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# Generic bifurcations of the twist coefficient

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*Abstract.* We study the behavior of the twist coefficient near an elliptic fixed point for a one-parameter family of area-preserving diffeomorphisms. By looking at the singularities near resonance we can explain the sign changes which are typically found in such a family.

## 1. *Introduction*

Let $\phi:\mathbf{R}^2\to\mathbf{R}^2$ be an area preserving diffeomorphism of the plane which fixes the origin. One calls the origin an elliptic fixed point if the matrix $D\phi(0,0)$ has eigenvalues of the form $\lambda=e^{\pm i\theta}$ for some $\theta\in(0,\pi)$ or $(\pi,2\pi)$. Thus the linear part of the mapping is essentially a rotation by angle $\theta$ around $(0,0)$. $\theta/2\pi$ will be called the *rotation number* of the fixed point. By means of an area preserving linear change of coordinates one can bring $\phi$ into the form:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\mathrm{O}(r^2).$$

Here $(x_1,y_1)$ denotes the image of $(x,y)$ under $\phi$ and $r^2=x^2+y^2$.

If the rotation number is irrational, then a sequence of nonlinear area preserving coordinate changes will bring $\phi$ into Birkhoff normal form to any given order:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}=\begin{bmatrix} \cos\Theta(r) & -\sin\Theta(r) \\ \sin\Theta(r) & \cos\Theta(r) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\mathrm{O}(r^{2N}) \tag{1.1}$$

where $\Theta(r)=\theta+\tau_1r^2+\tau_2r^4+\cdots+\tau_{N-1}r^{2N-2}$ is a polynomial in $r^2$. Thus, to the given order, $\phi$ preserves the family of circles around $(0,0)$ acting on each one as a rotation. However, if at least one of the coefficients $\tau_j$ is nonzero, the angle of rotation will vary from circle to circle. A radial line through the origin will experience a twist and for this reason the $\tau_j$ are sometimes called twist coefficients.

In this paper we will be concerned only with third order Birkhoff normal form. This is the case $N=2$ in formula (1.1). Thus we will have $\Theta(r)=\theta+\tau_1r^2$ and an error $\mathrm{O}(r^4)$. For simplicity we will drop the subscript and call $\tau=\tau_1$ the *twist coefficient.* To bring a mapping into normal form to this order it is only necessary

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to assume that $\theta \neq 0, \frac{1}{2}\pi, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, \frac{3}{2}\pi$. These values of $\theta$ will be called resonances. These are the values of $\theta$ such that $\lambda^j = 1$ for $j = 1, 2, 3, 4$; the value of $j$ is called the order of the resonance.

The main goal of this work is to study the relationship between the rotation number and the twist coefficient in a generic one parameter family of area preserving mappings. At first it seems that there is no connection between the two, for if we choose arbitrary functions $\theta(a)$, $\tau(a)$ of some parameter, $a$, we can use (1.1) to define a family of mappings $\phi_a$ with $\Theta_a(r) = \theta(a) + \tau(a)r^2$. This argument fails if $\theta(a)$ passes through the resonances. Generically, a mapping with a resonant rotation number cannot be brought into normal form. Since the family $\phi_a$ is in normal form even at resonances, it cannot be typical.

On the other hand, the one parameter family of area preserving Henon maps exhibit the phenomena of interest quite nicely. Define

$$\phi_a: \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a - y - x^2 \\ x \end{bmatrix}.$$

This map is an area preserving diffeomorphism of the plane for all values of the parameter $a$. When $a \in (-1, 3)$, there is an elliptic fixed point at $(\sqrt{1+a} - 1, \sqrt{1+a} - 1)$. A simple translation will carry the fixed point to $(0, 0)$. The rotation numbers satisfy $\cos(\theta(a)) = 1 - \sqrt{1+a}$. As $a$ varies from $-1$ to 3, $\theta$ varies monotonically from 0 to $\pi$. A computation eventually shows that the twist coefficient is given by:

$$\tau(a) = \frac{4\cos\theta + 1}{8\sin^2\theta(\cos\theta - 1)(2\cos\theta + 1)}$$

with $\theta = \theta(a)$. Figure 1 shows the graph of $\tau$ as a function of $\theta$. The singularities arise from the fact that the coordinate changes which are needed to put $\phi_a$ into normal form become singular at the resonances $\theta = 0, \frac{2}{3}\pi, \pi$.

