UNIQUE EXTENSION AND PRODUCT MEASURES

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Following **(2)** we say that a measure μ on a ring $\mathfrak{R}$ is *semifinite* if

$$\mu(E) = \text{lub}\{\mu(F); F \in \mathfrak{R}, F \subset E, \mu(F) < \infty\} \qquad \text{for every } E \in \mathfrak{R}.$$

Clearly every σ-finite measure is semifinite, but the converse fails.

In § 1 we present several reformulations of semifiniteness (Theorem 2), and characterize those semifinite measures μ on a ring $\mathfrak{R}$ that possess unique extensions to the σ-ring $\mathfrak{S}$ generated by $\mathfrak{R}$ (Theorem 3). Theorem 3 extends a classical result for σ-finite measures **(3**, 13.A**)**. Then, in § 2, we apply the results of § 1 to the study of product measures; in the process, we compare the "semifinite product measure" **(1**; **2**, pp. 127ff.**)** with the product measure described in **(4**, pp. 229ff.**)**, finding necessary and sufficient conditions for their equality; see Theorem 6 and, in relation to it, Theorem 7. Finally, in § 3, we investigate some extensions of the Fubini theory relative to semifiniteness.

1. The unique extension of a measure. Fix a set X, a ring $\mathfrak{R}$ of subsets of X, and a measure μ on $\mathfrak{R}$. Let $\mathfrak{S}$ be the σ-ring generated by $\mathfrak{R}$. We write μ^* for the outer measure induced by μ on the hereditary σ-ring $\mathfrak{H}$ generated by $\mathfrak{R}$, and $\bar{\mu}$ for the restriction of μ^* to $\mathfrak{S}$. Then $\bar{\mu}$ is a measure that extends μ **(3**, 10.A, 11.C, and 12.A**)**. Following **(2)** we write $\mathfrak{R}_\phi$ for the class of sets in $\mathfrak{R}$ of finite μ-measure; thus

$$\mathfrak{R}_\phi = \{E \in \mathfrak{R}; \mu(E) < \infty\} = \{E \in \mathfrak{R}; \bar{\mu}(E) < \infty\}.$$

Clearly $\mathfrak{R}_\phi$ is a ring of sets.

Our interest will not only involve *extensions* of μ to $\mathfrak{S}$ but also measures on $\mathfrak{S}$ which agree with μ on $\mathfrak{R}_\phi$ (called *semi-extensions* of μ). Clearly every extension of μ is a semi-extension of μ. We shall show later (Theorem 1) that $\bar{\mu}$ is the maximal semi-extension of μ (hence also the maximal extension of μ). We now seek the minimal semi-extension of μ. For each $P \in \mathfrak{R}_\phi$, the set function $\bar{\mu}_P$ defined by $\bar{\mu}_P(E) = \bar{\mu}(P \cap E)$ for all $E \in \mathfrak{S}$ is a finite measure on $\mathfrak{S}$; the family $\{\bar{\mu}_P; P \in \mathfrak{R}_\phi\}$ is increasingly directed in the obvious sense, and, defining $\tilde{\mu} = \text{lub}\{\bar{\mu}_P; P \in \mathfrak{R}_\phi\}$, we obtain a measure on $\mathfrak{S}$ **(2**, Theorem 1, p. 32**)**. Since

$$(1) \qquad \tilde{\mu}(E) = \text{lub}\{\bar{\mu}(P \cap E); P \in \mathfrak{R}_\phi\} \qquad \text{for all } E \in \mathfrak{S},$$

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it is readily seen that $\tilde{\mu} = \bar{\mu}$ on $\{P \cap E; P \in \mathfrak{R}_\phi, E \in \mathfrak{S}\}$. As a consequence, we have $\tilde{\mu} = \bar{\mu} = \mu$ on $\mathfrak{R}_\phi$ (so that $\tilde{\mu}$ is a semi-extension of μ) and

$$\tilde{\mu}(E) = \text{lub}\{\bar{\mu}(P \cap E); P \in \mathfrak{R}_\phi\} \qquad \text{for all } E \in \mathfrak{S}. \tag{2}$$

From (2) it follows that (i) $\tilde{\mu}$ is semifinite and (ii) the restriction of $\tilde{\mu}$ to $\mathfrak{R}$ (denoted $\tilde{\mu}/\mathfrak{R}$) is a semifinite measure on $\mathfrak{R}$.

