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On twisted group ring isomorphism problem for p-groups

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Abstract
In this article, we explore the problem of determining isomorphisms between the twisted complex group algebras of finite p-groups. This problem bears similarity to the classical group algebra isomorphism problem and has been recently examined by Margolis-Schnabel. Our focus lies on a specific invariant, referred to as the generalized corank, which relates to the twisted complex group algebra isomorphism problem. We provide a solution for non-abelian p-groups with generalized corank at most three.

1. Introduction

The group ring RG, where G is a finite group and R is a commutative ring, holds significant importance in representation theory. Over the past few decades, there has been considerable interest in decoding information about a group G from its group ring RG. One particularly challenging problem in this context is the isomorphism problem, which investigates whether a group ring uniquely determines its corresponding group. Specifically, if RG and RH are isomorphic as R-rings, does it imply that the groups G and H are isomorphic as well? For the current status of this problem, one can refer to [2, 13, 15, 16, 27]. The solution to this problem depends mainly upon the ring under consideration. For example, all the finite abelian groups of a given order have isomorphic complex group algebras, whereas the rational group algebras of any two non-isomorphic abelian groups are always non-isomorphic (see [25]). In 1971, Dade [3] constructed an example demonstrating the existence of two non-isomorphic metabelian groups that possess isomorphic group algebras over any field. Subsequently, Hertweck [9] presented a counterexample of this phenomenon for integral group rings, showcasing two non-isomorphic groups of even order whose integral group rings are isomorphic. However, the problem of determining whether integral group rings of groups with odd order are isomorphic remains an open question. Additionally, investigating this problem in the context of modular group rings, in particular, for the group rings of finite p-groups over a field of characteristic p has been of significant interest (see [26]).

In recent times, a variant of the classical isomorphism problem known as the twisted group ring isomorphism problem has gained considerable attention. The problem was initially introduced in [17] and has been further explored by the authors in [18, 19]. In order to explain this version of the isomorphism problem, we start by introducing some notation.

Let R be a commutative ring with unity and $R^\times$ be the unit group of R. Following [14], we denote the set of 2-cocyles of G by $Z^2(G, R^\times)$ and the second cohomology group of G over $R^\times$ by $\mathrm{H}^2(G, R^\times)$. For a 2-cocycle α, let $R^\alpha[G]$ be the α-twisted group ring of G over R, that is $R^\alpha[G] = \{\sum_{g\in G} x_g e_g \mid x_g \in R\}$ is an R-algebra with the following operations:

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- (addition) $(\sum_{g\in G} x_g e_g) + (\sum_{g\in G} y_g e_g) = \sum_{g\in G}(x_g + y_g)e_g$.
- (multiplication) $e_g.e_h = \alpha(g,h)e_{gh}$ for all $g,h \in G$, and defined distributively for all other elements of $R^\alpha[G]$.

For an element $\alpha \in Z^2(G,\mathbb{C}^\times)$, the corresponding element of $\mathrm{H}^2(G,\mathbb{C}^\times)$ will be denoted by $[\alpha]$. It is well known that $R^\alpha[G] \cong R^\beta[G]$ as R-algebras for $\alpha,\beta \in Z^2(G,R^\times)$ such that $[\alpha]=[\beta]$.

Given groups G, H and a ring R, we write $G \sim_R H$ if there exists an isomorphism $\psi : \mathrm{H}^2(G,R^\times) \to \mathrm{H}^2(H,R^\times)$ such that $R^\alpha[G] \cong R^{\psi(\alpha)}[H]$ for every $[\alpha] \in \mathrm{H}^2(G,R^\times)$. The twisted group ring isomorphism problem is to determine the equivalence classes of groups of order n, under the relation $\sim_R$. We call these equivalence classes the R-twist isomorphism classes.

Throughout this article, our focus lies on the $\mathbb{C}$-twist isomorphism classes of finite p-groups. The second cohomology group of a finite group G over $\mathbb{C}^\times$, that is $\mathrm{H}^2(G,\mathbb{C}^\times)$, is commonly referred to as the Schur multiplier of G. The order of the group, the structure of the Schur multiplier, and the structure of the complex group algebra remain invariant under $\mathbb{C}$-twist isomorphism.

In [17, Theorem 4.3], Margolis and Schnabel determined the $\mathbb{C}$-twist isomorphism classes of groups of order p^4, where p is a prime. In the same article, they proved (see [17, Lemma 1.2]) that any equivalence class of a finite abelian group with respect to $\sim_\mathbb{C}$ is a singleton. Hence, it is sufficient to focus on the classification of the $\mathbb{C}$-twist isomorphism classes of non-abelian finite groups. In this article, we continue this line of investigation of the $\mathbb{C}$-twist isomorphism classes of finite non-abelian p-groups by fixing the order of the second cohomology group over $\mathbb{C}^\times$. Green [6] proved that the order of the Schur multiplier of a p-group G of order p^n is at most $p^{\frac{n(n-1)}{2}}$. Niroomand [20] improved this bound for non-abelian p-groups and proved that $|\mathrm{H}^2(G,\mathbb{C}^\times)| \le p^{\frac{(n-1)(n-2)}{2}+1}$ for any non-abelian group G of order p^n. In view of this result, a finite non-abelian p-group G is said to have generalized corank $s(G)$ if $|\mathrm{H}^2(G,\mathbb{C}^\times)| = p^{\frac{(n-1)(n-2)}{2}+1-s(G)}$.

We study the $\mathbb{C}$-twist isomorphism classes of finite non-abelian p-groups by fixing their generalized corank. In particular, we describe the $\mathbb{C}$-twist isomorphism classes of all p-groups with $s(G) \le 3$. The classification of all non-isomorphic p-groups with $s(G) \le 3$ is known in the literature by the work of Niroomand [24] and Hatui [7]. We use this classification along with the structure of the corresponding twisted group algebras to obtain our results. We use the following notation:

- C_{p^n} denotes the cyclic group of order p^n.
- $C_{p^n}^m$ or $C_{p^n}^{(m)}$ denote the direct product of m-copies of the cyclic group of order p^n.
- H_m^1 and H_m^2 denote the extraspecial p-groups of order p^{2m+1} and of exponent p and p^2, respectively.
- $H.K$ denotes the central product of the groups H and K.
- $E(r) = E.C_{p^r}$, where E is an extraspecial p-group.

We now list the main results of this article. Our first result describes the $\mathbb{C}$-twist isomorphism classes of groups with generalized corank zero or one.

Lemma 1.1. *For non-abelian groups G of order p^n with $s(G) \in \{0,1\}$, every $\mathbb{C}$-twist isomorphism class is a singleton, that is consists of only one group up to isomorphism.*

In our next result, we describe all non-singleton $\mathbb{C}$-twist isomorphism classes of finite p-groups with $s(G) = 2$.

Theorem 1.2. *All non-singleton $\mathbb{C}$-twist isomorphism classes of non-abelian groups of order p^n and generalized corank two are as follows:*

(1) for any $n \ge 4$, $\mathbb{Q}_8 \times C_2^{(n-3)} \sim_\mathbb{C} E(2) \times C_2^{(n-4)}$;
(2) for an odd prime p, $E(2) \sim_\mathbb{C} H_1^2 \times C_p \sim_\mathbb{C} \langle a,b \mid a^{p^2}=1, b^p=1, [a,b,a]=[a,b,b]=1\rangle$;

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$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Hom}(G_2', \mathbb{C}^\times) & \xrightarrow{\mathrm{tra}_2} & \mathrm{H}^2(G_2/G_2', \mathbb{C}^\times) & \xrightarrow{\mathrm{inf}_2} & \mathrm{H}^2(G_2, \mathbb{C}^\times) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \tilde{\delta}} & & \downarrow{\scriptstyle \tilde{\sigma}} & & & & \\ 1 & \longrightarrow & \mathrm{Hom}(G_1', \mathbb{C}^\times) & \xrightarrow{\mathrm{tra}_1} & H^2(G_1/G_1', \mathbb{C}^\times) & \xrightarrow{\mathrm{inf}_1} & H^2(G_1, \mathbb{C}^\times) & \longrightarrow & 1. \end{array}$$

Figure 1. *Relation between Hochschild-Serre spectral sequences of G_1 and G_2.*

(3) for $n \geq 5$, $E(2) \times C_p^{(n-4)} \sim_{\mathbb{C}} H_1^2 \times C_p^{(n-3)}$;
(4) for $n = 2m + 1$ and $m \geq 2$, $H_m^1 \times C_p^{(n-2m-1)} \sim_{\mathbb{C}} H_m^2 \times C_p^{(n-2m-1)}$;
(5) for $n \geq 6$ and $1 < m \leq n/2 - 1$, $E(2) \times C_p^{(n-2m-2)} \sim_{\mathbb{C}} H_m^1 \times C_p^{(n-2m-1)} \sim_{\mathbb{C}} H_m^2 \times C_p^{(n-2m-1)}$.

