Proceedings of the Edinburgh Mathematical Society: page 1 of 24
doi:10.1017/S0013091524000257

A VARIATIONAL PRINCIPLE OF AMENABLE RANDOM METRIC MEAN DIMENSIONS

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(Received 14 April 2023)

Abstract In the context of random amenable group actions, we introduce the notions of random upper metric mean dimension with potentials and the random upper measure-theoretical metric mean dimension. Besides, we establish a variational principle for the random upper metric mean dimensions. At the end, we study the equilibrium state for random upper metric mean dimensions.

Keywords: random dynamical systems; amenable group actions; metric mean dimension; measure-theoretic metric mean dimension; variational principle; equilibrium states; potentials

2020 Mathematics subject classification: 37A35; 28D20

1. Introduction

Mean dimensions are quantities that measure the complexity of dynamical systems with infinite entropy. The concept of mean dimension was first introduced by Gromov in [20], which is a topological invariant of dynamical systems to count the average number of parameters needed per iteration for describing a point. Later, Lindenstrauss and Weiss in [39] introduced the metric mean dimension and they related metric mean dimension to the problem that whether a dynamical system can be embedded in the shift system. The metric mean dimension of a dynamical system is not a topological invariant, as it depends on the specific metric chosen for the phase space X; however, there exists an intriguing invariant property for metrizable topological spaces. Specifically, the infimum of metric dimensions over all metrics on X that induce its topology is invariant under topological conjugacy. In general, the metric mean dimension of a system may not exceed its mean dimension. However, Lindenstrauss and Tsukamoto presented an exciting result in [38]. They demonstrated that systems possessing the marker property can be endowed with a metric ρ compatible with the topology of the phase space. Remarkably, the metric mean dimension of ρ coincides with the mean dimension of the system. Besides, in [37], Lindenstrauss and Tsukamoto delved into the concept of mean dimension alongside rate distortion theory, a field that investigates the lossy data compression of stochastic processes under distortion constraints. Within this study, they established a variational principle connecting rate distortion function to metric mean dimension. Later, Gutman and Śpiewak [19] demonstrated that in the Lindenstrauss–Tsukamoto variational principle, it suffices to consider the supremum over ergodic measures. In a more recent study, Shi [46] established variational principles linking metric mean dimension with Shapira's entropy, Katok's entropy and Brin-Katok's entropy.

The fundamental theory of amenable group actions in the analysis of dynamical systems was established by Ornstein and Weiss in [43], which provides an approach to generalize vast majority of entropic and ergodic theorems known for the actions of $\mathbb{Z}$ such as [1, 36, 45]. As the core techniques in the analysis of dynamical systems with amenable group actions, the ε-tiling and the ε-quasi-tiling were introduced in [42, 43]. The ε-tiling has been further developed by Weiss in [48], where he showed that countable amenable groups from a large class admit a precise tiling by only one monotile belonging to a selected Følner sequence. The ε-quasi-tiling has been developed by Downarowicz et al. [17]. After Gromov–Lindenstrauss–Weiss' fundamental works on mean dimension theory, there are sequences of studies on mean dimensions in the context of amenable group actions, for instance [10, 12, 13, 27, 30, 47].

Scholars showed interest in the ergodic theory of random transformations since 1980s, which emerged from Kifer [24], Crauel [9], Ledrappier and Young [40], Bogenschutz [4, 5], etc. For the classic $\mathbb{Z}_+$ or $\mathbb{Z}$-system, the random transformations can be regarded as skew-product transformations rather than iterations of just one map. The study within the framework of bundle random dynamical systems (RDSs) with amenable group actions was conducted by Dooley and Zhang in [16]. They introduced the notions of random local pressures as well as entropies and a concept of (factor) excellent or good covers as a crucial technique to establish the variational principle for random local pressures. We refer to the elegant literatures [2, 26] for more information on the theory of RDSs and [21, 29, 33–35, 41, 51–53] for the recent progresses on the entropy theory in RDSs.

An approach rooted in convex analysis arises as a method to investigate the thermodynamic formalism of dynamical systems, along with the study of the metric mean dimension. In [8], Cioletti, Silva and Stadlbauer considered the pressure functional $P(\phi)$, which is convex for the bounded continuous potentials ϕ defined on the sequence space $X = E^{\mathbb{N}}$, where E is a general standard Borel space. They obtained the existence of equilibrium states as finitely additive probability measures for any bounded continuous potential. Biś, Carvalho, Mendes and Varandas in [3] established an abstract variational principle for the so-called *pressure functions* acting on a Banach space of potentials of a compact metric space, for which equilibrium states always exist. This approach was later adapted by Yang, Chen and Zhou, as reported in [54], where they successfully proved a variational principle for the upper metric mean dimension, incorporating the concept of potential. In [7], Carvalho, Pessil and Varandas introduced the concept of the upper metric mean dimension for a one-parameter family of scaled pressure functions. With the approach supported by convex analysis, they established a corresponding variational principle. Additionally, they examined the computability of this measure-theoretic map and presented several examples.