For another example consider the family:

$$\phi_a: \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -1 - \sqrt{1 + \sin^2\theta} \\ -1 + \sqrt{1 + \sin^2\theta} & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y + x^2 + 2x^3 \end{bmatrix}.$$

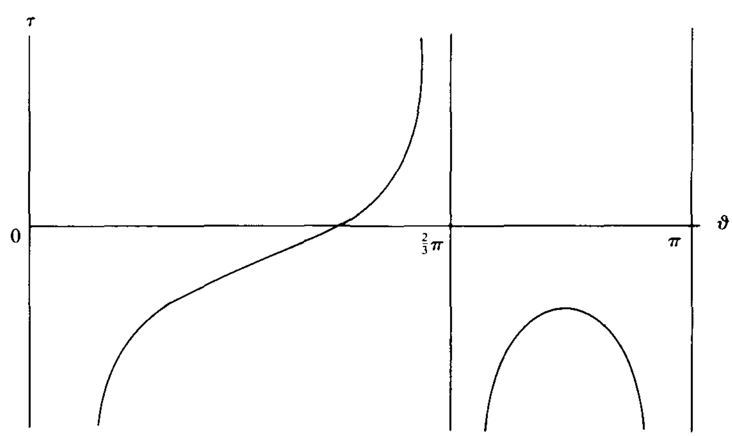


FIGURE 1. Twist versus rotation for the Henon maps.

The parameter is $\theta$ itself and the origin is an elliptic point for $\theta \in (0, \pi)$. The unusual form of the matrix will become clearer in § 3. A plot of $\tau$ versus $\theta$ based on numerical computations is shown in figure 2. Once again we find singularities at the resonances $\theta = 0, \frac{2}{3}\pi, \pi$. The character of the singularities at 0 and $\frac{2}{3}\pi$ is identical in the two examples but different at $\theta = \pi$.

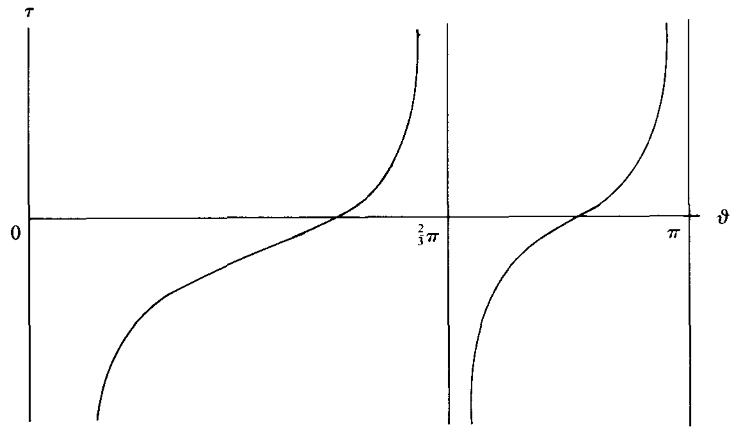


FIGURE 2. Twist versus rotation for the second example map.

A third example arises in the restricted three body problem. This problem, as is well known, concerns the motion of a point particle of negligible mass under the gravitational influence of two more massive bodies moving in a circular orbit. The problem contains one parameter, namely, the ratio of the masses of the two large bodies. A certain determinant $D$ is relevant to the question of stability of the triangular Lagrange equilibrium points; it can be interpreted as a constant multiple of the twist coefficient of an area preserving mapping [**Mos**]. This determinant has been computed explicitly [**Dep**]. When plotted as a function of the parameter it is strikingly similar to figure 1. In fact, explaining the similarity of these two figures was one of the motivations for the present paper.

We will show that in a generic family of area preserving mappings for which the rotation number varies over $(0, \frac{1}{2})$ the twist coefficient behaves like one of these two examples near the resonances. The behavior is similar as the rotation number varies over $(\frac{1}{2}, 1)$. To formulate the result more precisely we introduce a definition.

*Definition.* A function $f(x)$ has a *positive pole of order* $k$ at $x = \xi$ if for some positive number $C$:

$$f(\xi+\varepsilon) = \frac{C}{\varepsilon^k} + O\left(\frac{1}{\varepsilon^{k-1}}\right) \quad \text{as } \varepsilon \to 0.$$

We define the idea of a negative pole by the same condition with $C$ negative. Also, we say that $f$ has a pole of a certain type at $x = \xi_+$ or at $x = \xi_-$ if the condition above holds for the appropriate one sided limits.