$\tilde{\mu}$ need not be an extension of μ; e.g., it would not be if the range of μ is $\{0, \infty\}$. (Later we shall see that $\tilde{\mu}$ is an extension of μ if, and only if, μ is semifinite.)

One of the keys to our results is a mildly strengthened version of (**4**, 12.10). For that result and what follows it will be convenient to let $\mathfrak{S}_\phi$ denote $\{E \in \mathfrak{S}; \bar{\mu}(E) < \infty\}$; thus $\mathfrak{S}_\phi$ is simply the class of all sets $E \in \mathfrak{S}$ such that $E \subset \bigcup F_n$ for some sequence $F_n \in \mathfrak{R}$ with $\sum \mu(F_n) < \infty$.

LEMMA 1. *Let ν be a semi-extension of μ. Then $\nu = \bar{\mu}$ on $\mathfrak{S}_\phi$; in particular, $\nu \leqslant \bar{\mu}$.*

The proof of Lemma 1 is not hard to deduce from that of (**4**, 12.10).

THEOREM 1. *Let ν be a semi-extension of μ. Then*
(i) $\tilde{\mu} = \nu = \bar{\mu}$ *on* $\mathfrak{S}_\phi$,
(ii) $\tilde{\mu} \leqslant \nu \leqslant \bar{\mu}$.

Proof. In view of Lemma 1 and the fact that $\tilde{\mu}$ is a semi-extension of μ, it suffices to show that $\tilde{\mu} \leqslant \nu$ whenever ν is a semi-extension of μ.

Now $\tilde{\mu} = \nu$ on $\{P \cap E; P \in \mathfrak{R}_\phi, E \in \mathfrak{S}\} \subset \mathfrak{S}_\phi$ by Lemma 1. It follows from (2) that

$$\begin{aligned} \tilde{\mu}(E) &= \text{lub}\{\nu(P \cap E); P \in \mathfrak{R}_\phi\} \\ &\leqslant \nu(E) \qquad \text{for all } E \in \mathfrak{S}. \end{aligned}$$

We now obtain some necessary and sufficient conditions for μ to be semifinite:

THEOREM 2. *The following are equivalent:*
(i) μ *is semifinite.*
(ii) $\tilde{\mu}$ *is an extension of* μ.
(iii) *Every semi-extension of* μ *is an extension of* μ.

Proof. (i) $\Rightarrow$ (iii): Let ν be a semi-extension of μ. Then $\nu/\mathfrak{R} \leqslant \bar{\mu}/\mathfrak{R} = \mu$ by Theorem 1 (ii). On the other hand, since μ is semifinite and $\nu = \mu$ on $\mathfrak{R}_\phi$, it follows that for every $E \in \mathfrak{R}$,

$$\mu(E) = \text{lub}\{\nu(F); F \in \mathfrak{R}_\phi, F \subset E\} \leqslant \nu(E).$$

Hence $\nu/\mathfrak{R} = \mu$, so ν is an extension of μ.

That (iii) implies (ii) and (ii) implies (i) is obvious.

Unfortunately, the semifiniteness of μ is not enough to ensure that of $\bar{\mu}$ (see the last paragraph of § 2). What is true along these lines is contained in the following theorem, which, moreover, generalizes the unique extension theorem (**3**, 13.A). In particular, we find that the semifiniteness of $\bar{\mu}$ is sufficient to guarantee uniqueness of the extension of μ.

THEOREM 3. *The following are equivalent:*

(i) $\bar{\mu}$ *is semifinite.*

(ii) μ *is semifinite and there exists a unique extension of* μ.