The present work is dedicated to explore the notion of the random metric mean dimension with potentials in the context of amenable group actions, which is mainly inspired by the work on an abstract variational principle for upper semi-continuous affine

https://doi.org/10.1017/S0013091524000257 Published online by Cambridge University Press

entropy-like map [3] and the work on random metric mean dimension for skew product transformations [55]. First, in §3.1, we introduce a notion of random upper metric mean dimension with potentials for amenable group actions and study its basic properties. Second, in §3.2, we introduce random upper measure-theoretic metric mean dimension and present its several characterizations. Third, in §4, we establish a variational principle between these two kinds of mean dimensions. Finally, in §5, we we introduce a notion of equilibrium state for amenable random upper metric mean dimension with potentials and discuss its properties.

2. Preliminaries and main result

In this section, we collect some basic notions and notations in literature and then state the main result. An *action* on a set X induced by a group G (or we say G *acts* on X) is a set of maps $f_G = \{f_g : X \to X | g \in G\}$ satisfying:

(1) $f_e = \mathrm{id}_X$, where e is the unit element of G,
(2) $f_{g_2} \circ f_{g_1} = f_{g_2 g_1}$ for every $g_1, g_2 \in G$.

Note that f_g is a bijection for each $g \in G$. For simplicity, we identify the the group G with its action f_G by writing $gx := f_g(x)$ for every $g \in G$ and $x \in X$. A group action G of a measurable space $(X, \mathscr{B}, \mu)$ is said to be *measure preserving* (or we say μ is G-invariant), if $\mu(gB) = \mu(B)$ for every $g \in G$ and $B \in \mathscr{B}$. A probability measure μ on a measurable space $(X, \mathscr{B})$ with a group action G is said to be G-ergodic, if for any given $B \in \mathscr{B}$, we have $\mu(gB \Delta B) = 0$ for every $g \in G$ if and only if $\mu(B) = 1$ or $\mu(B) = 0$.

The setup of bundle RDS or RDS consists of a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ together with a group G acting on Ω measure preservingly, a compact metic space (X, d) with corresponding Borel σ-algebra $\mathscr{B}$, and a bundle $\mathcal{E}$ that is a measurable subset of $\Omega \times X$ with respect to the product σ-algebra $\mathscr{F} \times \mathscr{B}$ such that the fibres $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$ for all $\omega \in \Omega$ are compact and non-empty. A *bundle RDS associated to* $(\Omega, \mathscr{F}, \mathbb{P}, G)$ is a set of maps $\mathbf{F} = \{\mathbf{F}_{g,\omega} : \mathcal{E}_\omega \to \mathcal{E}_{g\omega} | g \in G, \omega \in \Omega\}$ satisfying

(α) $\mathbf{F}_{e,\omega} = \mathrm{id}_{\mathcal{E}_\omega}$ for every $\omega \in \Omega$, where e is the unit element of G;
(β) for any given $g \in G$, the map $(\mathcal{E}, \mathscr{F} \times \mathscr{B}|_\mathcal{E}) \to (X, \mathscr{B})$ given by $(\omega, x) \mapsto \mathbf{F}_{g,\omega}(x)$ is measurable, where $\mathscr{F} \times \mathscr{B}|_\mathcal{E} = \{A \cap \mathcal{E} : A \in \mathscr{F} \times \mathscr{B}\}$;
(γ) $\mathbf{F}_{g_2, g_1\omega} \circ \mathbf{F}_{g_1,\omega} = \mathbf{F}_{g_2 g_1,\omega}$ for every $g_1, g_2 \in G$ and $\omega \in \Omega$.

If for $\mathbb{P}$-a.e. $\omega \in \Omega$, and for every $g \in G$, the random transformation $\mathbf{F}_{g,\omega}$ is continuous (and is therefore a homeomorphism), then $\mathbf{F}$ is called a *continuous bundle RDS associated to* $(\Omega, \mathscr{F}, \mathbb{P}, G)$. The RDS $\mathbf{F}$ naturally induces an action $\tilde{f}_G = \{\tilde{f}_g : g \in G\}$ on $\mathcal{E}$, given by $\tilde{f}_g(\omega, x) = (g\omega, \mathbf{F}_{g,\omega}(x))$ for every $g \in G$ and $(\omega, x) \in \mathcal{E}$; in the absence of ambiguity, we still identify $\tilde{f}_G$ with G by denoting $g(\omega, x) := \tilde{f}_g(\omega, x)$ for every $g \in G$, $\omega \in \Omega$ and $x \in \mathcal{E}_\omega$. According to [11, Chapter III], the map $\omega \mapsto \mathcal{E}_\omega$ is measurable with respect to the Borel σ-algebra induced by the Hausdorff topology on the space $\mathcal{K}(X)$ of compact subsets of X. This is equivalent to that for any given $x \in X$ the distance function $\omega \mapsto d(x, \mathcal{E}_\omega)$ is measurable for $\mathscr{F}$. We denote $\mathcal{M}_\mathbb{P}(\mathcal{E})$ the set of all probability measures μ on $(\mathcal{E}, \mathscr{F} \times \mathscr{B}|_\mathcal{E})$ admitting marginal