In the case of the Henon map (figure 1) the singularities can be studied explicitly using the formula for $\tau(\theta)$; one finds a negative pole of order 4 at $0_+$, a positive pole of order 1 at $\frac{2}{3}\pi$, and a negative pole of order 2 at $\pi_-$. The following theorem asserts that these features are typical.

THEOREM. *Let $\phi_a$ be a smooth family of area preserving diffeomorphisms with elliptic fixed points of rotation number $\theta(a)$. Then generically one can parametrize by $\theta$ near a resonance and the twist coefficient $\tau(\theta)$ has a singularity of the following type:*
(i) *at $\theta = 0_+$, a negative pole of order 4; at $\theta = 2\pi_-$, a positive pole of order 4*
(ii) *at $\theta = \frac{2}{3}\pi$ or $\frac{4}{3}\pi$, a negative pole of order 1*
(iii) *at $\theta = \pi_+$ or $\pi_-$, a pole of order 2.*

There is no singularity of $\tau(\theta)$ at the fourth order resonances $\theta = \frac{1}{2}\pi$ and $\frac{3}{2}\pi$. The problem there is not with $\tau$ but rather with the elimination of the other terms of order 3 to obtain the $O(r^4)$ error estimate. In what follows, we will ignore these resonances. The large order of the poles at 0 and $\pi$ is due in part to the use of $\theta$ as a parameter. Generically the original parameter, $a$, will behave like $\cos\theta$ rather than $\theta$ and so near $\theta = 0$, $\pi$ or $2\pi$ it is of order $\theta^2$. Thus the function $\tau(a)$ will have poles of orders 2, 1, and 1 rather than 4, 1, and 2.

The singularities of the theorem have no direct dynamical significance since they are merely artifacts of the singular coordinate changes used to put the family into normal form. In fact, there are other, more complicated, normal forms for families of mappings passing through resonances which are valid even at the resonant parameter values. The remarkable part of the theorem is the fact that the signs of the poles at $0_+$, $2\pi_-$, $\frac{2}{3}\pi$ and $\frac{4}{3}\pi$ are generically determined. Some interesting consequences are described in the following corollary which is a kind of global bifurcation theorem.

COROLLARY. *In a generic one parameter family of area preserving mappings with elliptic fixed points the twist coefficient changes sign (from negative to positive) as the rotation number varies from 0 to $\frac{1}{3}$ or from $\frac{2}{3}$ to 1. In particular, the twist coefficient vanishes for some intermediate parameter value.*

What happens between rotation numbers $\frac{1}{3}$ and $\frac{1}{2}$ will depend on whether the pole at $\theta = \pi_-$ is positive or negative. If it is positive as in figure 2 then another sign change will occur. Similarly, the behavior between rotation numbers $\frac{1}{2}$ and $\frac{2}{3}$ depends on the sign of the pole at $\theta = \pi_+$.

The sign of the twist coefficient tells whether the rate of rotation around the fixed point is an increasing or decreasing function of distance from the fixed point. The corollary says, for example, that as the rotation number approaches $\frac{1}{3}$ from below, there will generically be positive twist so that points farther from the fixed point will be rotating faster than points nearer to it. On the other side of the third order resonance the situation is reversed. This harmonizes well with the generic picture of the dynamics near resonance shown in figure 3 [**Ar**, **Mey**]. Approaching the resonance from below, a period three orbit approaches the fixed point. Such a point rotates around the fixed point with rotation number $\frac{1}{3}$. Since the rotation number

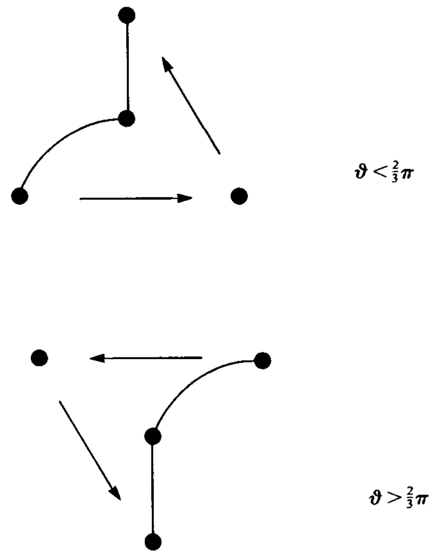


FIGURE 3. Generic bifurcation at the third order resonance.

at the fixed point is less than $\frac{1}{3}$, a radial line from the fixed point to the periodic point must experience a positive twist. After passing through the resonance, a period three orbit moves away from the fixed point. Since the rotation number at the fixed point is now greater than $\frac{1}{3}$, a radial line from the fixed point to the periodic point now experiences a negative twist.

