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NEAREST POINTS TO CLOSED SETS AND DIRECTIONAL DERIVATIVES OF DISTANCE FUNCTIONS

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We investigate the circumstances under which the distance function to a closed set in a Banach space having a one-sided directional derivative equal to 1 or -1 implies the existence of nearest points. In reflexive spaces we show that at a dense set of points outside a closed set the distance function has a directional derivative equal to 1.

1. Introduction

Let K be a closed nonempty subset of a Banach space X. The distance function

$$d(x) = \inf\{\|x - z\| : z \in K\}$$

is Lipschitzian of rank 1 so that for $\|y\| = 1$ we have

$$-1 \leqslant \liminf_{t \to 0+} \frac{d(x+ty) - d(x)}{t} \leqslant \limsup_{t \to 0+} \frac{d(x+ty) - d(x)}{t} \leqslant 1.$$

If the one-sided directional derivative

$$d'_+(x)(y) = \lim_{t \to 0+} \frac{d(x+ty) - d(x)}{t}$$

exists, then $\left|d'_+(x)(y)\right| \leqslant 1$ if $\|y\| = 1$. In this note we investigate the circumstances under which $d'_+(x)\left(\vec{x}\right)$ can equal 1 or -1 for some unit vector $\vec{x}$.

As shown by our previous work [**4, 5**] and by Zajicek [**10**], differentiability properties of d are related to nonemptiness and continuity of the metric projection

$$P(x) = \{z \in K : \|z - x\| = d(x)\}.$$

In Section 2 we give a geometric condition on the Banach space X and a unit vector $\vec{x}$ which is necessary and sufficient for $\left|d'_+(x)\left(\vec{x}\right)\right| = 1$ to imply that $P(x)$ is nonempty. It is not possible to deduce continuity of P at x from $d'_+(x)\left(\vec{x}\right) = -1$ but if the norm is locally uniformly convex at $\vec{x}$ and $d'_+(x)\left(\vec{x}\right) = 1$ then P is continuous at x.

In Section 3 we show that if X is reflexive then there is a dense subset D of $X \setminus K$ such that if $v \in D$ there is $\vec{v} \in X$ with $\left\|\vec{v}\right\| = 1$ and $d'_+(v)\left(\vec{v}\right) = 1$ and $d'_+(v)\left(-\vec{v}\right) = -1$.

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2. Existence of nearest points

If $P(x)$ is nonempty and $x \in X \setminus K$ then there is $\overrightarrow{x} \in X$ with $\left\| \overrightarrow{x} \right\| = 1$ and $d'_+(x)\left(\overrightarrow{x}\right) = -1$: simply take any $\overrightarrow{x}$ with $x + d(x)\overrightarrow{x} \in P(x)$. The following calculation is useful for constructing examples.

Lemma 2.1. *If $\|y_n\| = 1$ and $\lim_{n\to\infty} \left\| \overrightarrow{x} + y_n \right\| = 2$ and $\left\| \overrightarrow{x} \right\| = 1$ let $K = \{z_n : b \in \mathbb{N}\}$ where $z_n = \left(1 + \frac{1}{n}\right) \left\| \overrightarrow{x} + y_n \right\|^{-1} \left(\overrightarrow{x} + y_n\right)$. Then $d'_+(0)\left(\overrightarrow{x}\right) = -1 = -d'_+(0)\left(-\overrightarrow{x}\right)$.*

Proof: If $\alpha_N = \left(1 + \frac{1}{n}\right) \left\| \overrightarrow{x} + y_n \right\|^{-1}$, we have by convexity of the norm $t^{-1}\left(\left\| z_n - t\overrightarrow{x} \right\| - \|z_n\|\right) \leqq \alpha_n^{-1}\left(\left\| z_n - \alpha_n \overrightarrow{x} \right\| - \|z_n\|\right)$ whenever $1 \leqq \alpha_n$. Thus

$$\begin{aligned}
-1 \leqq \liminf_{t\to 0} \frac{d\left(t\overrightarrow{x}\right) - d(0)}{t} &\leqq \limsup_{t\to 0} \frac{d\left(t\overrightarrow{x}\right) - d(0)}{t} \\
&= \limsup_{t\to 0} t^{-1} \inf_n \left(\left\| t\overrightarrow{x} - z_n \right\| - 1\right) \\
&= \limsup_{t\to 0} t^{-1} \inf_n \left(\left\| z_n - t\overrightarrow{x} \right\| - \|z_n\| + \frac{1}{n}\right) \\
&\leqq \limsup_{n\to\infty} \alpha_n^{-1} \left(\left\| z_n - \alpha_n \overrightarrow{x} \right\| - \|z_n\|\right) \\
&= \lim_{n\to\infty} \left(\|y_n\| - \left\| \overrightarrow{x} + y_n \right\|\right) = -1
\end{aligned}$$

so that $d'_+(0)\left(\overrightarrow{x}\right) = -1 = -d'_+(0)\left(-\overrightarrow{x}\right)$. ∎