See Section 3 for the proof of Lemma 1.1 and Theorem 1.2. The next result describes the non-singleton $\mathbb{C}$-twist isomorphism classes for the groups of order p^n with $s(G) = 3$. A complete classification of these groups was given by Hatui [7, Theorem 1.1]. We refer the reader to Theorem 4.1 for the details and for the notation appearing in our next result.

Theorem 1.3. *All non-singleton $\mathbb{C}$-twist isomorphism classes of non-abelian groups of order p^n and generalized corank three are as follows:*

1. $\phi_3(211)a \sim_{\mathbb{C}} \phi_3(211)b_1 \sim_{\mathbb{C}} \phi_3(211)b_{r_p}$;
2. $\phi_2(2111)c \sim_{\mathbb{C}} \phi_2(2111)d$.

A proof of the above result is included in Section 4.

Remark 1.4. *By [4], a group G of order p^n is said to have corank $t(G)$ if its Schur multiplier has order $p^{\frac{n(n-1)}{2} - t(G)}$. A classification of all finite p-groups G with corank of G at most 6 is also known in literature, see [1, 4, 21, 23, 29]. By definition, $t(G) \leq 6$ for any non-abelian group G of order p^n implies $n \leq 8$ and $s(G) \leq 5$. Further, $s(G) \in \{4, 5\}$ gives $n \leq 4$. Therefore, our above description of $\mathbb{C}$-twist isomorphism classes along with the known results from literature also gives $\sim_{\mathbb{C}}$ classes of groups with $t(G) \leq 6$.*

To prove Theorem 1.2, we use the following general result. This is a helpful tool to prove the $\mathbb{C}$-twist isomorphism between groups G_1 and G_2 and may be of an independent interest.

Theorem 1.5. *Suppose G_1 and G_2 are two groups with isomorphisms $\delta : G_1' \to G_2'$, $\sigma : G_1/G_1' \to G_2/G_2'$ and the following short exact sequences for $i \in \{1, 2\}$:*

$$1 \longrightarrow \mathrm{Hom}(G_i', \mathbb{C}^\times) \xrightarrow{\mathrm{tra}_i} H^2(G_i/G_i', \mathbb{C}^\times) \xrightarrow{\mathrm{inf}_i} H^2(G_i, \mathbb{C}^\times) \longrightarrow 1,$$

where tra_i and inf_i are the transgression and inflation homomorphisms. If $\tilde{\delta} : \mathrm{Hom}(G_2', \mathbb{C}^\times) \to \mathrm{Hom}(G_1', \mathbb{C}^\times)$ and $\tilde{\sigma} : \mathrm{H}^2(G_1/G_1', \mathbb{C}^\times) \to \mathrm{H}^2(G_2/G_2', \mathbb{C}^\times)$ are the induced isomorphisms such that the Figure 1 is commutative, then $G_1 \sim_{\mathbb{C}} G_2$.

We refer the reader to Section 2 for the definition of transgression and inflation homomorphisms as well as for a proof of the above result.

2. Preliminaries

In this section, we fix notation and include a few preliminary results that we use throughout this article. We refer the reader to Karpilovsky [14] for most of the results included in this section.

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Let G be a finite group. Recall, for a group G, $Z^2(G, \mathbb{C}^\times)$ consists of all functions $f : G \times G \to \mathbb{C}^\times$ such that $f(x, 1) = f(1, x) = 1$ and $f(x, y)f(xy, z) = f(x, yz)f(y, z)$ for all $x, y, z \in G$. We shall call elements of $Z^2(G, \mathbb{C}^\times)$ as 2-cocycles (or sometimes just cocycles when it is clear from the context). Then $\mathrm{H}^2(G, \mathbb{C}^\times) = Z^2(G, \mathbb{C}^\times)/B^2(G, \mathbb{C}^\times)$, where $B^2(G, \mathbb{C}^\times)$ is the set of 2-coboundaries of G, is called the second cohomology group of G or the Schur multiplier of G. The elements of $\mathrm{H}^2(G, \mathbb{C}^\times)$ are called the cohomology classes of G. For an element $\alpha \in Z^2(G, \mathbb{C}^\times)$, the corresponding element of $\mathrm{H}^2(G, \mathbb{C}^\times)$ will be denoted by $[\alpha]$. For 2-cocycles $\alpha, \beta \in Z^2(G, \mathbb{C}^\times)$ we say α is cohomologous to β, whenever $[\alpha] = [\beta]$.

A central extension,

$$1 \to A \to G \to G/A \to 1 \tag{2.1}$$

is called a *stem* extension, if $A \subseteq Z(G) \cap G'$. For a given stem extension (2.1), the Hochschild-Serre spectral sequence [10, Theorem 2, p. 129] yields the following exact sequence

$$1 \to \mathrm{Hom}(A, \mathbb{C}^\times) \xrightarrow{\mathrm{tra}} \mathrm{H}^2(G/A, \mathbb{C}^\times) \xrightarrow{\mathrm{inf}} \mathrm{H}^2(G, \mathbb{C}^\times),$$

where $\mathrm{tra} : \mathrm{Hom}(A, \mathbb{C}^\times) \to \mathrm{H}^2(G/A, \mathbb{C}^\times)$ given by $f \mapsto [\mathrm{tra}(f)]$, where

$$\mathrm{tra}(f)(\bar{x}, \bar{y}) = f(\mu(\bar{x})\mu(\bar{y})\mu(\bar{x}\bar{y})^{-1}), \;\; \bar{x}, \bar{y} \in G/A,$$

for a section $\mu : G/A \to G$, denotes the transgression homomorphism and the inflation homomorphism, $\inf : \mathrm{H}^2(G/A, \mathbb{C}^\times) \to \mathrm{H}^2(G, \mathbb{C}^\times)$ is given by $[\alpha] \mapsto [\inf(\alpha)]$, where $\inf(\alpha)(x, y) = \alpha(xA, yA)$.

Let G be a finite group and V be a finite-dimensional complex vector space. A mapping $\rho : G \to \mathrm{GL}(V)$ is called a (finite-dimensional and complex) *projective representation* of G if thhere xists a mapping $\alpha : G \times G \to \mathbb{C}^\times$ such that

- $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for all $x, y \in G$,
- $\rho(1) = \mathrm{Id}_V$.

In this case, we say ρ is an α-representation of G on V. For $\alpha \in Z^2(G, \mathbb{C}^\times)$, we use $\mathrm{Irr}^\alpha(G)$ to denote the set of equivalence classes of irreducible α-representations of G. For $\alpha = 1$, we use $\mathrm{Irr}(G)$ instead of $\mathrm{Irr}^\alpha(G)$ and call this the set of ordinary irreducible representations of G. For $\rho \in \mathrm{Irr}(G)$, at times, we shall also omit the word "ordinary" and call ρ to be a representation of G. For $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$, the complex group algebra $\mathbb{C}^\alpha G$ is semisimple and

$$\mathbb{C}^\alpha[G] \cong \prod_{\rho \in \mathrm{Irr}^\alpha(G)} M_{\dim(\rho)}(\mathbb{C}), \tag{2.2}$$

where $M_n(\mathbb{C})$ denotes the $n \times n$ matrix algebra over $\mathbb{C}$. Hence, the information of projective representations of G is also helpful to study the twisted group algebra isomorphism problem. To this end, the representation group (also called a covering group) of a group G will play an important role. We now recall the definition of the representation group.

