

ODD MULTIPERFECT NUMBERS

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Abstract

A natural number n is called multiperfect or k -perfect for integer $k \geq 2$ if $\sigma(n) = kn$, where $\sigma(n)$ is the sum of the positive divisors of n . In this paper, we establish a theorem on odd multiperfect numbers analogous to Euler's theorem on odd perfect numbers. We describe the divisibility of the Euler part of odd multiperfect numbers and characterise the forms of odd perfect numbers $n = \pi^\alpha M^2$ such that $\pi \equiv \alpha \pmod{8}$, where π^α is the Euler factor of n . We also present some examples to show the nonexistence of odd perfect numbers of certain forms.

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1. Introduction

Let n be a positive integer and $\sigma(n)$ be the sum of the positive divisors of n . We say that n is perfect if $\sigma(n) = 2n$ and that n is *multiperfect* or *k-perfect* if $\sigma(n) = kn$ for some integer $k \geq 2$. For example, 6 is a perfect number and 120 is a 3-perfect number. One can refer to Flammencamp [4] or Hagsis and McDaniel [5] for a survey of perfect and multiperfect numbers.

Even perfect numbers were completely classified by Euclid and Euler, but odd perfect numbers remain utterly mysterious: no odd k -perfect numbers are known for any $k \geq 2$. But it is known that an odd perfect number must satisfy various conditions. One of the important structural results on odd perfect numbers is due to Euler, who showed that if n is perfect then $n = \pi^\alpha M^2$, where π is prime, $(\pi, M) = 1$ and $\pi \equiv \alpha \equiv 1 \pmod{4}$. We call π^α the Euler factor. Euler's theorem was extended by Broughan and Zhou recently to 4-perfect numbers, and then to 2^k -perfect numbers [1]. In this paper, we will establish a precise result on odd k -perfect numbers. It turns out that an odd k -perfect number n has the form $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} M^2$, $(p_i, M) = 1$, where the odd primes p_i and odd positive integers e_j satisfy some congruence restrictions.

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(See Theorem 2.2 for details.) We call $\Pi := p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ the *Euler part* of the k -perfect number n analogous to the Euler factor of odd perfect numbers.

It is interesting to study the Euler part of k -perfect numbers. It turns out that the properties of the Euler part of k -perfect numbers can be used to prove the nonexistence of odd k -perfect numbers of certain forms. For instance, Starni [11] recently proved that if an odd perfect number n has the form $n = \pi^\alpha 3^{2\beta} Q^{2\beta}$ with $(3, Q) = 1$, then $3^{2\beta} \mid \sigma(\pi^\alpha)$. As a corollary, he showed that if $\pi \equiv 1 \pmod{12}$ and $\alpha \equiv 1$ or $9 \pmod{12}$, then there do not exist odd perfect numbers $n = \pi^\alpha 3^{2\beta} M^2$. This result was extended to odd 2^k -perfect numbers by Broughan and Zhou in [1]. We will show that the prime 3 can be replaced by any Fermat prime under suitable conditions (see Corollary 3.2).

An early result of Starni [10] on odd perfect numbers is that there is no odd perfect number decomposable into primes, all congruent to 1 (mod 4), if $n = \pi^\alpha M^2$ and $\pi \not\equiv \alpha \pmod{8}$. In Section 3, we will characterise the forms of odd perfect numbers for which $\pi \equiv \alpha \pmod{8}$. As a consequence, we extend Starni's results.

2. Structure of multiperfect numbers

Recently, Broughan and Zhou [1] generalised Euler's structure theorem on odd perfect numbers to odd 4-perfect numbers, and then to odd 2^k -perfect numbers. To prove the structural result of odd 4-perfect numbers, they employed the following theorem, which is of independent interest.

THEOREM (Broughan and Zhou [1, Theorem 2.2]). *For all odd primes p , powers $j \geq 1$ and odd exponents $e > 0$,*

$$2^j \parallel \sigma(p^e) \quad \text{if and only if } 2^{j+1} \parallel (p+1)(e+1).$$

This theorem was proved by discussing many cases and constructing some polynomials. Here we will give a simple proof. More generally, we have the following theorem.