Theorem 2.2. *Let X be a Banach space and $\overrightarrow{x} \in X$ with $\left\| \overrightarrow{x} \right\| = 1$. The following statements are equivalent:*

(a) *if K is nonempty closed subset of X and $x \in X \setminus K$ with $d'_+(x)\left(\overrightarrow{x}\right) = -1$ then x has a nearest point in K;*

(b) *if K is a nonempty closed subset of X and $x \in X \setminus K$ with $\liminf_{t\to 0+} \left(d\left(x + t\overrightarrow{x}\right) - d(x)\right)/t = -1$ then x has a nearest point in K;*

(c) *if $\|y_n\| = 1$ and $\lim_{n\to\infty} \left\| \overrightarrow{x} + y_n \right\| = 2$ then (y_n) has a convergent subsequence.*

Proof: Clearly (b) implies (a). Suppose (c) holds and, to prove (b), let $t_n \to 0+$ with $\lim_{n\to\infty} \left(d\left(x + t_n \overrightarrow{x}\right) - d(x)\right)/t_n = -1$. Choose $z_n \in K$ with $\left\| x + t_n \overrightarrow{x} - z_n \right\| <$

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$d\left(x+t_n \vec{x}\right)+t_n^2$. Then

$$\left(d\left(x+t_n\vec{x}\right)-d(x)\right)/t_n > \left(\left\|x+t_n\vec{x}-z_n\right\|-\|x-z_n\|\right)/t_n - t_n$$
$$\geqq -t_n+\left(\|x-z_n\|-\|x-z_n-\|x-z_n\|\vec{x}\|\right)/\|x-z_n\| = -t_n+1-\left\|\vec{x}+y_n\right\|$$

where $y_n = -\|x-z_n\|^{-1}(x-z_n)$. Thus $\|y_n\|=1$ and $\left\|\vec{x}+y_n\right\|\to 2$ so (y_n) has a convergent subsequence $\left(y_{n_j}\right)$, $\left(z_{n_j}\right)$ converges to a point $z\in K$ (since K is closed) and $\|x-z\|=d(x)$. Finally suppose there is a sequence (y_n) with $\|y_n\|=1$ and $\left\|\vec{x}+y_n\right\|\to 2$ but (y_n) has no convergent subsequence. Then $K=\{(1+\frac{1}{n})\left\|\vec{x}+y_n\right\|^{-1}\left(\vec{x}+Y_n\right): n\in \mathbb{N}\}$ is a closed set and 0 has no nearext point in K. However Lemma 2.1 shows that $d'_+(0)\left(\vec{x}\right)=-1$, contradicting (a). ∎

If $d'_+(x)\left(\vec{x}\right)=1$ and $\left\|\vec{x}\right\|=1$ we can get a similar result or we can show continuity of P at x under a stronger hypothesis. We say that (z_n) is a *minimising sequence for* x if $z_n\in K$ and $\lim\limits_{n\to\infty}\|x-z_n\|=d(x)$.

PROPOSITION 2.3. *Suppose* $x\in X\setminus K$ *and* $\vec{x}\in X$ *with* $\left\|\vec{x}\right\|=1$ *and* $\limsup\limits_{t\to 0+}\left(d\left(x+t\vec{x}\right)-d(x)\right)/t = 1$. *If* (z_n) *is a minimising sequence for* x *and* $y_n=\|x-z_n\|^{-1}(x-z_n)$ *then* $\left\|\vec{x}+y_n\right\|\to 2$.

PROOF: Let $r_n\to 0+$ so that $\lim\limits_{n\to\infty}\left(d\left(x+t_n\vec{x}\right)-d(x)\right)/t_n=1$. We may assume that $t_n<d(x)$ and $t_n^2>\|x-z_n\|-d(x)$. Now

$$t_n^{-1}\left(d\left(x+t\vec{x}\right)-d(x)\right)\leqslant t_n^{-1}\left(\left\|z+t_n\vec{x}-z_n\right\|-\|x-z_n\|+t_n^2\right)$$
$$\leqq \|x-z_n\|^{-1}\left(\|x-z_n+\|x-z_n\|\vec{x}\|-\|x-z_n\|\right)+t_n$$
$$=\left\|\vec{x}+y_n\right\|-1+t_n.$$

Thus $\liminf\limits_{n\to\infty}\left\|\vec{x}+y_n\right\|=2$. Since $\left\|\vec{x}+y_n\right\|\leqq 2$ we have $\left\|\vec{x}+y_n\right\|\to 2$. ∎

We use this to get the analogue of Theorem 2.2.