Definition 2.1. *A group G^* is called a representation group of the group G if the following conditions are satisfied:*

1. *There exists a central extension* $1 \to A \to G^* \to G \to 1$ *such that* $\mathrm{Hom}(A, \mathbb{C}^\times) \cong \mathrm{H}^2(G, \mathbb{C}^\times)$.
2. *For every projective representation ρ of G, thhere xists an ordinary representation $\tilde{\rho}$ of G^* such that $\rho(g) = \tilde{\rho}(s(g))$ for all $g \in G$ and for some section $s : G \to G^*$.*

In [28], Schur proved that representation group of a finite group always exists (see also [14, Chapter 3 (Corollary 3.3)]). From above, it is clear that to determine the projective representations of a group G, it is enough to determine a representation group of G (there may be many non-isomorphic ones) and its ordinary representations. We now recall a result that elaborates the above correpondence between the projective representations of G and the ordinary representations of $G^\star$.

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Let N be a normal subgroup of G and $\chi \in \mathrm{Irr}(N)$. Let $\mathrm{Irr}(G \mid \chi)$ denote the set of inequivalent ordinary irreducible representations of G lying above χ, that is $\rho \in \mathrm{Irr}(G \mid \chi)$ if and only if $\mathrm{Hom}_N(\rho|_N, \chi)$ is non-trivial. The following well-known result relates the projective representations of G and the ordinary ones of $G^\star$, see [14, Chapter 3, Lemma 3.1] and [8, Theorem 3.2] for its proof.

Theorem 2.2. *Let α be a 2-cocycle of G. Let $\chi \in \mathrm{Hom}(A, \mathbb{C}^\times)$ be such that $\mathrm{tra}(\chi) = [\alpha]$. There is a bijective correspondence between*

$$\mathrm{Irr}^\alpha(G) \leftrightarrow \mathrm{Irr}(G^\star \mid \chi)$$

obtained via lifting a projective representation of G to an ordinary representation of $G^\star$. In particular, we obtain the following:

$$\bigcup_{[\alpha]\in \mathrm{H}^2(G,\mathbb{C}^\times)} \mathrm{Irr}^\alpha(G) \leftrightarrow \mathrm{Irr}(G^\star).$$

To understand the ordinary representations of a representation group, we also require a few general results from the theory of the ordinary characters of a finite group. These results are usually known by the name of Clifford's theory and they provide an important connection between the complex representations of a finite group G and its normal subgroups. Recall that $\mathrm{Irr}(G)$, for a finite group G, denotes the set of all inequivalent irreducible representations of G. For an abelian group A, we also use $\widehat{A}$ to denote $\mathrm{Irr}(A)$ and call this to be the set of characters (or one-dimensional representations) of A. Let N be a normal subgroup of G and $\rho \in \mathrm{Irr}(N)$ be an irreducible representation of N. The representation obtained by inducing ρ from N to G will be denoted by $\mathrm{Ind}_N^G(\rho)$. We say $\rho \in \mathrm{Irr}(N)$ has an *extension* to G if thhere xists $\tilde{\rho} \in \mathrm{Irr}(G)$ such that $\tilde{\rho}|_N = \rho$.

The group G acts on $\mathrm{Irr}(N)$ via conjugation action of G on N. For $\rho \in \mathrm{Irr}(N)$ and $g \in G$, define $\rho^g \in \mathrm{Irr}(N)$ by $\rho^g(x) = \rho(gxg^{-1})$ for all $x \in N$. For $\rho, \rho' \in \mathrm{Irr}(N)$, we use $\rho \cong \rho'$ to denote that ρ and ρ' are equivalent representations of N. For $\rho \in \mathrm{Irr}(N)$, let $I_G(\rho) = \{g \in G \mid \rho^g \cong \rho\}$ denote the *stabilizer* (or the *inertia*) group of ρ in G. We will use the following results of Clifford's theory:

Theorem 2.3.

(i) ([11], Theorem 6.11) The map

$$\theta \mapsto \mathrm{Ind}_{I_G(\rho)}^G(\theta)$$

is a bijection of $\mathrm{Irr}(I_G(\rho) \mid \rho)$ *onto* $\mathrm{Irr}(G \mid \rho)$.

(ii) ([11], Theorem 6.16) Let H be a subgroup of G containing N, and suppose that ρ is an irreducible representation of N which has an extension $\tilde{\rho}$ to H. Then the representations $\delta \otimes \tilde{\rho}$ for $\delta \in \mathrm{Irr}(H/N)$ are irreducible, distinct for distinct δ and

$$\mathrm{Ind}_N^H(\rho) = \oplus_{\delta \in \mathrm{Irr}(H/N)} \delta' \otimes \tilde{\rho},$$

where δ' is obtained by composing δ with the natural projection from H onto H/N.

(iii) ([11], Corollary 11.22) Suppose G/N is cyclic. Let $\rho \in \mathrm{Irr}(N)$ such that $I_G(\rho) = G$. Then ρ has an extension to G.

We conclude this section by including a proof of Theorem 1.5.

Proof of Theorem 1.5. Our goal is to define an isomorphism $\gamma : \mathrm{H}^2(G_2, \mathbb{C}^\times) \to \mathrm{H}^2(G_1, \mathbb{C}^\times)$ that gives $\mathbb{C}$-twist isomorphism between G_2 and G_1. It follows from [14, Theorem 2.9] and [8, Theorem 3.2] that the projective representations of G_i are obtained from those of G_i/G_i' via inflation and

$$\mathbb{C}^\alpha[G_i] \cong \prod_{\mathrm{inf_i}([\beta])=[\alpha]} \mathbb{C}^\beta[G_i/G_i']. \tag{2.3}$$

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The map $\tilde{\sigma}$ is an induced isomorphism obtained from σ. Hence, $\mathbb{C}^{\beta}[G_1/G_1'] \cong \mathbb{C}^{\tilde{\sigma}(\beta)}[G_2/G_2']$. Therefore, it is sufficient to define an isomorphism $\gamma : \mathrm{H}^2(G_2, \mathbb{C}^\times) \to \mathrm{H}^2(G_1, \mathbb{C}^\times)$ such that Figure 1 is commutative. Indeed, such a γ is obtained by defining

$$\gamma([\alpha]) = \inf_1(\tilde{\sigma}([\alpha_0])) \text{ for } [\alpha] \in H^2(G_2, \mathbb{C}^\times),$$

where $[\alpha_0] \in H^2(G_2/G_2', \mathbb{C}^\times)$ is any element such that $\inf_2([\alpha_0]) = [\alpha]$. □

3. p-groups with $s(G) \leq 2$

In this section, we study the $\mathbb{C}$-twist isomorphism classes of p-groups with $s(G) \leq 2$. We first deal with the case of $s(G) \in \{0, 1\}$.

Proof of Lemma 1.1. Niroomand [22, Theorem 21, Corollary 23] proved the following classification of finite non-abelian p-groups G with $s(G) \in \{0, 1\}$:

(a) $s(G) = 0$ if and only if G is isomorphic to $H_1^1 \times C_p^{(n-3)}$.
(b) $s(G) = 1$ if and only if G is isomorphic to $D_8 \times C_2^{(n-2)}$ or $C_p^{(4)} \rtimes C_p$ $(p \neq 2)$.

We remark that in [22], Niroomand uses the corank of a group G (denoted $t(G)$) instead of the generalized corank of G. We have used the well-known relation $t(G) = s(G) + (n-2)$ for any non-abelian p-group G to use the results of [22]. We observe that among the groups mentioned in (a) and (b), thhere xists at most one group for any given order and fixed $s(G)$. Since both order and $s(G)$ are invariant of $\mathbb{C}$-twist isomorphism, our result follows. □

The rest of this section is devoted to the $s(G) = 2$ case.

Proposition 3.1.