https://doi.org/10.1017/S0013091524000257 Published online by Cambridge University Press

measure $\mathbb{P}$, denote $\mathcal{M}_{\mathbb{P}}^{G}(\mathcal{E}) = \{\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}) : \mu \circ g = \mu \text{ for every } g \in G\}$ the set of all the G-invariant measures in $\mathcal{M}_{\mathbb{P}}(\mathcal{E})$. According to [25, Lemma 2.1], $\mathcal{M}_{\mathbb{P}}^{G}(\mathcal{E})$ is compact with respect to weak*-topology, that is, the topology induced by the convergency that a sequence $(\sigma_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathbb{P}}(\mathcal{E})$ converges to $\sigma \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$ if and only if $\lim_{n\to\infty} \int f \,\mathrm{d}\sigma_n = \lim_{n\to\infty} \int f \,\mathrm{d}\sigma$ for every $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ (see definition of $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ in §3.1). To see more backgrounds and details about random dynamical systems, we refer to [2, 4, 5, 24–26].

We collect some examples of continuous bundle RDSs below.

Example 2.1. Among interesting examples of continuous bundle RDSs are random sub-shifts. In the case where $G = \mathbb{Z}$, these are treated in detail in [6, 23, 25]. We present a brief recall of some of their properties. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a Lebesgue space and $\vartheta : (\Omega, \mathscr{F}, \mathbb{P}) \to (\Omega, \mathscr{F}, \mathbb{P})$ an invertible measure-preserving transformation. Set $X = \{(x_i : i \in \mathbb{Z}) : x_i \in \mathbb{N} \cup \{+\infty\}, i \in \mathbb{Z}\}$, a compact metric space equipped with the metric

$$d((x_i : i \in \mathbb{Z}), (y_i : i \in \mathbb{Z})) = \sum_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} |x_i^{-1} - y_i^{-1}|,$$

and let $F : X \to X$ be the translation $(x_i : i \in \mathbb{Z}) \mapsto (x_{i+1} : i \in \mathbb{Z})$. Then, the integer group $\mathbb{Z}$ acts on $(\Omega \times X, \mathscr{F} \times \mathscr{B}_X)$ measurably with $(\omega, x) \mapsto (\vartheta^i \omega, F^i x)$ for each $i \in \mathbb{Z}$, where $\mathscr{B}_X$ denotes the Borel σ-algebra of the space X. Now let $\mathcal{E} \in \mathscr{F} \times \mathscr{B}_X$ be an invariant subset of $\Omega \times X$ (under the $\mathbb{Z}$-action) such that $\emptyset \neq \mathcal{E}_\omega \subset X$ is compact for $\mathbb{P}$-a.e. $\omega \in \Omega$. This defines a continuous bundle RDS where, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $F_{i,\omega}$ is just the restriction of F^i over $\mathcal{E}_\omega$ for every $i \in \mathbb{Z}$.

A very special case is when the subset $\mathcal{E}$ is given as follows. Let k be a random $\mathbb{N}$-valued random variable satisfying

$$0 < \int_\Omega \log k(\omega) \,\mathrm{d}\mathbb{P}(\omega) < \infty,$$

and, for $\mathbb{P}$-a.e. $\omega \in \Omega$, let $M(\omega)$ be a random matrix $(m_{i,j}(\omega) : i = 1, \ldots, k(\omega), j = 1, \ldots, k(\vartheta\omega))$ with entries 0 and 1. Then, the random variable k and the random matrix M generate a random sub-shift of finite type, where

$$\mathcal{E} = \{(\omega, (x_i : i \in \mathbb{Z})) : \omega \in \Omega, 1 \le x_i \le k(\vartheta^i \omega), m_{x_i, x_{i+1}}(\vartheta^i \omega) = 1, i \in \mathbb{Z}\}.$$

It is not hard to see that this is a continuous bundle RDS. In [16], Dooley and Zhang studied the local variational principle of general group G action RDS.