A similar geometrical explanation is possible for the resonances at rotation numbers 0, $\frac{1}{2}$, and 1. The ambiguity of the sign of the pole at $\theta = \pi_-$ or $\pi_+$ is explained by the fact that there are two generic bifurcations possible there [**Mey**]. Approaching the bifurcation leads to positive twist in one case and to negative twist in the other.

A final remark concerning the zero of $\tau(\theta)$ is in order. To prove the existence of invariant curves, Mather sets and periodic orbits of diverse rotation numbers near an elliptic fixed point, a twist condition of some sort is required. Thus a zero of $\tau(\theta)$ is a definite inconvenience. However, it does not mean that these features are absent. In fact, generically, at a zero of $\tau_1$ the higher twist coefficients will not vanish.

## 2. *Proof of the theorem*

The proof is based on an explicit formula for $\tau(\theta)$ first computed by Wan in the context of Hopf bifurcation theory. For completeness, we will present a proof of the formula in § 3 using only area preserving coordinate changes.

We begin with an area preserving mapping $\phi: \mathbf{R}^2 \to \mathbf{R}^2$ with $\phi(0,0) = (0,0)$ and such that $D\phi(0,0)$ has eigenvalues of the form $\lambda = e^{\pm i\theta}$ for some $\theta \in [0, 2\pi)$. The third order Birkhoff normal form of $\phi$ depends only on the three-jet of $\phi$ which we write in the form:

$$J_3\phi = A(I + \phi_2 + \phi_3), \tag{2.1}$$

where $A$ is a $2\times 2$ matrix with $\det(A)=1$, $I$ is the identity map of $\mathbf{R}^2$, and $\phi_j$ denotes a homogeneous polynomial mapping of degree $j$ ($A\phi_j$ is the $j$th order part of the Taylor expansion of $\phi$). To facilitate the computation of the normal form it is convenient to introduce complex coordinates such that the linear part of $\phi$ becomes the diagonal matrix:

$$\Lambda=\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}.$$

Thus we view the three-jet of $\phi$ as a polynomial mapping $J_3\phi:\mathbf{C}^2\to\mathbf{C}^2$ preserving $\mathbf{R}^2$. Now define complex coordinates $z=x+iy$ and $w=x-iy$ where $x, y, z, w\in\mathbf{C}$. Thus $\mathbf{R}^2$ is given by the linear equation $w=\bar{z}$. If $\theta\neq 0$ or $\pi$, it will be possible to find new complex area-preserving coordinates diagonalizing $A$ and such that the equation for $\mathbf{R}^2$ is unchanged. In a one parameter family $\phi_a$ we can introduce such a coordinate change smoothly in $a$ except at the parameter values where $\theta(a)=0$ or $\pi$. After making this preliminary change of variables, the three-jet of $\phi$ takes the form:

$$\begin{aligned} z_1&=\lambda(z+c_{20}z^2+c_{11}zw+c_{02}w^2+c_{30}z^3+c_{21}z^2w+c_{12}zw^2+c_{03}w^3)\\ w_1&=\bar{\lambda}(w+\overline{c_{20}}w^2+\overline{c_{11}}wz+\overline{c_{02}}z^2+\overline{c_{30}}w^3+\overline{c_{21}}w^2z+\overline{c_{12}}wz^2+\overline{c_{03}}z^3).\end{aligned}\tag{2.2}$$

The fact that the coefficients of the formula for $w_1$ are the conjugates of the coefficients of $z_1$ follows from the fact that the mapping is real (preserves $\mathbf{R}^2$). Conversely, any mapping of this form is real. Also it is easy to show that the area preserving property of $\phi$ implies that $c_{11}=-2\overline{c_{20}}$, $|c_{20}|=|c_{02}|$, and $c_{21}+\overline{c_{21}}=0$.