(i) (a) $E(2) \times C_p^{(n-4)} \sim_{\mathbb{C}} H_1^2 \times C_p^{(n-3)}$, *for* $p \neq 2$
(b) $E(2) \times C_2^{(n-4)} \sim_{\mathbb{C}} Q_8 \times C_2^{(n-3)}$
(ii) (a) For $n = 2m+1$ *and* $m \geq 2$, $H_m^1 \times C_p^{(n-2m-1)} \sim_{\mathbb{C}} H_m^2 \times C_p^{(n-2m-1)}$
(b) For $n \geq 6$ *and* $m \geq 2$, $E(2) \times C_p^{(n-2m-2)} \sim_{\mathbb{C}} H_m^1 \times C_p^{(n-2m-1)} \sim_{\mathbb{C}} H_m^2 \times C_p^{(n-2m-1)}$.

Proof. We proceed to prove (i). The proof of (ii) is similar so we only give essential ingredients there and omit the details.

(i)(a) For simplification of notations, we denote $E(2) \times C_p^{(n-4)}$ and $H_1^2 \times C_p^{(n-3)}$ by G_1 and G_2, respectively. The groups G_1 and G_2 have following presentations:

$$G_1 = < x_1, y_1, z_1, \gamma_1, a_1, a_2, \cdots a_{n-4} \mid [x_1, y_1] = z_1 = \gamma_1^p, x_1^p = y_1^p = a_i^p = 1, \gamma_1^{p^2} = 1 >$$

$$G_2 = < x_2, y_2, z_2, b_1, \cdots, b_{n-3} \mid [x_2, y_2] = x_2^p = z_2, x_2^{p^2} = y_2^p = z_2^p = b_i^p = 1 > .$$

Therefore, $G_i' \cong C_p$ and $G_i/G_i' \cong C_p^{n-1}$ for $i \in \{1, 2\}$. Also, in view of Proposition 1.3 of [14],

$$\mathrm{H}^2(G_i/G_i', \mathbb{C}^\times) \cong C_p^{\frac{(n-1)(n-2)}{2}}, \ \ \mathrm{H}^2(G_i, \mathbb{C}^\times) \cong C_p^{\frac{(n-1)(n-2)}{2}-1}.$$

This yields the following short exact sequences for $i \in \{1, 2\}$:

$$1 \longrightarrow \mathrm{Hom}(G_i', \mathbb{C}^\times) \xrightarrow{\mathrm{tra}_i} H^2(G_i/G_i', \mathbb{C}^\times) \xrightarrow{\inf_i} H^2(G_i, \mathbb{C}^\times) \longrightarrow 1.$$

We now define $\delta : G_1' \to G_2'$ and $\sigma : G_1/G_1' \to G_2/G_2'$ such that the Figure 1 is commutative. Define δ by $\delta(z_1) = z_2$ and σ by

$$\sigma(x_1 G_1') = x_2 G_2', \sigma(y_1 G_1') = y_2 G_2', \sigma(\gamma_1 G_1') = b_{n-3} G_2', \sigma(a_i G_1') = b_i G_2',$$

for all $i \in \{1, \ldots, n-4\}$. We now describe transgression maps for these groups.

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Define a section $s_1 : G_1/G_1' \to G_1$ by

$$s_1(x_1^i y_1^j \gamma_1^k a_1^{r_1} \cdots a_{n-4}^{r_{n-4}} G_1') = x_1^i y_1^j \gamma_1^k a_1^{r_1} \cdots a_{n-4}^{r_{n-4}}$$

For $u = x_1^i y_1^j \gamma_1^k a_1^{r_1} \cdots a_{n-4}^{r_{n-4}} G_1'$ and $v = x_1^{i'} y_1^{j'} \gamma_1^{k'} a_1^{r_1'} \cdots a_{n-4}^{r_{n-4}'} G_1'$, we have

$$s_1(u)s_1(v)s_1(uv)^{-1} = \gamma_1^{-pji'}.$$

Hence, a representative of $[\mathrm{tra}_1(\chi)]$ is given by $\mathrm{tra}_1(\chi)(u, v) = \chi(z_1^{-ji'})$ for $\chi \in \mathrm{Hom}(G_1', \mathbb{C}^\times)$. Define a section $s_2 : G_2/G_2' \to G_2$ by

$$s_2(x_2^i y_2^j b_1^{r_1} \cdots b_{n-3}^{r_{n-3}} G_2') = x_2^i y_2^j b_1^{r_1} \cdots b_{n-3}^{r_{n-3}}$$

For $u = x_2^i y_2^j b_1^{r_1} \cdots b_{n-3}^{r_{n-3}} G_2'$ and $v = x_2^{i'} y_2^{j'} b_1^{r_1'} \cdots b_{n-3}^{r_{n-3}'} G_2'$, we have

$$s_2(u)s_2(v)s_2(uv)^{-1} = z_2^{-ji'}.$$

Therefore, a representative of $[\mathrm{tra}_2(\chi)]$ is given by $\mathrm{tra}_2(\chi)(u, v) = \chi(z_2^{-ji'})$ for $\chi \in \mathrm{Hom}(G_2', \mathbb{C}^\times)$. This combined with the given isomorphisms δ and σ gives the commutativity of Figure 1. Now, the result follows as a direct consequence of Theorem 1.5.

For $(i)(b)$, proof is along the same lines as that of $(i)(a)$ with only difference that $Q_8 \times C_2^{(n-3)}$ has the following presentation:

$$\langle a, b, c, b_1, \cdots, b_{n-3} \mid a^4 = 1, a^2 = b^2 = c, b^{-1}ab = ca, b_i^2 = 1 \ \forall\ 1 \le i \le (n-3)\rangle.$$

We leave the rest of the details for the reader.

(ii) We denote the groups $E(2) \times C_p^{(n-2m-2)}$, $H_m^1 \times C_p^{(n-2m-1)}$ and $H_m^2 \times C_p^{(n-2m-1)}$ by G_1^m, G_2^m and G_3^m, respectively. Note that if $m \ne m'$, then for any $i, j \in \{1, 2, 3\}$, the complex group algebras of G_i^m and $G_j^{m'}$ are not isomorphic. Further, observe that for any $m \ge 2$, the order of G_1^m is p^n such that $n \ge 6$. Therefore, for some $n \ge 6$, if G_i^m is $\mathbb{C}$-twist isomorphic to $G_j^{m'}$, then it implies that $m = m'$. Similarly, when $n = 5$, a necessary condition for the $\mathbb{C}$-twist isomorphism of G_1^m and $G_2^{m'}$ is that $m = m'$. Therefore, from now onwards, we fix m and prove the result.

The commutator subgroup of G_i^m is central and is isomorphic to C_p; and $G_i^m/(G_i^m)' \cong C_p^{(n-1)}$. Further, since for any $i \in \{1, 2, 3\}$, $\mathrm{H}^2(G_i^m, \mathbb{C}^\times) \cong C_p^{\frac{n^2-3n}{2}}$, we get the following short exact sequences:

$$1 \longrightarrow \mathrm{Hom}(G_i', \mathbb{C}^\times) \xrightarrow{\mathrm{tra}_i} \mathrm{H}^2(G_i/G_i', \mathbb{C}^\times) \xrightarrow{\mathrm{inf}_i} \mathrm{H}^2(G_i, \mathbb{C}^\times) \longrightarrow 1.$$

As in (i), the proof of $\mathbb{C}$-twist isomorphism follows by considering the image of the tra_i for $i \in \{1, 2, 3\}$ and by proving the commutativity of Figure 1. This is obtained by using the following presentation of groups G_i^m.

$$\begin{aligned}
G_1^m &= < x_1, \cdots, x_m, y_1, \cdots y_m, z, \gamma, a_1, a_2, \cdots a_{n-2m-2} \mid [x_i, y_i] = z = \gamma^p, \\
&\quad x_i^p = y_i^p = a_i^p = 1, \gamma^{p^2} = 1 > . \\
G_2^m &= < x_1, \cdots, x_m, y_1, \cdots y_m, z, a_1, a_2, \cdots, a_{n-2m-1} \mid [x_i, y_i] = z, \\
&\quad x_i^p = y_i^p = a_i^p = 1 > . \\
G_3^m &= < x_1, \cdots, x_m, y_1, \cdots y_m, z, a_1, a_2, \cdots, a_{n-2m-1} \mid [x_i, y_i] = z = x_m^p = y_m^p, \\
&\quad x_i^p = y_i^p = z^p = 1 (1 \le i \le m-1), a_i^p = 1 > .
\end{aligned}$$