Example 2.2. There are many other interesting examples of continuous bundle RDSs coming from smooth ergodic theory, see, for example [28, 31], where one considers not only the action of $\mathbb{Z}$ or $\mathbb{Z}_+$ on a compact metric state space but also on Riemannian manifolds. Let M be a C^∞ compact connected Riemannian manifold without boundary and $C^r(M, M)$, $r \in \mathbb{Z}_+ \cup \{+\infty\}$ the space of all C^r maps from M into itself endowed with the usual C^r topology and the Borel σ-algebra. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a Lebesgue space and $\{\phi_t : \Omega \to C^r(M, M)\}_{t\ge 0}$ be a stochastic flow of $C^r(M, M)$ diffeomorphisms. It is well

https://doi.org/10.1017/S0013091524000257 Published online by Cambridge University Press

known that every smooth stochastic differential equation (SDE) in a finite dimensional compact manifold has a stochastic flow of diffeomorphisms as its solution flow. When the SDE is non-degenerate, it has a unique stationary measure, which is ergodic and equivalent to Lebesgue measure. Hence, Pesin's entropy formula holds true, which can be viewed as a sharp contradiction with the deterministic dynamical systems.

A measure space $(M, \mathcal{B}, m)$ is called *Lebesgue space* or *standard probability space* if there is an invertible map $f : M \to \Delta$, where $\Delta \subset \mathbb{R}$ is composed of an interval $I \subset \mathbb{R}$ and at most countable set of points $\{x_j \in \mathbb{R} : j \in J \subset \mathbb{N}\}$, such that f, f^{-1} both are measurable and measure preserving, and m on I takes the usual Lebesgue measure and $m(x_j) = m_j$ such that $m(\Delta) = m(I) + \sum_{j\in J} m_j = 1$. Any Lebesgue space $(M, \mathcal{B}, m)$ is complete and countably separated; the latter means that there is $(A_n)_{n\in\mathbb{N}} \subset \mathcal{B}$ such that $\{n \in \mathbb{N} : x \in A_n\} = \{n \in \mathbb{N} : y \in A_n\}$ implies $x = y$ for every $x, y \in M$.

For G a group, we denote $\mathcal{F}_G$ the set of all non-empty finite subsets of G. For each $K \in \mathcal{F}_G$ and $\delta > 0$, we say $F \in \mathcal{F}_G$ is *(K,δ)-invariant*, if $\frac{|\partial_K F|}{|F|} < \delta$, where $|\cdot|$ is the cardinality of a set and $\partial_K F = \{g \in G : Kg \cap F \neq \emptyset, Kg \cap F^c \neq \emptyset\}$ is the K-boundary of F. Moreover, a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_G$ is called *Følner sequence* of G if

$$\lim_{n\to\infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0, \text{ for every } g \in G.$$

A group G is said to be *amenable* if for any given $K \in \mathcal{F}_G$ and $\delta > 0$, there is $F \in \mathcal{F}_G$ such that $\frac{|kF\Delta F|}{|F|} \leq \delta$ for every $k \in K$. According to [43, p. 11], if G is a countable group, then G is amenable if and only if G admits a Følner sequence. To see more details about amenable group action, we refer to [17, 22, 43, 49].

Throughout the rest of this paper, let $(\Omega, \mathscr{F}, \mathbb{P})$ always be a Lebesgue space together with G, a measure-preserving countable discrete amenable group action, and $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G. Moreover, let (X, d) always denote a compact metric space with corresponding Borel σ-algebra $\mathscr{B}$. Fix $\mathcal{E} \in \mathscr{F} \times \mathscr{B}$ such that for each $\omega \in \Omega$ the fibre $\mathcal{E}_\omega$ is non-empty and compact and $\mathbf{F} = \{\mathbf{F}_{g,\omega} : \mathcal{E}_\omega \to \mathcal{E}_{g\omega} | g \in G, \omega \in \Omega\}$ a continuous bundle RDS associated with $(\Omega, \mathscr{F}, \mathbb{P}, G)$. We emphasize under these settings $\mathcal{M}_{\mathbb{P}}(\mathcal{E}) \neq \emptyset$ (see, for example [25, Lemma 2.1 (i)]) and $\mathcal{M}_{\mathbb{P}}^G(\mathcal{E}) \neq \emptyset$ (see [16, p. 30]).

The main results of this work are as follows:

Theorem 2.3. (Variational principle). *Suppose Ω admits a compact metric, $\mathscr{F}$ is the corresponding Borel σ-algebra, and G acts ergodicly on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathcal{E} = \Omega \times X$, if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$ then*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \sup_{\mu\in\mathcal{M}_{\mathbb{P}}^G(\mathcal{E})} \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d),$$

where $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)$ is the random upper metric mean dimension defined in Definition 3.3 and $\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d)$ is the random upper measure-theoretic mean dimension defined in Definition 3.16.

Let $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ denote the set of equilibrium states of a function $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, as defined in Definition 5.1. Let $\mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ denote the set of tangent functionals at

https://doi.org/10.1017/S0013091524000257 Published online by Cambridge University Press

$f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, as defined in Definition 5.2. The second main result of our study establishes both the existence and uniqueness of the equilibrium states for $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$. Furthermore, it demonstrates that the set $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ of equilibrium states for $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ precisely coincides with the set $\mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ of tangent functionals at $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$.