THEOREM 2.1. *Let p be a prime and e be a positive integer. Let $v_2(m)$ be the highest power of 2 dividing the integer m . Then*

$$v_2(\sigma(p^e)) = \begin{cases} v_2(p+1) + v_2\left(\frac{e+1}{2}\right) & \text{if } p > 2 \text{ and } e \equiv 1 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

$$v_2(\sigma(p^e) - 1) = \begin{cases} 0 & \text{if } p > 2 \text{ and } e \equiv 1 \pmod{2}, \\ v_2(p+1) + v_2\left(\frac{e}{2}\right) & \text{if } p > 2 \text{ and } e \equiv 0 \pmod{2}, \\ 1 & \text{if } p = 2. \end{cases} \quad (2.2)$$

PROOF. It is obvious that $\sigma(p^e) = 1 + p + \cdots + p^e \equiv 1 \pmod{2}$ when $p = 2$ or when p is odd and e is even. Now we assume that p and e are both odd. We

write $r = v_2((e + 1)/2)$ and $(e + 1)/2 = 2^r s$ for some odd integer s . Then

$$\begin{aligned} \sigma(p^e) &= \frac{p^{e+1} - 1}{p - 1} = \frac{(p^2)^{(e+1)/2} - 1}{p - 1} \\ &= \frac{(p^2)^{2^r s} - 1}{p - 1} = (p + 1) \frac{(p^{2s})^{2^r} - 1}{p^2 - 1} \\ &= (p + 1)((p^{2s})^{2^{r-1}} + 1)((p^{2s})^{2^{r-2}} + 1) \cdots (p^{2s} + 1) \frac{p^{2s} - 1}{p^2 - 1}. \end{aligned}$$

Since p is odd,

$$(p^{2s})^{2^{r-i}} + 1 = (p^{s2^{r-i}})^2 + 1 \equiv 2 \pmod{4}, \quad 1 \leq i \leq r.$$

Note that

$$\frac{p^{2s} - 1}{p^2 - 1} = 1 + p^2 + p^4 + \cdots + (p^2)^{s-1} \equiv s \equiv 1 \pmod{2}.$$

Therefore

$$\begin{aligned} v_2(\sigma(p^e)) &= v_2(p + 1) + \sum_{i=1}^r v_2((p^{2s})^{2^{r-i}} + 1) + v_2\left(\frac{p^{2s} - 1}{p^2 - 1}\right) \\ &= v_2(p + 1) + r \\ &= v_2(p + 1) + v_2\left(\frac{e + 1}{2}\right). \end{aligned}$$

Formula (2.2) follows from (2.1) and the fact that

$$v_2(\sigma(p^e) - 1) = v_2(p\sigma(p^{e-1})) = v_2(p) + v_2(\sigma(p^{e-1})).$$

This completes the proof of Theorem 2.1. □

We remark that Theorem 2.1 can be shown to follow from [9, Theorems 94 and 95].

As an application of Theorem 2.1, we will establish an explicit structure theorem for odd k -perfect numbers.

THEOREM 2.2. *Let n be odd and k -perfect with $v_2(k) \geq 1$ and s be any integer satisfying $1 \leq s \leq v_2(k)$. Then n has the shape*

$$n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} M^2, \tag{2.3}$$

where M is a positive integer, the p_i are primes with $(p_i, M) = 1$, and the e_j are odd positive integers. If $v_2(k) - s$ has a nonnegative partition

$$v_2(k) - s = a_1 + a_2 + \cdots + a_s + b_1 + b_2 + \cdots + b_s, \quad a_i \geq 0, b_j \geq 0, \tag{2.4}$$

then the primes p_1, \dots, p_s satisfy

$$p_i \equiv 2^{a_i+1} - 1 \pmod{2^{a_i+2}}$$

and the exponents e_1, \dots, e_s satisfy

$$e_j \equiv 2^{b_j+1} - 1 \pmod{2^{b_j+2}}.$$

Before proving the theorem, we give the definition of the Euler part of an odd k -perfect number.

DEFINITION 2.3. The Euler part of an odd k -perfect number n with the shape (2.3) is denoted by

$$\Pi := p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}.$$

REMARK.

- (i) Theorem 2.2 shows that if n_1 is k_1 -perfect, n_2 is k_2 -perfect and $v_2(k_1) = v_2(k_2)$, then n_1 and n_2 have the same shapes. Therefore we only consider 2^k -perfect numbers in many situations.
- (ii) Note that there are $v_2(k)$ shapes of an odd k -perfect number n since s can take $v_2(k)$ values and each s gives a shape of n as in (2.3).