(iii) $\tilde{\mu} = \bar{\mu}$.

(iv) *There exists a unique semi-extension of* μ.

(v) $\bar{\mu}(E) = \text{lub}\{\bar{\mu}(P \cap E); P \in \mathfrak{R}_\phi\}$ *for all* $E \in \mathfrak{S}$.

Proof. (i) $\Rightarrow$ (iii): Since $\bar{\mu} = \tilde{\mu}$ on $\mathfrak{S}_\phi$ (Theorem 1 (i)) and $\bar{\mu}$ is semifinite, one has

$$\bar{\mu}(E) = \text{lub}\{\bar{\mu}(F); F \subset E, F \in \mathfrak{S}_\phi\} \leqslant \tilde{\mu}(E) \qquad \text{for all } E \in \mathfrak{S}.$$

Hence $\tilde{\mu} = \bar{\mu}$ since $\tilde{\mu} \leqslant \bar{\mu}$ in general (Theorem 1 (ii)).

(iii) $\Leftrightarrow$ (iv): Theorem 1 (ii).

(iii) $\Leftrightarrow$ (v): Equation (1).

(iii) $\Rightarrow$ (ii): Since $\mu = \bar{\mu}/\mathfrak{R} = \tilde{\mu}/\mathfrak{R}$, μ is semifinite. The proof is completed by noting that any extension of μ is also a semi-extension of μ, hence is unique by the criterion (iv) (which is equivalent to (iii)).

(ii) $\Rightarrow$ (i): Since μ is semifinite, $\tilde{\mu}$ is an extension of μ by Theorem 2. But the extension of μ is unique so that $\bar{\mu} = \tilde{\mu}$, a semifinite measure.

COROLLARY 1. *If* $\bar{\mu}$ *is* σ*-finite, then* μ *has a unique extension.*

Proof. Every σ-finite measure is semifinite.

COROLLARY 2. (*Unique Extension Theorem*). *If* μ *is* σ*-finite, then* μ *has a unique extension.*

Proof. Obviously $\bar{\mu}$ is σ-finite.

Actually, Corollary 1 can be deduced from Corollary 2, since it is easy to show that μ is σ-finite if, and only if, $\bar{\mu}$ is σ-finite. That it does not suffice in Corollary 1 to simply assume that some extension of μ is σ-finite is illustrated by (**3**, Exercise 5, p. 57).

Finally, it follows from Theorem 3 that since the semifiniteness of μ is not sufficient to ensure that of $\bar{\mu}$, it is likewise not sufficient to ensure that the extension of μ is unique.

2. Application to product measures. The application of § 1 to product measures follows. For the remainder of the paper, let $(X, \mathbf{S}, \mu)$, $(Y, \mathbf{T}, \nu)$ be two fixed arbitrary measure spaces in the sense of (**3**, p. 73). (Indeed, what follows will be of interest only when at least one of $(X, \mathbf{S}, \mu)$ and $(Y, \mathbf{T}, \nu)$ is not σ-finite.) We shall use $\mathbf{S}_\phi$ and $\mathbf{T}_\phi$ to denote $\{P \in \mathbf{S}; \mu(P) < \infty\}$ and

$\{Q \in \mathbf{T}; \nu(Q) < \infty\}$, respectively. Clearly $\mathbf{S}_\phi$ and $\mathbf{T}_\phi$ are rings. The set of all finite, disjoint unions of measurable rectangles is a ring $\mathfrak{R}$ of subsets of $X \times Y$ (**3**, 33.E). As is customary, we use $\mathbf{S} \times \mathbf{T}$ to denote the σ-ring generated by $\mathfrak{R}$.