Theorem 2.4. *Suppose* $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$ *then*

(1) $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ *is non-empty, convex and compact for every* $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$*;*
(2) *for every* $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$,

$$\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) = \mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) = \bigcap_{n \geq 1} \overline{M_n},$$

where $M_n = \{\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}) : \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) + \int f \, \mathrm{d}\mu > \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \frac{1}{n}\}$ *for every* $n \in \mathbb{N}$*;*
(3) *there exists a dense subset* $\mathscr{L} \subset \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ *such that* $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ *is a singleton for every* $f \in \mathscr{L}$.

3. Random metric mean dimension

3.1. Random upper metric mean dimension with potentials

In this subsection, we present the notion and several properties of random mean dimension with random potentials.

Let $L^1_{\mathcal{E}}(\Omega, C(X))$ be the set of all the measurable functions $f : \mathcal{E} \to \mathbb{R}$ such that

(α) the function $f_\omega = f(\omega, \cdot) : \mathcal{E}_\omega \to \mathbb{R}$ is continuous for $\mathbb{P}$-a.e. $\omega \in \Omega$,
(β) $||f||_{\mathbb{P}} := \int ||f_\omega||_\infty \, \mathrm{d}\mathbb{P} < \infty$,

where $||f_\omega||_\infty = \sup_{x \in \mathcal{E}_\omega} |f_\omega(x)|$. Let $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ be the quotient set $L^1_{\mathcal{E}}(\Omega, C(X))$ mod $||\cdot||_{\mathbb{P}}$, that is, the set of all the equivalent classes of elements in $L^1_{\mathcal{E}}(\Omega, C(X))$ in the sense that identify $f, g \in L^1_{\mathcal{E}}(\Omega, C(X))$ if $||f - g||_{\mathbb{P}} = 0$. Note that $(\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)), ||\cdot||_{\mathbb{P}})$ is a Banach space.

Definition 3.1. *A random potential on* $\mathcal{E}$ *is a map* $\phi : \mathcal{F}_G \to \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$*. The set of all random potentials on* $\mathcal{E}$ *is denoted by* $\mathfrak{P}(\mathcal{E})$*.*

For the sake of simplicity, for every $\phi, \psi \in \mathfrak{P}(\mathcal{E})$,

1. let $\phi_F := \phi(F)$ for every $F \in \mathcal{F}_G$;
2. $\phi_{F,\omega}$ denotes the function $\phi_F(\omega, \cdot) : \mathcal{E}_\omega \to \mathbb{R}$, for every $\omega \in \Omega$ and $F \in \mathcal{F}_G$.
3. $\phi \leq \psi$ denotes that $\phi_F \leq \psi_F$ for every $F \in \mathcal{F}_G$;
4. $\phi = 0$ denotes that $\phi_F(\omega, x) = 0$ for every $F \in \mathcal{F}_G$, $\omega \in \Omega$, and $x \in \mathcal{E}_\omega$.

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For each $\omega \in \Omega$, $\phi \in \mathfrak{P}(\mathcal{E})$, $F \in \mathcal{F}_G$ and $\varepsilon > 0$, define

$$\Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F, d, \varepsilon) := \sup_{E \in \mathfrak{sep}(\mathcal{E}_\omega, F, d, \varepsilon)} \sum_{x \in E} \mathrm{e}^{|\log \varepsilon| \phi_F(\omega, x)},$$

where $\mathfrak{sep}(\mathcal{E}_\omega, F, d, \varepsilon)$ is the set of all (F, ε)-separated sets of $\mathcal{E}_\omega$; i.e. $E \in \mathfrak{sep}(\mathcal{E}_\omega, F, d, \varepsilon)$ if and only if $E \subset \mathcal{E}_\omega$ and $d_F^\omega(x, y) := \max_{g \in F} d(\mathbf{F}_{g,\omega} x, \mathbf{F}_{g,\omega} y) > \varepsilon$ for every distinct $x, y \in E$.

With a slight modification to the proof of [25, Lemma 1.2], it is easy to see that for any given $\phi \in \mathfrak{P}(\mathcal{E})$, $F \in \mathcal{F}_G$, and $\varepsilon > 0$, the functions $\omega \mapsto \Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F, d, \varepsilon)$ and $\omega \mapsto s(\mathcal{E}_\omega, F, d, \varepsilon)$ are measurable with respect to $\mathscr{F}$, where $s(\mathcal{E}_\omega, F_n, d, \varepsilon) = \max\{\mathrm{card}\, E : E \in \mathfrak{sep}(\mathcal{E}_\omega, F_n, d, \varepsilon)\}$. Moreover, there is the following lemma:

Lemma 3.2. *Let $\varepsilon > 0$ and $\phi \in \mathfrak{P}(\mathcal{E})$, then the function $\omega \mapsto \log \Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F, d, \varepsilon)$ belongs to $L^1_{\mathbb{P}}(\Omega)$ for every $F \in \mathcal{F}_G$.*

Proof. The statement follows from that

$$\mathrm{e}^{-||\phi_{F,\omega}||_\infty |\log \varepsilon|} \leq \Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F, d, \varepsilon) \leq s(\mathcal{E}_\omega, F, d, \varepsilon) \mathrm{e}^{||\phi_{F,\omega}||_\infty |\log \varepsilon|}$$

for every $F \in \mathcal{F}_G$. □

This means that the following definition makes sense:

$$\Lambda_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d, \varepsilon) := \limsup_{n \to \infty} \frac{1}{|F_n|} \int \log \Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F_n, d, \varepsilon) \, \mathrm{d}\mathbb{P}.$$

Definition 3.3. *Let $\phi \in \mathfrak{P}(\mathcal{E})$, the quantity*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d) := \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \Lambda_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d, \varepsilon)$$

is called the random upper metric mean dimension of $\mathbf{F}$ with random potential ϕ on $\mathcal{E}$ with respect to the Følner sequence $\mathfrak{s}$. In addition, the quantity

$$\begin{aligned} \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) : &= \limsup_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \limsup_{n \to \infty} \frac{1}{|F_n|} \int \log s(\mathcal{E}_\omega, F_n, d, \varepsilon) \, \mathrm{d}\mathbb{P} \\ &= \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, 0, \mathfrak{s}, d) \end{aligned}$$

is called the random upper metric mean dimension of $\mathbf{F}$ on $\mathcal{E}$ with respect to the Følner sequence $\mathfrak{s}$.

Remark 3.4. By the definition of random upper metric mean dimensions, it is possible that the values of $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)$ and $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d)$ are influenced by the choice of the Følner sequence $\mathfrak{s}$. However, if (X, d) has tame growth of covering numbers, similar arguments as in [10, Proposition 3.4] also give that $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)$ and $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d)$ are independent of the choice of the Følner sequence $\mathfrak{s}$.

We present several basic properties of random mean dimension with random potentials. We omit or only give a sketch of the proofs that can be directly deduced by the definition or are obvious.

Fact 3.5. (Monotonicity of mean dimension). Let $\phi \leq \psi$ be random potentials, then

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \psi, \mathfrak{s}, d).$$

Fact 3.6. If the function $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \cdot, \mathfrak{s}, d) : \mathfrak{P}(\mathcal{E}) \to \mathbb{R} \cup \{\infty\}$ takes finite value for every $\phi \in \mathfrak{P}(\mathcal{E})$, then $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \cdot, \mathfrak{s}, d)$ is convex.

Proof. Let $\phi, \psi \in \mathfrak{P}(\mathcal{E})$. For any $\omega \in \Omega$, $t \in [0, 1]$, $n \in \mathbb{N}$ and $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}
\Lambda_{\mathcal{E}}(\omega, t\phi + (1-t)\psi, F_n, d, \varepsilon) &= \sup_{E \in \mathfrak{sep}(\mathcal{E}_\omega, F_n, d, \varepsilon)} \sum_{x \in E} (\frac{1}{\varepsilon})^{t\phi_{F_n} + (1-t)\psi_{F_n}(\omega, x)} \\
&= \sup_{E \in \mathfrak{sep}(\mathcal{E}_\omega, F_n, d, \varepsilon)} \sum_{x \in E} (\frac{1}{\varepsilon})^{t\phi_{F_n}(\omega, x)} (\frac{1}{\varepsilon})^{(1-t)\psi_{F_n}(\omega, x)} \\
&\leq \sup_{E \in \mathfrak{sep}(\mathcal{E}_\omega, F_n, d, \varepsilon)} (\sum_{x \in E} (\frac{1}{\varepsilon})^{\phi_{F_n}(\omega, x)})^t (\sum_{x \in E} (\frac{1}{\varepsilon})^{\psi_{F_n}(\omega, x)})^{1-t} \\
&\leq (\Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F_n, d, \varepsilon))^t (\Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \psi, F_n, d, \varepsilon))^{1-t},
\end{aligned}$$

where the first inequality follows from the Hölder's inequality. This shows the statement. □

Fact 3.7. (Subadditivity of mean dimension). For any $\phi, \psi \in \mathfrak{P}(\mathcal{E})$, we have

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi + \psi, \mathfrak{s}, d) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d) + \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \psi, \mathfrak{s}, d).$$

Proof. The statement follows from that

$$\sum_{x \in E} \mathrm{e}^{|\log \varepsilon| \phi_{F_n} + \psi_{F_n}(\omega, x)} \leq \sum_{x \in E} \mathrm{e}^{|\log \varepsilon| \phi_{F_n}(\omega, x)} \times \sum_{x \in E} \mathrm{e}^{|\log \varepsilon| \psi_{F_n}(\omega, x)},$$

for any $\omega \in \Omega$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $E \in \mathfrak{sep}(\mathcal{E}_\omega, F_n, d, \varepsilon)$. □