If we assume that $\theta\neq\frac{2}{3}\pi$ or $\frac{4}{3}\pi$ we can make a further area preserving coordinate change which eliminates all of the quadratic terms in the mapping. If also $\theta\neq\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ all of the third order terms except the $z^2w$ term in $z_1$ and the $w^2z$ term in $w_1$ can be eliminated. It is these terms then which determine the twist coefficient, $\tau$. Carrying out this procedure leads to the following formula:

$$\tau=\frac{1}{i}\left(c_{21}+2|c_{20}|^2\left[\frac{2\lambda+1}{\lambda-1}+\frac{1}{\lambda^3-1}\right]\right).\tag{2.3}$$

The quantity in parentheses is always purely imaginary so the formula yields a real number for $\tau$. This formula can be derived from a formula of Wan [**W**] by making use of the relations $c_{11}=-2\overline{c_{20}}$ and $|c_{20}|=|c_{02}|$. However, we will outline the derivation in § 3.

If we apply this formula to a one parameter family $\phi_a$ we obtain $\tau(a)$ as a function of the rotation number and of the coefficients $c_{20}(a)$, $c_{02}(a)$ and $c_{21}(a)$ which will vary smoothly except at parameter values such that $\theta(a)=0$ or $\pi$ where the transformation diagonalizing the linear part of the map becomes singular. We will need to understand the behavior of these coefficients to prove the theorem. Since a generic family will pass through the resonances simply we can assume that $\theta$ itself is the parameter. Then we can prove:

LEMMA. *For a generic family, $|c_{20}|^2$ and $|c_{02}|^2$ have poles of order 3 at $\theta=0_+$, $\pi_-$, $\pi_+$ and $2\pi_-$ while $|c_{21}|$ has a pole of order 2.*

Since $|c_{20}|^2$ and $|c_{02}|^2$ are always non-negative, the signs of the poles will be positive for the limits from the right and negative for the limits from the left. This lemma will be proved in § 3. It allows us to identify the dominant term in the formula for $\tau$ at each resonance.

An exercise in trigonometry yields:

$$\tau(\theta)=\operatorname{im}(c_{21})+\frac{2|c_{20}|^2 \sin\theta(4\cos\theta+1)}{(\cos\theta-1)(2\cos\theta+1)}.$$

The theorem can now be proved simply by checking the behavior of $\tau(\theta)$ at each resonance, using the lemma to understand the behavior of the $c_{ij}$.

We begin with the first order resonance $\theta=0_+$. Setting $\theta=\varepsilon>0$ and keeping only dominant terms in the expansions of the trigonometric functions gives:

$$\tau(\varepsilon)=\operatorname{im}(c_{21}(\varepsilon))+\frac{\varepsilon 10|c_{20}|^2}{-3\varepsilon^2/2}+\cdots.$$

Using the lemma we find that the second term dominates the first and produces a negative pole of order 4 as claimed. We can use the same formula to study the case $\theta=2\pi_-$. This is equivalent to $\theta=0_-$ so we can just take $\varepsilon<0$ above. This leads to a positive pole of order 4.

Next we turn to the second order resonances at $\theta=\pi_-$ and $\pi_+$. Setting $\theta=\pi+\varepsilon$ we find:

$$\tau(\pi+\varepsilon)=\operatorname{im}(c_{21}(\pi+\varepsilon))+3\varepsilon|c_{20}|^2+\cdots.$$

Here both terms have poles of order 2. The nature of the bifurcation will depend on which term dominates. For example, since the Henon map is quadratic, $c_{21}=0$ and we get a negative pole of order 2 at $\pi_-$ in this case (recall $\varepsilon<0$). It is clear that generically one of the two terms will dominate and the whole expression will have a pole of order 2 of indeterminate sign.

Finally we consider the resonances of order 3 at $\theta=\frac{2}{3}\pi$ and $\frac{4}{3}\pi$. We treat only the first case, setting $\theta=\frac{2}{3}\pi+\varepsilon$. This gives:

$$\tau(\tfrac{2}{3}\pi+\varepsilon)=\operatorname{im}(c_{21}(\tfrac{2}{3}\pi+\varepsilon))+\frac{-2|c_{20}|^2}{3\varepsilon}+\cdots.$$

Since the coefficients $c_{ij}$ are smooth near this resonance we see that provided $|c_{02}|\neq 0$ there will be a negative pole of order 1; clearly this condition is generic although it can fail in the presence of symmetries. □

This completes the proof of the theorem.