It is a routine exercise to show that there exists a unique measure π on $\mathfrak{R}$ such that $\pi(E \times F) = \mu(E)\nu(F)$ for all $E \in \mathbf{S}$, $F \in \mathbf{T}$. (The essentials needed to verify this are contained in (**4**, pp. 229ff.) although the approach there is slightly different.) Applying the framework of § 1 to π (i.e., letting π play the role of μ there), we find that (i) the product measure introduced in (**4**, pp. 229ff.) is a "completion" of $\tilde{\pi}$ to all π^*-measurable sets and that (ii) $\tilde{\pi}$ is the "semifinite product measure" (**1**; **2**, pp. 127ff., in particular, Theorem 1, p. 129). That is, $\tilde{\pi}$ is the unique measure on $\mathbf{S} \times \mathbf{T}$ satisfying

$$\text{(I)} \qquad \tilde{\pi}(P \times Q) = \mu(P)\nu(Q) \qquad \text{for all } P \in \mathbf{S}_\phi, Q \in \mathbf{T}_\phi$$

and

$$\text{(II)} \qquad \tilde{\pi}(M) = \text{lub}\{\tilde{\pi}[(P \times Q) \cap M]; P \in \mathbf{S}_\phi, Q \in \mathbf{T}_\phi\}$$

for all $M \in \mathbf{S} \times \mathbf{T}$.

(I) is a consequence of the following facts: (i) $\tilde{\pi} = \pi$ on $\mathfrak{R}_\phi$ and (ii)

$$\{P \times Q; P \in \mathbf{S}_\phi, Q \in \mathbf{T}_\phi\} \subset \mathfrak{R}_\phi.$$

The latter fact also yields

$$\text{lub}\{\tilde{\pi}[(P \times Q) \cap M]; P \in \mathbf{S}_\phi, Q \in \mathbf{T}_\phi\} \leqslant \text{lub}\{\tilde{\pi}(R \cap M); R \in \mathfrak{R}_\phi\} \qquad \text{for all } M \in \mathbf{S} \times \mathbf{T}.$$

Thus according to § 1, Equation (2), in order to verify (II) it will suffice to show the opposite inequality. Let $R = \bigcup(E_i \times F_i)$ be a representation of $R \in \mathfrak{R}_\phi$ as a finite, disjoint union of measurable rectangles, each of which is necessarily of finite $\tilde{\pi}$-measure; then

$$\bigcup{}'(E_i \times F_i) \subset (\bigcup{}' E_i) \times (\bigcup{}' F_i)$$

where $\bigcup'$ in each case denotes the union over all i such that $\tilde{\pi}(E_i \times F_i)$ is non-zero (equivalently, all i such that $\mu(E_i)$ and $\nu(F_i)$ are non-zero and finite). Letting $P = \bigcup' E_i$, $Q = \bigcup' F_i$, we have $P \in \mathbf{S}_\phi$, $Q \in \mathbf{T}_\phi$, and

$$\tilde{\pi}(R \cap M) = \tilde{\pi}[(\bigcup{}'(E_i \times F_i)) \cap M] \leqslant \tilde{\pi}[(P \times Q) \cap M] \qquad \text{for all } M \in \mathbf{S} \times \mathbf{T}.$$

A measure λ on $\mathbf{S} \times \mathbf{T}$ is *multiplicative* on a measurable rectangle $E \times F$ if $\lambda(E \times F) = \mu(E)\nu(F)$. Again referring to the framework of § 1, we see that:

(i) The set of all semi-extensions of π is the set of all measures on $\mathbf{S} \times \mathbf{T}$ which are multiplicative on every measurable rectangle $E \times F$ satisfying $\mu(E)\nu(F) < \infty$ (equivalently, multiplicative on every measurable rectangle

$E \times F$ such that either (a) both E and F are of σ-finite measure or (b) $\mu(E) = 0$ or $\nu(F) = 0$).

(ii) The set of all extensions of π is the set of all measures on $\mathbf{S} \times \mathbf{T}$ which are multiplicative on every measurable rectangle.

In view of the preceding, we shall call an extension of π a *product measure* on $\mathbf{S} \times \mathbf{T}$ and a semi-extension of π a *pseudo-product measure* on $\mathbf{S} \times \mathbf{T}$.