PROOF OF THEOREM 2.2. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ be the standard factorisation of n . Since n is k -perfect,

$$\sigma(n) = \sigma(p_1^{e_1})\sigma(p_2^{e_2}) \cdots \sigma(p_r^{e_r}) = kn.$$

Since $v_2(k) \geq 1$, it follows from (2.1) of Theorem 2.1 that some e_i must be odd. Without loss of generality, we can assume that e_1, \dots, e_s are odd and e_{s+1}, \dots, e_r are even. So

$$n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \cdot p_{s+1}^{e_{s+1}} p_{s+2}^{e_{s+2}} \cdots p_r^{e_r} = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} M^2,$$

where $M^2 = p_{s+1}^{e_{s+1}} \cdots p_r^{e_r}$. By Theorem 2.1,

$$\begin{aligned} v_2(k) &= v_2(kn) = v_2(\sigma(n)) \\ &= \sum_{i=1}^s v_2(\sigma(p_i^{e_i})) + \sum_{j=s+1}^r v_2(\sigma(p_j^{e_j})) \\ &= \sum_{i=1}^s \left(v_2(p_i + 1) + v_2\left(\frac{e_i + 1}{2}\right) \right) \\ &= s + \sum_{i=1}^s \left(v_2\left(\frac{p_i + 1}{2}\right) + v_2\left(\frac{e_i + 1}{2}\right) \right). \end{aligned}$$

Therefore

$$v_2(k) - s = \sum_{i=1}^s v_2\left(\frac{p_i + 1}{2}\right) + \sum_{i=1}^s v_2\left(\frac{e_i + 1}{2}\right).$$

Given a nonnegative partition of $v_2(k) - s$ such that

$$v_2(k) - s = a_1 + a_2 + \cdots + a_s + b_1 + b_2 + \cdots + b_s, \quad a_i \geq 0, b_j \geq 0,$$

we can take $v_2((p_i + 1)/2) = a_i$ and $v_2((e_i + 1)/2) = b_i$, $1 \leq i \leq s$. The desired congruences for p_i and e_i follow immediately. This completes the proof of Theorem 2.2. □

We give two examples to recover Euler’s theorem on odd perfect numbers and Broughan and Zhou’s structure theorem on odd 4-perfect numbers [1, Theorem 2.1].

EXAMPLE 2.4. Let n be an odd perfect number. Then $v_2(2) = 1$, $s = 1$, and n has the unique form $n = \pi^\alpha M^2$ with π prime, α odd and $(\pi, M) = 1$. By (2.4), $a_1 = b_1 = 0$. So $\pi \equiv \alpha \equiv 1 \pmod{4}$.

EXAMPLE 2.5. Let n be an odd 4-perfect number. Then $v_2(4) = 2$ and $s = 1$ or 2 .

- (1) If $s = 1$, then $n = p^e M^2$. By (2.4), $a_1 = 1, b_1 = 0$ or $a_1 = 0, b_1 = 1$. Therefore $p \equiv 3 \pmod{8}$, $e \equiv 1 \pmod{4}$ or $p \equiv 1 \pmod{4}$, $e \equiv 3 \pmod{8}$.
- (2) If $s = 2$, then $n = p_1^{e_1} p_2^{e_2} M^2$. By (2.4), $a_1 = a_2 = b_1 = b_2 = 0$. Therefore $p_1 \equiv p_2 \equiv e_1 \equiv e_2 \equiv 1 \pmod{4}$.

3. The Euler part of a 2^k -perfect number

Based on a result of McDaniel [7], Starni [11] proved that if an odd perfect number n has the form $n = \pi^\alpha 3^{2\beta} Q^{2\beta}$ with $(3, Q) = 1$, then $3^{2\beta} \mid \sigma(\pi^\alpha)$. This result was generalised to odd 2^k -perfect numbers by Broughan and Zhou [2, Theorem 2.6]. Recall that Π is the Euler part of an odd k -perfect number. In the following theorem we will prove necessary and sufficient conditions on the divisibility of $\sigma(\Pi)$.

THEOREM 3.1. Let $n = \Pi q^{2\beta} \prod_{i=1}^s p_i^{2\beta_i}$ be an odd 2^k -perfect number, where Π is the Euler part of n , and q and the p_i are distinct odd primes. Then

$$q^{2\beta} \mid \sigma(\Pi) \quad \text{if and only if} \quad \begin{cases} 2\beta_i + 1 \not\equiv 0 \pmod{\text{ord}_q(p_i)} & \text{if } p_i \not\equiv 1 \pmod{q}, \\ 2\beta_i + 1 \not\equiv 0 \pmod{q} & \text{otherwise,} \end{cases}$$

where $\text{ord}_q(m)$ is the order of m in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$.