## 3. *Details*

In this section we will derive the formula (2.3) for the twist coefficient used in § 2 and prove the lemma stated there. We will begin with the lemma. The singularities of the coefficients $c_{ij}(a)$ are caused by the attempt to diagonalize the linear part of the map $\phi_a$ at parameter values for which it is not diagonalizeable, namely, where $\theta(a)=0$ or $\pi$. We will consider the case $\theta(a)=0$ but the other case is entirely similar.

Consider a one parameter family of matrices $A(a)$ with $\det(A(a))=1$, i.e., $A(a)$ lies in $Sl(2, \mathbf{R})$. Since the determinant is always 1 the eigenvalues $\lambda(a)$ are completely determined by the traces $\operatorname{tr}(A(a))$. Let $\Sigma = \{A \in \mathrm{Sl}(2, \mathbf{R}) \colon \operatorname{tr}(A) = 2\}$. These are just the matrices with a repeated eigenvalue of 1. We assume that the curve $A(a)$ in $\mathrm{Sl}(2, \mathbf{R})$ crosses $\Sigma$ transversely when $a=0$, that $A(0) \neq I$, and that for $a>0$ we have $\operatorname{tr}(A(a))<2$. The second hypothesis asserts that $A(0)$ is not semisimple (diagonalize-able); this is a generic condition. The last hypothesis means that for $a>0$, the eigenvalues of $A(a)$ are of the form $\lambda = e^{\pm i\theta(a)}$ where $\cos\theta(a) = \frac{1}{2}\operatorname{tr}(A(a))$.

It is convenient to have a picture of $\Sigma$ inside $\mathrm{Sl}(2, \mathbf{R})$ and to this end we introduce coordinates $s$, $t$, $u$, $v$ on the set of $2\times 2$ matrices by setting:

$$A = \begin{bmatrix} \dfrac{t+s}{2} & \dfrac{v+u}{2} \\ \dfrac{v-u}{2} & \dfrac{t-s}{2} \end{bmatrix}.$$

Thus $t = \operatorname{tr}(A)$. In these coordinates, $\mathrm{Sl}(2, \mathbf{R}) = \{A \colon t^2 - s^2 + u^2 - v^2 = 4\}$. In the region of interest we can parametrize $\mathrm{Sl}(2, \mathbf{R})$ by $(s, u, v)$. Figure 4 shows the level surfaces of the trace $t$ in $\mathrm{Sl}(2, \mathbf{R})$. In particular, the level surface $t=2$, which is just $\Sigma$, appears as a cone in these coordinates.

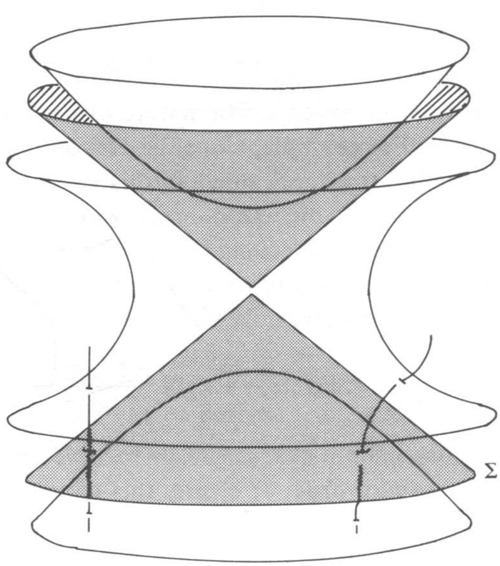


FIGURE 4. Sl (2, **R**) with families of matrices.

The vertex of the cone is the identity matrix, $I$, but the other matrices on the cone are non-semisimple. The family of matrices $A(a)$ under consideration is shown in the figure as the curve crossing the cone from the region $t>2$ where the level sets of $t$ are hyperboloids of one sheet to the region $t<2$ where the level sets are hyperboloids of two sheets. We will make a preliminary change of coordinates to smoothly conjugate the family of matrices to a family such as the one shown as a straight line segment crossing the cone in the same manner. That this is possible follows from the fact that all of the matrices in a given hyperboloid of one sheet or in a given component of a hyperboloid of two sheets or of the cone with the vertex deleted are conjugate in $\mathrm{Sl}(2, \mathbf{R})$. Specifically, we assume without loss of generality that for the family of matrices under consideration $s(a)=0$ and $v(a)=-2$ for all $a$ near $a=0$. This determines the vertical line in the figure. The remaining variables are related by $t^2+u^2=8$. When $a=0$ we will have $t=2$ and $u=-2$. We have already mentioned the fact that for $t<2$ the eigenvalues satisfy $\cos\theta(a)=\frac{1}{2}t$. Using $\theta$ as parameter, the normalized family of matrices is:

$$A(\theta)=\begin{bmatrix}\cos\theta & -1-\sqrt{1+\sin^2\theta}\\ -1+\sqrt{1+\sin^2\theta} & \cos\theta\end{bmatrix}$$

and in particular:

$$A(0)=\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}.$$

Now for $\theta\in(0,\pi)$, $A(\theta)$ is conjugate in $\mathrm{Sl}(2, \mathbf{R})$ to the matrix of rotation by $\theta$. In fact, if we set:

$$\alpha(\theta)=\sqrt{\frac{\sin\theta}{1+\sqrt{1+\sin^2\theta}}}$$

then the coordinate change $(x, y)\to(\alpha x, \alpha^{-1}y)$ conjugates $A(\theta)$ to:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

In the complex coordinates $z$, $w$ introduced in § 2, this matrix becomes the diagonal matrix $\Lambda$ and $\phi$ will take the form 2.2. Note that $\alpha(0)=0$ so that these coordinates are singular when $\theta=0$. To prove the lemma we will investigate how this singular coordinate change affects the nonlinear terms in the mapping. Beginning with (2.1) and making the coordinate change $(X, Y)=(\alpha x, \alpha^{-1}y)$ gives:

$$\begin{bmatrix}X_1\\ Y_1\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\left(\begin{bmatrix}X\\ Y\end{bmatrix}+\begin{bmatrix}\alpha f_2(\alpha^{-1}X, \alpha Y)\\ \alpha^{-1}g_2(\alpha^{-1}X, \alpha Y)\end{bmatrix}+\begin{bmatrix}\alpha f_3(\alpha^{-1}X, \alpha Y)\\ \alpha^{-1}g_3(\alpha^{-1}X, \alpha Y)\end{bmatrix}\right),$$

where $\phi_j(x, y)=(f_j(x, y), g_j(x, y))$. Now since $f_2$ and $g_2$ are quadratic polynomials and since $\alpha^{-1}$ has a pole of order $\frac{1}{2}$ the most singular of the terms of $f_2$ and $g_2$ has a pole of order $\frac{3}{2}$ (this will be the term involving $X^2$ in $g_2$). The most singular of the terms arising from $f_3$ and $g_3$ has a pole of order 2 (the term involving $X^3$ in $g_3$). Generically, these terms will not vanish. Converting to complex coordinates $z$, $w$ the mapping will take the form (2.2). The quadratic coefficients $c_{20}$, $c_{11}$, and $c_{02}$ are linear combinations of the quadratic terms above and it is easy to see that each of them inherits a pole of order $\frac{3}{2}$. Thus $|c_{20}|^2$ and $|c_{02}|^2$ will have poles of order 3 as claimed in the lemma. Similarly, the cubic terms of 2.2 are linear combinations of the cubic terms above and so inherit poles of order 2. This completes the proof of the lemma. □

The formula for $\tau$ is established by explicitly putting the mapping (2.1) into Birkhoff normal form. The first step of this process is the elimination of the quadratic terms by means of an area preserving change of coordinates under the hypothesis that $\theta \neq \frac{2}{3}\pi$ or $\frac{4}{3}\pi$. The next step would be the elimination of all cubic terms except the $z^2w$ term in $z_1$ and the $w^2z$ term in $w_1$ (which cannot be eliminated). It is not necessary to carry out this step because it does not affect the coefficients of the significant cubic terms and it is these which determine $\tau$. Thus after the quadratic terms are gone we can use the $z^2w$ term in $z_1$ and the $w^2z$ term in $w_1$ to compute $\tau$ simply ignoring the other cubic terms.