We now translate into their present context the results of § 1. In accordance with the notation there, $(\mathbf{S} \times \mathbf{T})_\phi$ signifies $\{M \in \mathbf{S} \times \mathbf{T};\ \bar{\pi}(M) < \infty\}$.

THEOREM 4. *Let* λ *be a pseudo-product measure on* $\mathbf{S} \times \mathbf{T}$. *Then*
(i) $\tilde{\pi} = \lambda = \bar{\pi}$ *on* $(\mathbf{S} \times \mathbf{T})_\phi$.
(ii) $\tilde{\pi} \leqslant \lambda \leqslant \bar{\pi}$.

THEOREM 5. *The following are equivalent*:
(i) π *is semifinite.*
(ii) $\tilde{\pi}$ *is a product measure on* $\mathbf{S} \times \mathbf{T}$.
(iii) *Every pseudo-product measure on* $\mathbf{S} \times \mathbf{T}$ *is a product measure on* $\mathbf{S} \times \mathbf{T}$.

THEOREM 6. *The following are equivalent*:
(i) $\bar{\pi}$ *is semifinite.*
(ii) π *is semifinite and there exists a unique product measure on* $\mathbf{S} \times \mathbf{T}$.
(iii) $\tilde{\pi} = \bar{\pi}$.
(iv) *There exists a unique pseudo-product measure on* $\mathbf{S} \times \mathbf{T}$.
(v) $\bar{\pi}(M) = \text{lub}\{\bar{\pi}[(P \times Q) \cap M];\ P \in \mathbf{S}_\phi,\ Q \in \mathbf{T}_\phi\}$ *for all* $M \in \mathbf{S} \times \mathbf{T}$.

The relationship between the semifiniteness of μ and ν and that of π is as follows:

THEOREM 7. (i) *If* μ *and* ν *are semifinite, then* π *is semifinite.*
(ii) *Conversely, if* $\mu, \nu \neq 0$ *and* π *is semifinite, then* μ *and* ν *are semifinite.*

Theorem 7 follows immediately from Theorem 5 and the following slight improvement of **(2**, Exercise 18, p. 133**)**:

LEMMA 2. (i) *If* μ *and* ν *are semifinite, then* $\tilde{\pi}$ *is a product measure on* $\mathbf{S} \times \mathbf{T}$.
(ii) *Conversely, if* $\mu, \nu \neq 0$ *and* $\tilde{\pi}$ *is a product measure on* $\mathbf{S} \times \mathbf{T}$, *then* μ *and* ν *are semifinite.*

Proof. (i) An application of property (II) of $\tilde{\pi}$ to each measurable rectangle, along with the semifiniteness of μ and ν, yields the result.

(ii) Since $\mu, \nu \neq 0$ and $\tilde{\pi}$ is a product measure on $\mathbf{S} \times \mathbf{T}$, it follows that $\tilde{\pi} \neq 0$. Hence neither μ nor ν has range $\{0, \infty\}$ since otherwise property (II) of $\tilde{\pi}$ would imply that $\tilde{\pi} = 0$. The result then follows quite easily from property (II) of $\tilde{\pi}$ and the fact that $\tilde{\pi}$ is multiplicative on every measurable rectangle.

If $\mu = 0$, then $\pi = 0$ whatever be ν; thus the assumption that $\mu, \nu \neq 0$ cannot be delected in Theorem 7 (ii) or in Lemma 2 (ii).

In the example of (**3**, Exercise 1, p. 145), π is semifinite since μ and ν are. It is easy to show, however, that $\bar{\pi}(D) = 0$ whereas $\tilde{\pi}(D) > 0$ (thus $\tilde{\pi}(D) = \infty$ by Theorem 4 (i)). Hence $\tilde{\pi}$ is not semifinite (Theorem 6).