Below we calculate tra_1 explicitly and leave the details for tra_2 and tra_3 as those are similar. By the given presentation of G_1^m, we have $(G_1^m)' = < \gamma^p >$. Define a section $s : G_1^m/(G_1^m)' \to G_1^m$ by

$$s(x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m} \gamma^k a_1^{r_1} \cdots a_{n-2m-2}^{r_{n-2m-2}} (G_1^m)') = x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m} \gamma^k a_1^{r_1} \cdots a_{n-2m-2}^{r_{n-2m-2}}$$

Note that for any two elements

$$u = x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_m^{j_m} \gamma^k a_1^{r_1} \cdots a_{n-2m-2}^{r_{n-2m-2}} (G_1^m)', v = x_1^{i_1'} \cdots x_m^{i_m'} y_1^{j_1'} \cdots y_m^{j_m'} \gamma^{k'} a_1^{r_1'} \cdots a_{n-2m-2}^{r_{n-2m-2}'} (G_1^m)'$$

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of $G_1^m/(G_1^m)'$, we have $s(u)s(v)s(uv)^{-1} = \gamma^{-p\sum_{l=1}^{m} j_l i_l'}$. Therefore, for any $\chi \in \mathrm{Hom}((G_1^m)', \mathbb{C}^\times)$, a representative of $[\mathrm{tra}_1(\chi)]$ is given by $\mathrm{tra}_1(\chi)(u, v) = \chi(z^{-\sum_{l=1}^{m} j_l i_l'})$. By a similar computation of tra_2 and tra_3, we obtain that for all $i \in \{1, 2, 3\}$, the groups G_i^m pairwise satisfy the hypothesis of Theorem 1.5 and hence are $\mathbb{C}$-twist isomorphic. □

We now complete the details regarding the $\mathbb{C}$-twist isomorphism classes for p-groups with $s(G) = 2$.

Proof of Theorem 1.2. It follows from [24, Theorem 11] that for any fixed p, thhere xists only one p-group of order p^3 with $s(G) = 2$ and so it forms a singleton $\mathbb{C}$-twist class. We now consider cases when $n \geq 4$.

$n = 4$: Any group of order p^4 with $s(G) = 2$ is isomorphic to one of the following:

- $E(2)$
- $H_1^2 \times C_p, p \neq 2$
- $Q_8 \times C_2$
- $\langle a, b \mid a^4 = 1, b^4 = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^2b^2 \rangle$
- $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$
- $\langle a, b \mid a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$
- $\langle a, b \mid a^9 = b^3 = 1, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle$
- $\langle a, b \mid a^p = 1, b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle$ $(p \neq 3)$.

Here, the group $\langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$ is isomorphic to $E(2)$. As mentioned in Theorem 4.3 in [17], any non-singleton $\mathbb{C}$-twist isomorphism class of groups of order p^4 consists of two groups, when $p = 2$; and of three groups, when p is an odd prime. Thus, comparing with the groups given in Tables 3 and 4 in [17], we obtain the following non-singleton $\mathbb{C}$-twist isomorphism classes of groups of order p^4 with generalized corank 2:

- $\mathbb{Q}_8 \times C_2 \sim_{\mathbb{C}} E(2)$, when $p = 2$
- $E(2) \sim_{\mathbb{C}} H_1^1 \times C_p \sim_{\mathbb{C}} \langle a, b \mid a^{p^2} = 1, b^p = 1, [a, b, a] = [a, b, b] = 1 \rangle$, when p is odd.

Thus, each of the remaining groups of order p^4 in the above list constitutes a $\mathbb{C}$-twist isomorphism class of size 1.

$n \geq 5$: Any group of order p^n with $n \geq 5$ and $s(G) = 2$ is isomorphic to one of the following:

- $E(2) \times C_p^{(n-2m-2)}$
- $H_1^2 \times C_p^{(n-3)}, p \neq 2$
- $Q_8 \times C_2^{(n-3)}$
- $H_m^1 \times C_p^{(n-2m-1)}$
- $H_m^2 \times C_p^{(n-2m-1)}$
- $C_p \times (C_p^4 \rtimes_\theta C_p)$, $p \neq 2$.

The derived subgroup of $C_p \times (C_p^4 \rtimes_\theta C_p)$, $p \neq 2$ is of order p^2; whereas the derived subgroup of the rest of the groups in the above list is of order p. Therefore, by comparing the complex group algebras, we obtain that for a fixed odd prime p, the group $C_p \times (C_p^4 \rtimes_\theta C_p)$ forms a singleton $\mathbb{C}$-twist isomorphism class. Finally, Proposition 3.1 completes the classification of the rest of the groups into $\mathbb{C}$-twist isomorphism classes. □

4. p-groups with $s(G) = 3$

In this section, we proceed with the determination of the $\mathbb{C}$-twist isomorphism classes of the p-groups with $s(G) = 3$. The following result, using the notations of [12], gives a complete list of p-groups with $s(G) = 3$.

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Theorem 4.1. *([7], Theorem 1.1) Let G be a non-abelian p-group of order p^n with $s(G) = 3$. Let r_p be the smallest positive integer which is a non-quadratic residue mod (p).*

(a) For an odd prime p, G is isomorphic to one of the following groups:

(i) $\phi_2(22) = \langle \alpha,\ \alpha_1,\ \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2,\ \alpha_1^{p^2} = \alpha_2^p = 1\rangle$
(ii) $\phi_3(211)a = \langle \alpha,\ \alpha_1, \alpha_2,\ \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2,\ [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$
(iii) $\phi_3(211)b_r = \langle \alpha,\ \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2,\ [\alpha_2, \alpha]^r = \alpha_1^{(p)} = \alpha_3^r,\ \alpha^p = \alpha_2^p = \alpha_3^p = 1\rangle$*, where r is either* 1 *or* r_p.
(iv) $\phi_2(2111)c = \phi_2(211)c \times C_p$*, where* $\phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2,\ \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1\rangle$
(v) $\phi_2(2111)d = ES_p(p^3) \times C_{p^2}$
(vi) $\phi_3(1^5) = \phi_3(1^4) \times C_p$*, where* $\phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1},\ \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1\ (i = 1, 2)\rangle$
(vii) $\phi_7(1^5) = \langle\ \alpha,\ \alpha_1, \alpha_2, \alpha_3, \beta \mid\ [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3,\ \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1\ (i = 1, 2)\rangle$
(viii) $\phi_{11}(1^6) = \langle\ \alpha_1,\ \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid\ [\alpha_1, \alpha_2] = \beta_3,\ [\alpha_2, \alpha_3] = \beta_1,\ [\alpha_3, \alpha_1] = \beta_2, \alpha_i^{(p)} = \beta_i^p\ (i = 1, 2, 3)\rangle$
(ix) $\phi_{12}(1^6) = ES_p(p^3) \times ES_p(p^3)$
(x) $\phi_{13}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_1, \alpha_{i+1}] = \beta_i,\ [\alpha_2, \alpha_4] = \beta_2,\ \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2)\rangle$
(xi) $\phi_{15}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_1, \alpha_{i+1}] = \beta_i,\ [\alpha_3, \alpha_4] = \beta_1,\ [\alpha_2, \alpha_4] = \beta_2^g,\ \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2)\rangle$*, where g is the smallest positive integer, which is a primitive root modulo p*
(xii) $(C_p^{(4)} \rtimes C_p) \times C_p^2$.

(b) For $p = 2$*, G is isomorphic to one of the following groups:*

(xiii) $C_2^4 \rtimes C_2$
(xiv) $C_2 \times ((C_4 \times C_2) \rtimes C_2)$
(xv) $C_4 \rtimes C_4$
(xvi) D_{16}*, the dihedral group of order 16.*

As mentioned earlier, the $\mathbb{C}$-twist isomorphism classes of the groups of order p^4 were described by Margolis-Schnabel [17]. We now consider the groups of order p^5 with $s(G) = 3$. Let H_1 and H_2 denote the groups $\phi_2(2111)d$ and $\phi_2(2111)c$, respectively. We proceed to prove that H_1 and H_2 are $\mathbb{C}$-twist isomorphic. We use the following general result to prove this.