PROOF. Since n is 2^k -perfect,

$$2^k n = 2^k \Pi q^{2\beta} \prod_{i=1}^s p_i^{2\beta_i} = \sigma(\Pi) \sigma(q^{2\beta}) \sigma\left(\prod_{i=1}^s p_i^{2\beta_i}\right) = \sigma(n).$$

Therefore $q^{2\beta} \mid \sigma(\Pi)$ is equivalent to $(q, \sigma(p_i^{2\beta_i})) = 1, i = 1, \dots, s$.

Note that if $p_i \equiv 1 \pmod{q}$, then

$$\sigma(p_i^{2\beta_i}) \equiv 2\beta_i + 1 \pmod{q}.$$

Hence $(q, \sigma(p_i^{2\beta_i})) = 1$ implies that $2\beta_i + 1 \not\equiv 0 \pmod{q}$.

If $p_i \not\equiv 1 \pmod{q}$, then $(q, \sigma(p_i^{2\beta_i})) = 1$ implies that $p_i^{2\beta_i+1} \not\equiv 1 \pmod{q}$. It follows that $2\beta_i + 1 \not\equiv 0 \pmod{\text{ord}_q(p_i)}$. This completes the proof of Theorem 3.1. □

COROLLARY 3.2. Let $n = \Pi q^{2\beta} \prod_{i=1}^s p_i^{2\beta_i}$ be an odd 2^k -perfect number as in Theorem 3.1. If q is a Fermat prime, that is, $q = 2^{2^t} + 1$ for some integer $t \geq 0$, and $\prod_{i=1}^s (2\beta_i + 1) \not\equiv 0 \pmod{q}$, then $q^{2\beta} \mid \sigma(\Pi)$.

PROOF. Note that $\phi(q) = 2^{2^t}$, where ϕ is the Euler function. Lagrange’s theorem implies that $\text{ord}_q(p_i) \mid \phi(q)$. Hence $\text{ord}_q(p_i)$ is a power of 2. The corollary follows by Theorem 3.1 immediately. \square

REMARK. If $n = \pi^\alpha q^{2\beta} \prod_{i=1}^s p_i^{2\beta}$ is an odd perfect number with $\alpha \equiv \pi \equiv 1 \pmod{4}$, then it is known that $\beta \neq 2$ [6], $\beta \neq 3$ [5], $\beta \neq 5, 12, 17, 24, 62$ [8], and $\beta \neq 6, 8, 11, 14, 18$ [2]. In [8] McDaniel and Hagis conjectured that there do not exist such odd perfect numbers for any positive integer β . Corollary 3.2 can be used to prove this conjecture in some special cases. For example, if $q = 5$, $(2\beta + 1, 5) = 1$, $\pi \equiv 1 \pmod{20}$, $\alpha \equiv 13 \pmod{20}$, then by Corollary 3.2, $5 \mid \sigma(\pi^\alpha)$, but $\sigma(\pi^\alpha) \equiv 1 + \alpha \equiv 4 \pmod{5}$. Therefore there do not exist such odd perfect numbers. In particular, $n = 41^{13} 5^{2\beta} \prod_{i=1}^s p_i^{2\beta}$ cannot be perfect for any $\beta \not\equiv 2 \pmod{5}$ and odd primes p_i .

Let $n = \pi^\alpha \prod_i p_i^{2\beta_i}$ be an odd perfect number, π^α with $\pi \equiv \alpha \equiv 1 \pmod{4}$ being Euler’s factor. Based on a result of Ewell [3], Starni [10] proved that $\pi \equiv \alpha \pmod{8}$ if each prime $p_i \equiv 1 \pmod{4}$. In the following Theorem 3.3, we will extend Starni’s results independently of Ewell’s result.