3. Remarks on the Fubini theory. No discussion of product measures is complete without at least a few words on the Fubini theory. By the preceding example (**3**, Exercise 1, p. 145), both the "integrable form" (**3**, 36.C; **2**, Theorem 1, p. 142) and the "non-negative form" (**3**, 36.B) of Fubini's theorem fail for $\tilde{\pi}$ and the latter is not valid for $\bar{\pi}$, even if μ and ν are semifinite. It follows from the discussion in (**4**, pp. 231–233) that the "integrable form" is valid for $\bar{\pi}$ in general. Using this, we can show that the "partial converse" (**2**, Theorem 2, p. 143) of the "integrable form" is valid if $\bar{\pi}$ is semifinite. The proof proceeds as follows with notation as in (**2** and **3**): Say $\iint h\, d\nu\, d\mu$ exists and is finite. Let

$$A_n = \{(x, y); h(x, y) \geqslant 1/n\}, \qquad n = 1, 2, \ldots .$$

Fix n. Since $\bar{\pi}$ is semifinite, there are sets $M_{mn} \in (\mathbf{S} \times \mathbf{T})_\phi$ such that $M_{mn} \subset A_n$ for all m and $\bar{\pi}(M_{mn}) \uparrow \bar{\pi}(A_n)$ as $m \uparrow \infty$. Then $(1/n)\chi_{M_{mn}} \leqslant h$ for all m so that $(1/n)\nu((M_{mn})_x) \leqslant \int h_x\, d\nu$ for all $x \in X$ and all m. It follows then from the "integrable form," since $\chi_{M_{mn}}$ is $\bar{\pi}$-integrable for all m, that

$$(1/n)\bar{\pi}(M_{mn}) \leqslant \iint h\, d\nu\, d\mu = K < \infty \qquad \text{for all } m.$$

Hence $\bar{\pi}(A_n) \leqslant nK < \infty$. Thus $\{(x, y); h(x, y) \neq 0\}$ is of σ-finite $\bar{\pi}$-measure and the rest of the proof follows that of (**2**, Theorem 2, p. 143).

One more question arises quite naturally: If $\bar{\pi}$ is semifinite, is the "non-negative form" of Fubini's theorem valid for $\bar{\pi}$ $(=\tilde{\pi})$? The problem is one of measurability: If the functions $f(x) = \nu(M_x)$ and $g(y) = \mu(M^y)$ are measurable for all $M \in \mathbf{S} \times \mathbf{T}$, then it follows easily from the "integrable form" and its "partial converse" that the answer is "yes." (Note that by (**4**, pp. 231–232) it is true in general that f and g are almost everywhere equal to measurable functions for all $M \in \mathbf{S} \times \mathbf{T}$ of σ-finite $\bar{\pi}$-measure and are, in fact, measurable for all such M if μ and ν are complete measures.) However, examples exist in which $\bar{\pi}$ is semifinite and μ and ν are complete measures; yet there is a set $M \in \mathbf{S} \times \mathbf{T}$ such that $f(x) = \nu(M_x)$ is not measurable. For example, we may take $X = Y = [0, 1]$, $\mathbf{S} = \mathbf{T} =$ Borel subsets of $[0, 1]$, and define $\mu = \nu$ as follows: Let F be a non-Borel subset of $[0, 1]$ and let μ be the unique measure on $\mathbf{S}$ determined by defining $\mu(\{x\}) = 2$ if $x \in F$ and $\mu(\{x\}) = 1$ if $x \notin F$. Then clearly $\bar{\pi}$ is semifinite and μ and ν are complete measures; but

$$\nu(D_x) = \chi_F(x) + 1$$

is not measurable, where $D = \{(x, y) \in X \times Y; x = y\}$ is the diagonal of $X \times Y$.

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References

1. S. K. Berberian, *The product of two measures*, Amer. Math. Monthly, *69* (1962), 961–968.
2. ——— *Measure and integration* (New York, 1965).
3. P. R. Halmos, *Measure theory* (New York, 1950).
4. H. L. Royden, *Real analysis* (New York, 1963).

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