THEOREM 2.4. *Let* X *be a Banach space and* $\vec{x}\in X$ *with* $\left\|\vec{x}\right\|=1$. *The following statements are equivalent:*

(a) *for each closed nonempty subset* K *of* X *and* $x\in X\setminus K$, *if* $d'_+(x)\left(\vec{x}\right)=1$ *then* K *has a nearest point to* x;

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(b) *for each closed nonempty subset* K *of* X *and* $x \in X \setminus K$*,if* $\limsup_{t\to 0+} \left(d\left(x + t\,\vec{x}\right) - d(x)\right)/t = 1$ *then every minimising sequence for* x *has a convergent subsequence;*

(c) *if* $\|y_n\| = 1$ *and* $\lim_{n\to\infty} \left\|\vec{x} + y_n\right\| = 2$ *then* (y_n) *has a convergent subsequence.*

PROOF: Clearly (b) implies (a). Assume (c) and suppose $x \in X \setminus K$ with $\limsup_{t\to 0+} \left(d\left(x + t\,\vec{x}\right) - d(x)\right)/t = 1$. Then any minimising sequence (z_n) has $\left\|\vec{x} + y_n\right\| \to 2$ where $y_n = \|x - z_n\|^{-1}(x - z_n)$, by Proposition 2.3. Therefore (y_n) has a convergent subsequence $\left(y_{n_j}\right)$ and $\left(z_{n_j}\right)$ is convergent because $\|x - z_n\| \to d(x) > 0$.

Finally, suppose (c) does not hold so there are y_n with $\|y_n\| = 1$ and $\lim_{n\to\infty} \left\|\vec{x} + y_n\right\| = 2$ but (y_n) has no convergent subsequence. Let $K = \{-(1 + \frac{1}{n}) \left\|\vec{x} + y_n\right\|^{-1} \left(\vec{x} + y_n\right) : n \in \mathbb{N}\}$. Then K is closed and 0 has no nearest point in K. But Lemma 2.1 shows that $d'_+(0)\left(\vec{x}\right) = 1$. ∎

Recall that X is *locally uniformly convex* at $\vec{x}$ with $\left\|\vec{x}\right\| = 1$ provided every sequence (y_n) with $\|y_n\| = 1$ and $\left\|\vec{x} + y_n\right\| \to 2$ has $\left\|\vec{x} - y_n\right\| \to 0$.

THEOREM 2.5. *Let* X *be a Banach space and* $\vec{x} \in X$ *with* $\left\|\vec{x}\right\| = 1$. *The following statements are equivalent:*

(a) *for each nonempty closed set* K *and* $x \in X \setminus K$, *if* $d'_+(x)\left(\vec{x}\right) = 1$ *then* $P(x)$ *has exactly one element;*

(b) *for each nonempty closed set* K *and* $x \in X \setminus K$, *if* $\limsup_{t\to 0+} \left(d\left(x + t\,\vec{x}\right) - d(x)\right)/t = 1$ *then every minimising sequence for* x *converges to* $x - d(x)\,\vec{x}$ *and* P *is continuous at* x*;*

(c) X *is locally uniformly convex at* $\vec{x}$.

PROOF: Clearly (b) implies (a). Assume (c) and let $\limsup_{t\to 0+} \left(d\left(x + t\,\vec{x}\right) - d(x)\right)/t = 1$ and $x \in X \setminus K$. Suppose (z_n) is a minimising sequence for x. By Proposition 2.3, $y_n = \|x - z_n\|^{-1}(x - z_n)$ has $\left\|\vec{x} + y_n\right\| \to 2$. Since $\|y_n\| = 1$ we have $\left\|\vec{x} - y_n\right\| \to 0$ so that $z_n \to x - d(x)\,\vec{x}$. The continuity of P at x follows immediately.

Finally, suppose X is not locally uniformly convex at $\vec{x}$. Then there are $y_n \in K$ with $\|y_n\| = 1$ and $\left\|\vec{x} + y_n\right\| \to 2$, but $\left\|\vec{x} - y_n\right\| \geqslant \delta > 0$ for all n. If (y_n) has no convergent subsequence we can use Theorem 2.4 to get a closed set K, and $x \in X \setminus K$ so that $d'_+(x)\left(\overrightarrow{x}\right) = 1$ and $P(x) = \emptyset$. Otherwise some subsequence (y_n) converges to a point y of X with $\left\|\overrightarrow{x} - y\right\| \geqslant \delta > 0$ and $\left\|\overrightarrow{x} + y\right\| = 2$. Since $\|y\| = 1$ for $K = \{-\overrightarrow{x}, -y\}$ we have $d'_+(0)\left(\overrightarrow{x}\right) = 1$, but $P(0) = \{-\overrightarrow{x}, y\}$ contradicting (a). ∎

We note that no geometric condition on the norm combined with $d'_+(x)\left(\overrightarrow{x}\right) = -1$ can give single-valuedness of the metric projection at x: let $x = 0$ and K be the unit sphere of X, for example.