THEOREM 3.3. *Let $n = \pi^\alpha M^2$ be an odd perfect number, with π prime, $(\pi, M) = 1$ and $\pi \equiv \alpha \equiv 1 \pmod{4}$. Then*

$$\sigma(M^2) \equiv 1 \pmod{4} \quad \text{if and only if } \pi \equiv \alpha \pmod{8}, \tag{3.1}$$

$$\sigma(M^2) \equiv 3 \pmod{4} \quad \text{if and only if } \pi \equiv \alpha + 4 \pmod{8}. \tag{3.2}$$

In particular, if $n = \pi^\alpha \prod_j q_j^{2\gamma_j}$ with $q_j \equiv 3 \pmod{4}$ then

$$\pi \equiv \alpha \pmod{8}.$$

PROOF. Using the fact that $\sigma(\pi^\alpha M^2) = 2\pi^\alpha M^2$ and M and α are odd, we find that

$$\pi^\alpha \equiv \pi \equiv \frac{\sigma(\pi^\alpha)}{2} \sigma(M^2) \pmod{8}. \tag{3.3}$$

Note that $\pi \equiv \alpha \equiv 1 \pmod{4}$ implies that $\pi^4 \equiv \alpha^4 \equiv 1 \pmod{16}$. It follows that

$$\begin{aligned} \sigma(\pi^\alpha) &= 1 + \pi + \dots + \pi^\alpha \\ &= (1 + \pi + \pi^2 + \pi^3)(1 + \pi^4 + \pi^8 + \dots + \pi^{\alpha-5}) + \pi^{\alpha-1}(1 + \pi) \\ &\equiv (1 + \pi)(1 + \pi^2) \frac{\alpha - 1}{4} + 1 + \pi \pmod{16}. \end{aligned}$$

Hence

$$\frac{\sigma(\pi^\alpha)}{2} \equiv (1 + \pi^2) \frac{1 + \pi}{2} \frac{\alpha - 1}{4} + \frac{1 + \pi}{2} \equiv \frac{1 + \pi}{2} \frac{1 + \alpha}{2} \pmod{8}.$$

Inserting this into (3.3),

$$\pi \equiv \frac{\pi + 1}{2} \frac{\alpha + 1}{2} \sigma(M^2) \pmod{8}.$$

Since $(1 + \pi)/2$ is odd,

$$\frac{\pi(\pi + 1)}{2} \equiv \left(\frac{\pi + 1}{2}\right)^2 \frac{\alpha + 1}{2} \sigma(M^2) \equiv \frac{\alpha + 1}{2} \sigma(M^2) \pmod{8}.$$

It follows that

$$\pi(\pi + 1) \equiv (\alpha + 1)\sigma(M^2) \pmod{16}. \quad (3.4)$$

If $\sigma(M^2) \equiv 1 \pmod{8}$, then

$$\pi(\pi + 1) \equiv \alpha + 1 \pmod{16}.$$

Recall that $\pi \equiv \alpha \equiv 1 \pmod{4}$. It is easy to find that the solutions $(\pi, \alpha) \pmod{16}$ are $(1, 1), (5, 13), (9, 9), (13, 5)$. In particular, we get

$$\pi \equiv \alpha \pmod{8}.$$

Similar arguments show that the solutions to (3.4) are

$$(\pi, \alpha) \pmod{16} = \begin{cases} (1, 9); (5, 5); (9, 1); (13, 13) & \text{if } \sigma(M^2) \equiv 5 \pmod{8}, \\ (1, 5); (5, 9); (9, 13); (13, 1) & \text{if } \sigma(M^2) \equiv 3 \pmod{8}, \\ (1, 13); (5, 1); (9, 5); (13, 9) & \text{if } \sigma(M^2) \equiv 7 \pmod{8}. \end{cases}$$

This implies (3.1) and (3.2).

If we write $M^2 = \prod_i p_i^{2\beta_i} \prod_j q_j^{2\gamma_j}$ where $p_i \equiv 1 \pmod{4}, q_j \equiv 3 \pmod{4}$, then by (2.2) of Theorem 2.1, $\sigma(q_j^{2\gamma_j}) \equiv 1 \pmod{4}$. Hence

$$\sigma(M^2) = \prod_i \sigma(p_i^{2\beta_i}) \prod_j \sigma(q_j^{2\gamma_j}) \equiv \prod_i (2\beta_i + 1) \pmod{4}. \quad (3.5)$$

Clearly, if $\beta_i = 0$ for all i , that is $M^2 = \prod_j q_j^{2\gamma_j}$, then $\sigma(M^2) \equiv 1 \pmod{4}$, and (3.1) implies that $\pi \equiv \alpha \pmod{8}$. \square

REMARK. By (3.2), (3.5) and (2.2) of Theorem 2.1, it is easy to see that $\pi \equiv \alpha + 4 \pmod{8}$ if and only if the number of prime factors p^e of M^2 with $p^e \parallel M^2, p \equiv 1 \pmod{4}$ and $e \equiv 2 \pmod{4}$ is odd.

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