Fact 3.8. If $t \geq 1$, then

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, t\phi, \mathfrak{s}, d) \leq t\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d);$$

if $0 \leq t \leq 1$, then

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, t\phi, \mathfrak{s}, d) \geq t\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d).$$

Proof. The statement follows from that the map $x \mapsto x^t$ on $\mathbb{R}_{\geq 0}$ is convex when $t \geq 1$ and concave when $t \leq 1$. □

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Fact 3.9. Let $\phi \in \mathfrak{P}(\mathcal{E})$, then

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, |\phi|, \mathfrak{s}, d).$$

Definition 3.10. *Let $\phi \in \mathfrak{P}(\mathcal{E})$, if there is $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ such that $\phi_F = S_F f := \sum_{g \in F} f \circ g$ for every $F \in \mathcal{F}_G$, then for every $\omega \in \Omega$, $F \in \mathcal{F}_G$, and $\varepsilon > 0$ denote*

$$\Lambda_{\mathcal{E}}(\omega, \mathbf{F}, f, F, d, \varepsilon) := \Lambda_{\mathcal{E}}(\omega, \mathbf{F}, \phi, F, d, \varepsilon),$$

$$\Lambda_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d, \varepsilon) := \Lambda_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d, \varepsilon),$$

and

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) := \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \phi, \mathfrak{s}, d).$$

Fact 3.11. Let $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ and $c \in \mathbb{R}$, then

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f + c, \mathfrak{s}, d) = \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) + c.$$

Fact 3.12. Let $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, then

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + \int \inf_{x \in \mathcal{E}_\omega} f(\omega, x) \,\mathrm{d}\mathbb{P} \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + ||f||_{\mathbb{P}}.$$

In particular,

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) - ||f||_{\mathbb{P}} \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + ||f||_{\mathbb{P}},$$

and

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + \inf_{\mathcal{E}} f \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + \sup_{\mathcal{E}} f,$$

where $\inf_{\mathcal{E}} f = \inf_{(\omega,x) \in \mathcal{E}} f(\omega, x)$ and $\sup_{\mathcal{E}} f = \sup_{(\omega,x) \in \mathcal{E}} f(\omega, x)$.

This immediately deduces the following as $||f||_{\mathbb{P}} < \infty$ for every $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$.

Fact 3.13. The function $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \cdot, \mathfrak{s}, d) : \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)) \to \mathbb{R} \cup \{\infty\}$ either takes finite values at all $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ or constantly ∞. Furthermore, $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) = \infty$ for all $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ if and only if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \infty$.

Fact 3.14. Suppose $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$. For any $f, g \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, we have

$$|\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, g, \mathfrak{s}, d)| \leq ||f - g||_{\mathbb{P}}.$$

In particular, the function

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \cdot, \mathfrak{s}, d) : \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)) \to \mathbb{R}$$

is continuous on $(\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)), || \cdot ||_{\mathbb{P}})$.

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Fact 3.15. (Invariance of mean dimension). Let $f, f^* \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, then for any $g \in G$ we have

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f + f^* \circ g - f^*, \mathfrak{s}, d) = \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d).$$

In particular,

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f \circ g, \mathfrak{s}, d) = \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d).$$

Proof. Let $g \in G$, $n \in \mathbb{N}$, $(\omega, x) \in \mathcal{E}$, $\varepsilon \in (0, 1)$, and $E \in \mathfrak{sep}(\mathcal{E}_\omega, F_n, d, \varepsilon)$, then

$$S_{F_n} f(\omega, x) - ||f^*_{g\omega}||_\infty - ||f^*_\omega||_\infty \le S_{F_n}(f + f^* \circ g - f^*(\omega, x) \le S_{F_n} f(\omega, x) + ||f^*_{g\omega}||_\infty + ||f^*_\omega||_\infty.$$

This shows that

$$\begin{aligned}\sum_{x\in E} \mathrm{e}^{|\log\varepsilon|(S_{F_n} f(\omega,x) - ||f^*_{g\omega}||_\infty - ||f^*_\omega||_\infty)} &\le \sum_{x\in E} \mathrm{e}^{|\log\varepsilon| S_{F_n}(f+f^*\circ g - f^*)(\omega,x)} \\ &\le \sum_{x\in E} \mathrm{e}^{|\log\varepsilon|(S_{F_n} f(\omega,x) + ||f^*_{g\omega}||_\infty + ||f^*_\omega||_\infty)}.\end{aligned}$$

Combining with the following equality

$$\sum_{x\in E} \mathrm{e}^{|\log\varepsilon| S_{F_n}(f+f^*\circ g - f^*)(\omega,x)} = \sum_{x\in E} \mathrm{e}^{|\log\varepsilon|(S_{F_n} f(\omega,x) + f^*(g\omega, \mathbf{F}_{g,\omega} x) - f^*(\omega,x))},$$

then we derive the first part of the statement. The rest is immediately obtained if we replace f^* with f. □

3.2. Random measure-theoretical metric mean dimension

In this subsection, we present the notion and characteristics of random upper measure-theoretical mean dimension.