Lemma 4.2. *Let G_1 and G_2 be two finite groups with $\widetilde{G}_1$ and $\widetilde{G}_2$ as their representation groups, respectively. For $i \in \{1, 2\}$, let A_i be a central subgroup of $\widetilde{G}_i$ such that $\widetilde{G}_i/A_i \cong G_i$ and the transgression maps $\mathrm{tra_i} : \mathrm{Hom}(A_i, \mathbb{C}^\times) \to \mathrm{H}^2(G_i, \mathbb{C}^\times)$ are isomorphisms. Let $\sigma : \mathrm{Hom}(A_1, \mathbb{C}^\times) \to \mathrm{Hom}(A_2, \mathbb{C}^\times)$ be an isomorphism such that the following sets are in a dimension preserving bijection for every $\chi \in \mathrm{Hom}(A_1, \mathbb{C}^\times)$:*

$$\mathrm{Irr}(\widetilde{G}_1 \mid \chi) \leftrightarrow \mathrm{Irr}(\widetilde{G}_2 \mid \sigma(\chi)).$$

Then G_1 and G_2 are $\mathbb{C}$-twist isomorphic.

Proof. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}(A_1, \mathbb{C}^\times) & \xrightarrow{\mathrm{tra}_1} & \mathrm{H}^2(G_1, \mathbb{C}^\times) \\ \downarrow \sigma & & \downarrow \\ \mathrm{Hom}(A_2, \mathbb{C}^\times) & \xrightarrow{\mathrm{tra}_2} & \mathrm{H}^2(G_2, \mathbb{C}^\times). \end{array}$$

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Define $\tilde{\sigma}: \mathrm{H}^2(G_1, \mathbb{C}^\times) \to \mathrm{H}^2(G_2, \mathbb{C}^\times)$ by $\tilde{\sigma}(\mathrm{tra}_1(\chi)) = \mathrm{tra}_2(\sigma(\chi))$ for $\chi \in \mathrm{Hom}(A_1, \mathbb{C}^\times)$. It is easy to see that $\tilde{\sigma}$ is a group isomorphism. By Theorem 2.2, the dimension preserving bijection

$$\mathrm{Irr}(\widetilde{G}_1 \mid \chi) \leftrightarrow \mathrm{Irr}(\widetilde{G}_2 \mid \sigma(\chi))$$

for any $\chi \in \mathrm{Hom}(A_1, \mathbb{C}^\times)$ gives

$$\mathbb{C}^{\alpha_1}[G_1] \cong \mathbb{C}^{\alpha_2}[G_2],$$

where $[\alpha_1] = \mathrm{tra}_1(\chi)$ and $[\alpha_2] = \mathrm{tra}_2(\sigma(\chi))$. Therefore, $\tilde{\sigma}$ gives the required $\mathbb{C}$-twist isomorphism between G_1 and G_2. □

Note that $H_1 = E_1 \times C_{p^2}$ and $H_2 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle \times \langle \alpha_3 \rangle$. Define the following groups:

$$\begin{aligned}\widetilde{H}_1 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, x, y, z, \alpha \mid\ & [x, y] = z, [x, z] = \alpha_1, [y, z] = \alpha_2, \\ & [x, \alpha] = \alpha_3, [y, \alpha] = \alpha_4, x^p = y^p = z^p = \alpha_i^p = \alpha^{p^2} = 1 \rangle.\end{aligned}$$

and

$$\begin{aligned}\widetilde{H}_2 = \langle x, y, z, w, \alpha, \alpha_1, \alpha_2, \alpha_3 \mid\ & [\alpha_1, \alpha] = \alpha_2, [\alpha_1, \alpha_2] = x, [\alpha, \alpha_2] = y, \\ & [\alpha_3, \alpha_1] = z, [\alpha_3, \alpha] = w, \alpha^{p^2} = \alpha_i^p = x^p = y^p = z^p = w^p = 1 \rangle.\end{aligned}$$

Lemma 4.3. *The groups $\widetilde{H}_1$ and $\widetilde{H}_2$ are representation groups of H_1 and H_2, respectively.*

Proof. Here, we give a proof for H_1. The proof for H_2 is along the same lines so we omit that part. The group H_1 has the following presentation:

$$H_1 = \langle x, y, z, \alpha \mid [x, y] = z, x^p = y^p = \alpha^{p^2} = 1 \rangle.$$

Consider the projection map from $\widetilde{H}_1$ onto H_1 obtained by mapping x, y, z, α to x, y, z, α, respectively, and all α_i to 1. Let K_1 be the kernel of this projection map. Then $|K_1| = |\mathrm{H}^2(H_1, \mathbb{C}^\times)| = p^4$ and $K_1 \subseteq Z(\widetilde{H}_1) \cap [\widetilde{H}_1, \widetilde{H}_1]$. Therefore, by [14, Theorem 3.7 (Chapter 3)], $\widetilde{H}_1$ is a representation group of H_1. □

Proposition 4.4. *The groups $\widetilde{H}_1$ and $\widetilde{H}_2$ satisfy the following:*

$$\mathbb{C}[\widetilde{H}_1] \cong \mathbb{C}[\widetilde{H}_2] \cong \mathbb{C}^{\oplus p^4} \oplus (\mathrm{M}_p(\mathbb{C}))^{\oplus (p^3+4p^2-p+1)p^2(p-1)} \oplus (\mathrm{M}_{p^2}(\mathbb{C}))^{\oplus p^3(p-1)^3(p+2)}.$$

Furthermore, for the subgroups $A = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ and $B = \langle x, y, z, w \rangle$ of $\widetilde{H}_1$ and $\widetilde{H}_2$ respectively, thehere ists an isomorphism $\sigma: \widehat{A} \to \widehat{B}$ such that the following sets are in a dimension preserving bijection for every $\chi \in \hat{A}$:

$$\mathrm{Irr}(\widetilde{H}_1 \mid \chi) \leftrightarrow \mathrm{Irr}(\widetilde{H}_2 \mid \sigma(\chi)).$$

Proof. **Representations of $\widetilde{H}_1$:** We first justify the representations of $\widetilde{H}_1$. By the definition of $\widetilde{H}_1$, the derived subgroup of $\widetilde{H}_1$ (denoted $\widetilde{H}_1'$) is $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, z \rangle$ and the center of $\widetilde{H}_1$ is $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha^p \rangle$. By considering the quotient group $\widetilde{H}_1/\widetilde{H}_1'$, we obtain that $\widetilde{H}_1$ has exactly p^4 one-dimensional representations.

We next consider the abelian normal subgroup $N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha, z \rangle$ of $\widetilde{H}_1$. The group N has order p^7. By Frobenius reciprocity and Theorem 2.3, every irreducible representation of $\widetilde{H}_1$ has dimension either $1, p$ or p^2. We have already justified all one-dimensional representations of $\widetilde{H}_1$. Our next goal is to determine all p and p^2 dimensional representations of $\widetilde{H}_1$.

Let $\chi \in \mathrm{Irr}(N)$ such that $\chi(\alpha_1) = \xi^{i_1}$, $\chi(\alpha_2) = \xi^{i_2}$, $\chi(\alpha_3) = \xi^{i_3}$ and $\chi(\alpha_4) = \xi^{i_4}$, where ξ is a primitive p^{th} root of unity and $0 \le i_1, i_2, i_3, i_4 \le (p-1)$. We determine the stabilizer (or inertia group) of χ in $\widetilde{H}_1$, denoted by $I_{\widetilde{H}_1}(\chi)$. Recall from Section 2, $I_{\widetilde{H}_1}(\chi) = \{g \in \widetilde{H}_1 \mid \chi^g = \chi\}$. By definition of $I_{\widetilde{H}_1}(\chi)$, $N \le I_{\widetilde{H}_1}(\chi)$. We will obtain the following information from $|I_{\widetilde{H}_1}(\chi)|$ and by the virtue of Theorem 2.3.

