \dots x_n^{j_n(i)})^{w_i}$$

for suitable integers w_i, $1 \leqq i \leqq n$.

Now consider the sum

$$\sum_\chi \left(\prod_{1 \leqq i \leqq n} a^{(i)}_{j_1(i),\dots,j_n(i)} \right) (\Theta(x_1))^{w_1} \dots (\Theta(x_n))^{w_n}$$

where this finite sum is taken over all possible collections χ of the type just described. Note that the ring elements $a^{(i)}_{j_1(i),\dots,j_n(i)}$ and the integers w_i in each term of the above sum are determined by the collection χ for that term.

Observe now that if $w_i \geqq 0$,

$$(\Theta(x_i))^{w_i} = \Sigma (a^{(i)}_{j_1, j_2, \dots, j_n})^{w_i} x_1^{j_1 w_i} \dots x_n^{j_n w_i} + N$$

where N is nilpotent in $z(R)G$.

Therefore

$$a^{(i)}_{j_1(i) \dots j_n(i)} (\Theta(x_i))^{w_i} = (a^{(i)}_{j_1(i),\dots,j_n(i)})^{w_i+1} (x_1^{j_1(i)} \dots x_n^{j_n(i)})^{w_i} + N'$$

where N' is nilpotent.

However, if $w_i < 0$, we can use the proof of Proposition 1 to conclude that

$$(\Theta(x_i))^{w_i} = ((\Theta(x_i))^{-1})^{-w_i}$$

$$= (u \Sigma a^{(i)}_{j_1, j_2, \dots, j_n})^{-w_i} (x_1^{-j_1} \dots x_n^{-j_n})^{-w_i} + M$$

where M is nilpotent in $z(R)G$ and u is a unit in $z(R)$.

Therefore

$$a^{(i)}_{j_1(i) \dots j_n(i)} (\Theta(x_i))^{w_i} = u^{-w_i} (a^{(i)}_{j_1(i),\dots,j_n(i)})^{-w_i+1} (x_1^{j_1(i)} \dots x_n^{j_n(i)})^{w_i} + M'$$

where M' is nilpotent.

Hence we conclude that

$$\left(\prod_{1\leqq i\leqq n} a^{(i)}_{j_1(i),\dots,j_n(i)}\right)(\Theta(x_1))^{w_1}\dots(\Theta(x_n))^{w_n}$$

$$=\left[u\prod_{1\leqq i\leqq n} a^{(i)}_{j_1(i),\dots,j_n(i)})^{|w_i|+1}\right]x_t+M''$$

where M'' is nilpotent in $z(R)G$ and u is a unit in $z(R)$.

We conclude that

$$\sum_{\chi}\left(\prod_{1\leqq i\leqq n}(a^{(i)}_{j_1(i),\dots,j_n(i)})\right)(\Theta(x_1))^{w_1}\dots(\Theta(x_n))^{w_n}$$

$$=\left[\sum_{\chi}u_\chi\prod_{1\leqq i\leqq n}(a^{(i)}_{j_1(i),\dots,j_n(i)})^{|w_i|+1}\right]x_t+N$$

where N is nilpotent in $z(R)G$ and u_χ is a unit in $z(R)$ in each case.

Now if P is any prime ideal of R, condition (*) guarantees us that exactly one of the terms $u_\chi\prod_{1\leqq i\leqq n}(a^{(i)}_{j_1(i),\dots,j_n(i)})^{|w_i|+1}$ in the above sum does not belong to P. It follows that the above coefficient is a unit in $z(R)$.

Let L be the R-subalgebra of RG generated by $R, \Theta(x_1)\dots\Theta(x_n)$.

We have shown that, if x_t is one of the given basis elements of G, then $x_t=a_t+b_t$ where $a_t\in L$ and b_t is nilpotent and central. It is easy to see that b_t must be of the form $\Sigma\alpha_g g$ where each α_g is nilpotent in $z(R)$. It follows that if I is the ideal of R generated by all the α_g (for all the b_t), then I is a nilpotent ideal.

Consequently, we can substitute the equations $x_t=a_t+b_t$ back into the various x_t terms appearing in b_t. We conclude that $x_t\in L$ and Θ is surjective as required.

Injective. Let us assume we can find finitely many ring elements $c_{w_1,w_2\dots w_n}$ such that

$$\Sigma c_{w_1,w_2,\dots,w_n}(\Theta(x_1))^{w_1}\dots(\Theta(x_n))^{w_n}=0.$$

We may assume that $c_{w_1\dots w_n}\in z(R)$, since if we can prove the map is an isomorphism on $z(R)G$, then it will be invertible on $z(R)G$ and we can prove the general case.

If P is a prime ideal of R, passing to $(R/P)G$ we obtain

$$\Sigma\bar{c}_{w_1,w_2,\dots,w_n}(\bar{\Theta}(x_1))^{w_1}\dots(\bar{\Theta}(x_n))^{w_n}=0.$$

Condition (*) now tells us that the group elements surviving in the above (one in each $\Theta(x_i)$) form a basis for G, so we conclude that the coefficients $c_{w_1,w_2,\dots,w_n}$ are nilpotent.

Let T be the nilpotent ideal of R generated by all $c_{w_1,w_2,\dots,w_n}$, all nilpotent $a^{(i)}_{j_1\dots j_n}$ and all products $a^{(i)}_{j_1,j_2,\dots,j_n}a^{(i)}_{k_1,k_2,\dots,k_n}$ where $j_s\neq k_s$ for some s.

Say each $c_{w_1,w_2,\dots,w_n}$ is in T^k but that some $c_{w_1,w_2,\dots,w_n}$ is not in T^{k+1}. In $(R/T^{k+1})G$, consider

$$\Sigma\bar{c}_{w_1,w_2\dots w_n}(\bar{\Theta}(x_1))^{w_1}\dots(\bar{\Theta}(x_n))^{w_n}=0.$$

If $\prod_{1 \leq i \leq n} \bar{a}^{(i)}_{j_1(i)\ldots j_n(i)}$ is not nilpotent, and we multiply both sides of the above equation by this, then identities like those used in the surjectivity part of the proof allow us to conclude that

$$\bar{c}_{w_1, w_2, \ldots, w_n}\left(\bar{u} \prod_{1 \leq i \leq n} (\bar{a}^{(i)}_{j_1(i)\ldots j_n(i)})^{|w_i|+1}\right) = 0$$

for each choice of $w_1 \ldots w_n$ where u is some central unit in R (dependent on $w_1 \ldots w_n$).

Therefore

$$\bar{c}_{w_1, w_2, \ldots, w_n}\left(\sum_{\chi} u_\chi \prod_{1 \leq i \leq n} (a^{(i)}_{j_1(i)\ldots j_n(i)})^{|w_i|+1}\right) = 0$$

where χ is defined in the same way as before. However, we saw earlier that the term in brackets is a unit in $z(R)$, forcing $\bar{c}_{w_1, w_2, \ldots, w_n} = 0$ for all choices of $w_1, w_2, \ldots, w_n$. We conclude that all $c_{w_1, w_2, \ldots, w_n} = 0$ and the map is injective, as required.

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