3. Dense sets of points with one-sided derivative 1 or −1

We start with an example to show that we need to consider reflexive Banach spaces.

Example 3.1. If X is a nonreflexive Banach space, let x^* be any element of X^* such that $\|x^*\| \cdot \|x\| > x^*(x)$ for all $x \neq 0$. These exist by James' Theorem [**7**]. Then $K = \ker x^*$ is closed subset of X with $|d'_+(x)(y)| < 1$ for all $x \in X \setminus K$ and $\|y\| = 1$.

Indeed, we have $d(x) = |x^*(x)|$ for every $x \in X$ so that $d'_+(x)(y) = \operatorname{sgn}(x^*(x))x^*(y)$.

Preiss [**9**] (see [**1**], p.523) has shown that any Lipschitzian function on a Banach space X which is an Asplund space is Fréchet differentiable at a dense set of points of X.

THEOREM 3.2. *If X is an Asplund space and K is a closed nonempty subset of X then there is a dense set of points in $X \setminus K$ at which d is Fréchet differentiable with derivative having norm 1.*

PROOF: Using Preiss' result we only need to show that if d is Fréchet differentiable at a point $x \in X \setminus K$ then $\|d'(x)\| = 1$. But we showed this in [**4**], Theorem 2.6. ∎

COROLLARY 3.3. *If X is a reflexive Banach space and K is a closed subset of X then there is a dense subset D of $X \setminus K$ such that for each $x \in D$ there is $\overrightarrow{x} \in X$ with $\left\|\overrightarrow{x}\right\| = 1$, $d'_+(x)\left(\overrightarrow{x}\right) = 1$ and $d'_+(x)\left(-\overrightarrow{x}\right) = -1$.*

PROOF: Let D be the dense set given in Theorem 3.2. For each $x \in D$ let $\overrightarrow{x}$ be any element with $\left\|\overrightarrow{x}\right\| = 1$ and $d'(x)\left(\overrightarrow{x}\right) = 1$. ∎

This corollary together with Theorem 2.4 (or Theorem 2.2) and Theorem 2.5 show that in a reflexive space with Kadec norm there is a dense set D of points in $X \setminus K$ such that each $x \in D$ has a nearest point K, and if the norm is locally uniformly convex then P is continuous at each point of D. However Lau [**8**] has shown that the sets of points with those properties in such reflexive spaces are residual in $X \setminus K$. Thus we ask the following question.

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Problem 3.4. If X is a reflexive Banach space and K a closed subset of X. Is the set

$$\{x \in X \setminus K : \left(\exists \overrightarrow{x} \in X\right) \quad \left(\left(\left\| \overrightarrow{x} \right\| = 1\right) \; \& \; \left(d'_{+}(x)\left(\overrightarrow{x}\right) = 1\right)\right)\}$$

residual in $X \setminus K$?

References

[1] J.M. Borwein and D. Preiss, 'A smooth variational principle with applications to subdifferentiability of convex functions', *Trans. Amer. Math. Soc.* **303** (1987), 517–527.

[2] J.M. Borwein and J.R. Giles, 'The proximal normal formula in Banach spaces', *Trans. Amer. Math. Soc.* **302** (1987), 371–381.

[3] J.M. Borwein, S.P. Fitzpatrick and J.R. Giles, 'The differentiability of real valued functions on normed linear spaces using generalised subgradients', *J. Math. Anal. Appl.* **128** (1987), 512–534.

[4] S. Fitzpatrick, 'Metric projections and the differentiability of distance functions', *Bull. Austral. Math. Soc.* **22** (1980), 291–312.

[5] S. Fitzpatrick, 'Differentiation of real valued functions and continuity of metric projections', *Proc. Amer. Math. Soc.* **91** (1984), 544–548.

[6] J.R. Giles, 'A distance function property implying differentiability', *Bull. Austral. Math. Soc.* (to appear).

[7] R.C. James, 'Weak compactness and reflexivity', *Israel Math. J.* **2** (1964), 101–119.

[8] Ka Sing Lau, 'Almost Chebyshev subsets in reflexive Banach spaces', *Indiana Univ. Math. J.* **2** (1978), 791–795.

[9] D. Preiss, 'Differentiability of Lipschitz functions on Banach spaces', (submitted).

[10] L. Zajicek, 'Differentiability of distance functions and points of multivaluedness of the metric projection in Banach spaces', *Czech. Math. J.* **33** (1983), 292–308.

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