The quadratic terms will be eliminated by means of an area preserving transformation of the form:

$$\begin{aligned} Z &= z + p_2(z, w) + \cdots = z + d_{20}z^2 + d_{11}zw + d_{02}w^2 + \cdots \\ W &= w - q_2(z, w) + \cdots = w + e_{20}z^2 + e_{11}zw + e_{02}w^2 + \cdots . \end{aligned} \tag{3.1}$$

The minus sign in the second formula will be convenient later. The quadratic coefficients $d_{ij}$ and $e_{ij}$ will be determined so as to eliminate the quadratic terms in the mapping. Let $r_2(z, w)$ denote the quadratic term of $z_1$ in 2.2: $r_2(z, w) = c_{20}z^2 + c_{11}zw + c_{02}w^2$. Substituting the coordinate change (3.1) into (2.2) one finds that the quadratic term of $Z_1$ is $\lambda r_2 - (\lambda p_2 - \hat{p}_2)$ where the circumflex denotes evaluation at $(\lambda Z, \bar{\lambda} W)$ instead of $(Z, W)$. Setting this to zero determines the quadratic part of $Z$:

$$d_{20} = \frac{c_{20}}{1-\lambda} \quad d_{11} = \frac{c_{11}}{1-\bar{\lambda}} \quad d_{02} = \frac{c_{02}}{1-\bar{\lambda}^3}.$$

Because the quadratic part of the $w_1$ equation is $\bar{r}_2(w, z)$ (the bar means that all of the coefficients are conjugated) the elimination of the quadratic terms in $W_1$ requires:

$$e_{20} = -\overline{d_{02}} \quad e_{11} = -\overline{d_{11}} \quad e_{02} = -\overline{d_{20}}. \tag{3.2}$$

Since $c_{11} = -2\overline{c_{20}}$ we have $d_{11} = -2\overline{d_{20}}$ and $e_{11} = -2\overline{e_{02}}$. Using these facts and (3.2) one finds that the quadratic polynomials $p_2$ and $q_2$ satisfy:

$$\frac{\partial p_2}{\partial z} = \frac{\partial q_2}{\partial w}$$

so there is a cubic polynomial $s_3(z, w)$ with:

$$p_2 = \frac{\partial s_3}{\partial w} \quad \text{and} \quad q_2 = \frac{\partial s_3}{\partial z}.$$

Define a generating function $S(z, W) = zW + s_3(z, W)$. Then, as usual, the formulas:

$$Z = \frac{\partial S(z, W)}{\partial W} \quad w = \frac{\partial S(z, W)}{\partial z}$$

define an area preserving coordinate change in some neighborhood of the origin and this coordinate change is of the form (3.1). Thus we really can eliminate the quadratic terms of (2.2) by an area preserving coordinate change. Furthermore, it is not difficult to check that the coordinate change is real.

It remains to carry out the coordinate change to third order and determine the $Z^2W$ term in $Z_1$ and the $W^2Z$ term in $W_1$. Since the mapping is real, these coefficients will be conjugates of one another so it suffices to compute one of them. A rather laborious computation, which we have the good taste to omit, leads to the following formula for the $Z^2W$ term in $Z_1$:

$$\lambda\left(c_{21}+2|c_{20}|^2\left[\frac{2\lambda+1}{\lambda-1}+\frac{1}{\lambda^3-1}\right]\right).$$

If we write this temporarily as $\lambda \mathrm{i} T$ then we can write the mapping as:

$$Z_1=\lambda(Z+\mathrm{i}TZ^2W)+\cdots=\lambda Z(1+\mathrm{i}TZW)+\cdots$$
$$W_1=\bar{\lambda}(W-\mathrm{i}\bar{T}W^2Z)+\cdots=\bar{\lambda}W(1-\mathrm{i}\bar{T}ZW)+\cdots.$$

The Jacobian determinant of this mapping at $(0,0)$ is $1+2\mathrm{i}ZW(T-\bar{T})+\cdots$ and so by area preservation we see that $T$ is always real. Recalling that $\lambda=e^{\mathrm{i}\theta}$ we find that the mapping can be written as:

$$Z_1=e^{\mathrm{i}\Theta}Z+\cdots$$
$$W_1=e^{-\mathrm{i}\Theta}W+\cdots,$$

where $\Theta=\theta+TZW$. Converting back to the real variables $X$, $Y$ with $Z=X+\mathrm{i}Y$ and $W=X-\mathrm{i}Y$ we find that the mapping is in third order Birkhoff normal form with twist coefficient $\tau=T$. This establishes formula (2.3).

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