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1. $\left|\frac{I_{\widetilde{H}_1}(\chi)}{N}\right| = 1$ implies every $\rho \in \mathrm{Irr}(\widetilde{H}_1 \mid \chi)$ satisfies $\dim(\rho) = p^2$.
2. $\left|\frac{I_{\widetilde{H}_1}(\chi)}{N}\right| = p$ implies every $\rho \in \mathrm{Irr}(\widetilde{H}_1 \mid \chi)$ satisfies $\dim(\rho) = p$.
3. $\left|\frac{I_{\widetilde{H}_1}(\chi)}{N}\right| = p^2$ implies every $\rho \in \mathrm{Irr}(\widetilde{H}_1 \mid \chi)$ satisfies $\dim(\rho) \in \{1, p\}$.

As we already have information regarding dimension one representations, we will easily obtain the rest of the information regarding $\mathbb{C}[\widetilde{H}_1]$ from the description of $I_{\widetilde{H}_1}(\chi)$. Consider $g = x^i y^j n$, with $n \in N$ and $i, j \in \{0, 1, \ldots, p-1\}$. Recall that every $m \in N$ satisfies $m = \alpha^e z^f h$ for some $h \in Z(\widetilde{H}_1)$. Therefore, we have

$$\chi^{x^i y^j}(\alpha^e z^f h) = \chi(\alpha^e z^f h)$$

if and only if $\chi(x^i y^j \alpha^e z^f y^{-j} x^{-i}) = \chi(\alpha^e z^f)$. Since

$$\begin{aligned} x^i(y^j\alpha^e)z^f y^{-j}x^{-i} &= x^i(\alpha^e y^j \alpha_4^{je})z^f y^{-j}x^{-i} = \alpha^e x^i \alpha_3^{ie} y^j \alpha_4^{je} z^f y^{-j} x^{-i} \\ &= \alpha^e x^i y^j z^f y^{-j} x^{-i} \alpha_3^{ie} \alpha_4^{je} = \alpha^e x^i z^f y^j \alpha_2^{if} y^{-j} x^{-i} \alpha_3^{ie} \alpha_4^{je} \\ &= \alpha^e z^f \alpha_1^{if} \alpha_2^{if} \alpha_3^{ie} \alpha_4^{je}, \end{aligned}$$

we obtain that $x^i y^j n \in I_{\widetilde{H}_1}(\chi)$ if and only if $\chi(\alpha_1^{if}\alpha_2^{if}\alpha_3^{ie}\alpha_4^{je}) = 1$. This is equivalent to saying that $\xi^{i_1 if + i_2 jf + i_3 ie + i_4 je} = 1$.

We now consider various cases of irreducible representations of N. Note that in each of the cases discussed below, all the computations for i and j are done modulo p.

- **Case I:** Consider $\chi \in \mathrm{Irr}(N)$ such that $i_1 = i_2 = i_3 = i_4 = 0$. These are exactly p^3 in number. Among these there are p^2 characters which act trivially on z and hence on $\widetilde{H}_1'$. These give p^4 one-dimensional characters of $\widetilde{H}_1$. The other $(p^3 - p^2)$ characters of N, by Theorem 2.3, of this case give irreducible characters of $\widetilde{H}_1$ of dimension p and therefore we obtain $p^4 - p^3$ many characters of $\widetilde{H}_1$ of dimension p.
- **Case II:** When any three of i_1, i_2, i_3, i_4 are 0 and the fourth one is non-zero, then $g = x^i y^j n \in I_{\widetilde{H}_1}(\chi)$ if and only if either i or j is 0. Thus, $|I_{\widetilde{H}_1}(\chi)| = p^8$.
- **Case III:** When any two of i_1, i_2, i_3, i_4 are 0 and the other two are non-zero, then i is a non-zero multiple of j or one of them is zero and other can take any value. Hence, $|I_{\widetilde{H}_1}(\chi)| = p^8$.
- **Case IV:** When any three of i_1, i_2, i_3, i_4 are non-zero and the fourth one is 0, then $i = j = 0$ and hence $I_{\widetilde{H}_1}(\chi) = N$.
- **Case V:** Assume that each i_t, where $t \in \{1, 2, 3, 4\}$, is non-zero. When $f = 0$ and $e = 1$, $i = \frac{-i_4 j}{i_3}$; and when $e = 0$ and $f = 1$, $i = \frac{-i_2 j}{i_1}$. Therefore, $\frac{-i_4 j}{i_3} = \frac{-i_2 j}{i_1}$, which holds if and only if $(i_4 i_1 - i_2 i_3)j = 0$. Now if $i_4 i_1 \neq i_2 i_3$, then $i = j = 0$ and hence $I_{\widetilde{H}_1}(\chi) = N$.
  Here, note that $(p-1)^3$ many characters of N satisfy $i_4 i_1 = i_2 i_3$ and their inertia group is of order p^8. Thus, the remaining $((p-1)^4 - (p-1)^3)$ characters have inertia group N.

Considering the case of $(p^7 - p^3)$ many characters of N, discussed in Cases II-V, the inertia group of $p^3(p-1)^3(p+2)$ characters is N, and for the other $p^3(p-1)(p^2 + 4p - 1)$ characters, it is of order p^8. Hence, we obtain the following description of the group algebra of $\widetilde{H}_1$.

$$\mathbb{C}[\widetilde{H}_1] \cong \mathbb{C}^{\oplus p^4} \oplus (\mathrm{M}_p(\mathbb{C}))^{\oplus(p^3+4p^2-p+1)p^2(p-1)} \oplus (\mathrm{M}_{p^2}(\mathbb{C}))^{\oplus p^3(p-1)^3(p+2)}.$$

Representations of $\widetilde{H}_2$: We have $\widetilde{H}_2' = \langle x, y, z, w, \alpha_2 \rangle$ and $Z(\widetilde{H}_2) = \langle \alpha^p, x, y, z, w \rangle$. Clearly, there are exactly p^4 number of linear characters of $\widetilde{H}_2$. Consider the subgroup

$$M = \langle \alpha_2, \alpha_3, x, y, z, w, \alpha^p \rangle$$

of $\widetilde{H}_2$. This is an abelian normal group of order p^7. Let $\chi \in \mathrm{Irr}(M)$ such that $\chi(x) = \xi^{i_1}$, $\chi(y) = \xi^{i_2}$, $\chi(z) = \xi^{i_3}$, and $\chi(w) = \xi^{i_4}$, where ξ is a primitive p^{th} root of unity and $0 \leq i_1, i_2, i_3, i_4 \leq (p-1)$. Let $g = \alpha^i \alpha_1^j m$, where $m \in M$, be an element of $I_G(\chi)$. Therefore, for any $m' = \alpha_2^k \alpha_3^l h \in M$, where $h \in Z(\widetilde{H}_1)$,

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we have $\chi^{\alpha^i\alpha_1^j}(\alpha_2^k\alpha_3^l h) = \chi(\alpha_2^k\alpha_3^l h)$. Computations, similar to $\widetilde{H}_1$, yield that $g \in I_{\widetilde{H}_2}(\chi)$ if and only if $\xi^{i_1jk+i_2ik+i_3(-jl)+i_4(-il)} = 1$ and hence the following cases arise:

- **Case I:** When any three of i_1, i_2, i_3, i_4 are 0 and the fourth one is non-zero, then either i or j is 0. Thus, $|I_{\widetilde{H}_2}(\chi)| = p^8$.
- **Case II:** When any two of i_1, i_2, i_3, i_4 are 0 and the other two are non-zero, then i is a non-zero multiple of j or one of them is zero and other can take any value. Hence, $|I_{\widetilde{H}_2}(\chi)| = p^8$.
- **Case III:** When any three of i_1, i_2, i_3, i_4 are non-zero and the fourth one is 0, then $i = j = 0$ and hence $I_{\widetilde{H}_2}(\chi) = M$.
- **Case IV:** Assume that each i_t, where $t \in \{1, 2, 3, 4\}$, is non-zero. When $l = 0$ and $k = 1$, $i = \frac{-i_1j}{i_2}$; and when $k = 0$ and $l = 1$, $i = \frac{-i_3j}{i_4}$. Therefore, $\frac{-i_3j}{i_4} = \frac{-i_1j}{i_2}$, which holds if and only if $(i_4i_1 - i_2i_3)j = 0$. Now if $i_4i_1 \neq i_2i_3$, then $i = j = 0$ and hence $I_G(\chi) = M$. On the other hand, if $i_4 = \frac{i_2i_3}{i_1}$, then $|I_G(\chi)| = p^8$.
  Therefore, in this case the inertia group of $(p-1)^3$ many characters is of order p^8 and for the remaining $((p-1)^4 - (p-1)^3)$ characters it is M.

Similar to the proof of $\widetilde{H}_1$, all the above cases along with Theorem 2.3 give

$$\mathbb{C}[\widetilde{H}_2] \cong \mathbb{C}^{\oplus p^4} \oplus (\mathrm{M}_p(\mathbb{C}))^{\oplus(p^3+4p^2-p+1)p^2(p-1)} \oplus (\mathrm{M}_{p^2}(\mathbb{C}))^{\oplus p^3(p-1)^3(p+2)}.$$

We now proceed to define required isomorphism $\sigma : \widehat{A} \to \widehat{B}$. For this, define an isomorphism $\sigma' : A \to B$ by $\sigma'(\alpha_1) = x, \sigma'(\alpha_2) = y, \sigma'(\alpha_3) = z, \sigma'(\alpha_4) = w$. This defines an isomorphism, denoted by σ, between $\widehat{A}$ and $\widehat{B}$. From above discussion, by considering various cases of i_j for $j \in \{1, \ldots, 4\}$, we obtain a dimension preserving bijection between the following sets for every $\chi \in \widehat{A}$:

$$\mathrm{Irr}(\widetilde{H}_1 \mid \chi) \leftrightarrow \mathrm{Irr}(\widetilde{H}_2 \mid \sigma(\chi)).$$

□

Proposition 4.5. *The groups H_1 and H_2 are $\mathbb{C}$-twist isomorphic.*

Proof. This follows from Lemmas 4.2, 4.3 and Proposition 4.4. □

Lemma 4.6. $\mathbb{C}[\phi_3(1^5)] \ncong \mathbb{C}[\phi_7(1^5)]$.

Proof. It follows from the presentations of $\phi_3(1^5)$ and $\phi_7(1^5)$ that the nilpotency class of both the groups is 3. Now, consider the abelian normal subgroup $N_1 = \langle \alpha_1, \alpha_2, \alpha_3, \beta \rangle$ of $\phi_3(1^5)$. Since it is of index p, each irreducible representation of $\phi_3(1^5)$ is of dimension at most p.

Now note that the derived subgroup of $\phi_7(1^5)$ is $\langle \alpha_2 \rangle \times \langle \alpha_3 \rangle$ and its center is $\langle \alpha_3 \rangle$. Consider the abelian normal subgroup $N_2 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ of $\phi_7(1^5)$. Let $\chi \in \mathrm{Irr}(N_2)$ such that $\chi(\alpha_1) = \zeta^{i_1}$, $\chi(\alpha_2) = \zeta^{i_2}$ and $\chi(\alpha_3) = \zeta^{i_3}$, where ζ is a primitive p-th root of unity and $0 \leq i_1, i_2, i_3 \leq (p-1)$. Assume that for some $0 \leq i, j \leq p-1$, $\alpha^i\beta^j$ stabilizes χ. Let $\alpha_1^a\alpha_2^b\alpha_3^c \in N_2$.

Since the group $\phi_7(1^5)$ is of nilpotency class 3,

$$\begin{aligned}
\alpha^i(\beta^j\alpha_1^a)\alpha_2^b\alpha_3^c\beta^{-j}\alpha^{-i} &= \alpha^i\alpha_1^a\beta^j\alpha_2^b\beta^{-j}\alpha^{-i}\alpha_3^{c-aj} \\
&= \alpha_2^{-ai}\alpha_1^a\alpha^i\beta^j\alpha_2^b\beta^{-j}\alpha^{-i}\alpha_3^{c-aj-a\binom{i}{2}} \\
&= \alpha_1^a\alpha_2^{b-ai}\alpha_3^{-ib+c-aj-a\binom{i}{2}} = \alpha_1^a\alpha_2^b\alpha_3^c\alpha_2^{-ai}\alpha_3^{-ib-aj-a\binom{i}{2}}.
\end{aligned}$$

Thus, $\alpha^i\beta^j$ stabilizes χ if, and only if, $\zeta^{-aii_2-(ib+aj+a\binom{i}{2})i_3} = 1$. When $i_2 = 0$ and $i_3 \neq 0$, then for $a = 0$ and $b = \frac{-1}{i_3}$, we have $\zeta^i = 1$; which implies that $i = 0$. Further, $a = \frac{-1}{i_3}$ (note that i_3 is invertible modulo p) gives $\zeta^j = 1$. Hence, $j = 0$ and it follows that the inertia group of χ in $\phi_7(1^5)$ is N_2. Therefore, by Theorem 2.3, the character of $\phi_7(1^5)$ induced from χ is irreducible of degree p^2. Since $\phi_3(1^5)$ has no

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irreducible representations of dimension p^2, the complex group algebras of $\phi_3(1^5)$ and $\phi_7(1^5)$ are not isomorphic. $\square$

Proof of Theorem 1.3. From Theorem 4.1, it is clear that every p-group with $s(G) = 3$ has order p^n where $n \in \{4, \ldots, 7\}$.

In the following, we separately consider the cases of n with $4 \le n \le 7$:

$n = 4$. For the proof of this case, one can refer to [17, Theorem 4.3].

$n = 5$. When p is an odd prime, it follows from Theorem 4.1 that the only groups of order p^5 with $s(G) = 3$ are $\phi_3(1^5)$, $\phi_7(1^5)$, H_1 and H_2. The derived subgroups of $\phi_3(1^5)$ and $\phi_7(1^5)$ are elementary abelian of order p^2 and of H_1 and H_2 are of order p. Thus, the groups $\phi_3(1^5)$ and $\phi_7(1^5)$ have p^3 linear characters; whehere H_1 and H_2 have p^4 linear characters. Therefore, no group in the set $\{\phi_3(1^5), \phi_7(1^5)\}$ can be $\mathbb{C}$-twist isomorphic to any group in $\{H_1, H_2\}$. Now it follows from Lemma 4.6 and Proposition 4.5 that in this case the only non-singleton $\mathbb{C}$-twist isomorphism class is constituted by H_1 and H_2.

When $p = 2$, $T_1 = C_2^4 \rtimes C_2$ and $T_2 = C_2 \times ((C_4 \times C_2) \rtimes C_2)$ are the only two 2-groups of order 32 that have generalized corank 3. The GAP ID of these groups is [32,27] and [32,22] and it can be checked using GAP [5] that the size of the derived subgroup of T_1 is 4 and that of T_2 is 2. Thus, the complex group algebras of these groups are not isomorphic and hence each of these groups constitutes a singleton $\mathbb{C}$-twist isomorphism class.

$n = 6$. The groups of order p^6 with $s(G) = 3$ are $\phi_{11}(1^6)$, $\phi_{12}(1^6)$, $\phi_{13}(1^6)$ and $\phi_{15}(1^6)$. Note that the size of the commutator subgroup of $\phi_{11}(1^6)$ is p^3 and of the rest of the groups is p^2.

It can be checked that $\phi_{12}(1^6) = ES_p(p^3) \times ES_p(p^3)$ has $2p^2(p-1)$ inequivalent irreducible representations of dimension p and $(p-1)^2$ of dimension p^2. Further, it follows from [12, Table 4.1] that $\phi_{13}(1^6)$ has $(p^3 - p^2)$ representations of dimension p; whereas $\phi_{15}(1^6)$ has no representation of dimension p. Thus, the complex group algebras of the groups of order p^6 with generalized corank 3 are not isomorphic. It establishes the desired result.

$n = 7$. There is a unique group of order p^7 with $s(G) = 3$ which is isomorphic to $C_p^{(4)} \rtimes C_p \times C_p^2$. Therefore, it constitutes a singleton $\mathbb{C}$-twist isomorphism class. $\square$

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