Define a set

$$\mathcal{A} = \left\{ f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)) : \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f, \mathfrak{s}, d) = 0 \right\}.$$

Note that $\mathcal{A} = \emptyset$ if and only if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \infty$. Indeed, if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$, then by fact 3.11, we have $\overline{\mathrm{Rmdim}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - f} \in \mathcal{A}$ for every $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$; conversely, according to fact 3.13, $\mathcal{A} = \emptyset$ if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \infty$.

In this paper, we use the conventions: any infimum taking over the empty set equals ∞, and $\inf \infty := \infty$.

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Definition 3.16. *For each $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, the quantity*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) = \inf_{f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))} \left(\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f \, \mathrm{d}\mu \right)$$

is called the random upper measure-theoretical metric mean dimension of μ.

It is obvious that $\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) = \infty$ if and only if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \infty$.

Proposition 3.17. *Let $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, then*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) = \inf_{f \in \mathcal{A}} \int f \, \mathrm{d}\mu.$$

Proof. Suppose $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$. Let $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ and $\varphi = \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - f$, then by fact 3.11 we have $\varphi \in \mathcal{A}$. Thus,

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f \mathrm{d}\mu = \int \varphi \, \mathrm{d}\mu \geq \inf_{\varphi \in \mathcal{A}} \int \varphi \, \mathrm{d}\mu.$$

Letting f range over $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ yields

$$\inf_{f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))} \left(\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f \mathrm{d}\mu \right) \geq \inf_{\varphi \in \mathcal{A}} \int \varphi \, \mathrm{d}\mu.$$

To see the inequality in opposite direction, let $f \in \mathcal{A}$, then

$$\inf_{f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))} \left(\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f d\mu \right) \leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f, \mathfrak{s}, d) + \int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

Therefore,

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) \leq \inf_{f \in \mathcal{A}} \int f \, \mathrm{d}\mu.$$

When $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \infty$, we have $\mathcal{A} = \emptyset$, so by the convention $\inf_{f \in \emptyset} \int f \, \mathrm{d}\mu = \infty$, the statement holds. □

For each $\mathscr{L} \subset \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, define

$$\mathcal{A}_{\mathscr{L}} = \left\{ f \in \mathscr{L} : \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f, \mathfrak{s}, d) \leq 0 \right\}.$$

Proposition 3.18. *For any dense subset $\mathscr{L} \subset \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ and $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, we have*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) = \inf_{f \in \mathcal{A}_{\mathscr{L}}} \int f \, \mathrm{d}\mu.$$

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Proof. Fix $\mathscr{L}$ dense in $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ and $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$. Set

$$\mathcal{A}' = \{f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)) : \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f, \mathfrak{s}, d) \leq 0\},$$

then by Proposition 3.17, we have the inequalities

$$\begin{aligned}\inf_{f\in\mathcal{A}'} \int f \,\mathrm{d}\mu &\leq \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d)\\ &\leq \inf_{f\in\mathbf{L}^1_{\mathcal{E}}(\Omega,C(X))} \left(\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f, \mathfrak{s}, d) + \int f \,\mathrm{d}\mu\right)\\ &\leq \inf_{f\in\mathcal{A}'} \int f \,\mathrm{d}\mu.\end{aligned}$$

This shows $\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) = \inf_{f\in\mathcal{A}'} \int f \,\mathrm{d}\mu$, and what remains is to prove

$$\inf_{f\in\mathcal{A}_{\mathscr{L}}} \int f \,\mathrm{d}\mu \leq \inf_{f\in\mathcal{A}'} \int f \,\mathrm{d}\mu.$$

To this end, fix $f \in \mathcal{A}'$ and by fact 3.11 we can find an $\varepsilon > 0$ such that

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -(f+\varepsilon), \mathfrak{s}, d) < 0.$$

As for any $f_1, f_2 \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, we have

$$|\int f_1 \,\mathrm{d}\mu - \int f_2 \,\mathrm{d}\mu| \leq \int_\Omega \int_{\mathcal{E}_\omega} |f_1(\omega, x) - f_2(\omega, x)| \,\mathrm{d}\mu_x \, d\mathbb{P} \leq ||f_1 - f_2||_{\mathbb{P}};$$

hence, the map $f \mapsto \int f \,\mathrm{d}\mu$ is continuous on $(\mathbf{L^1}_{\mathcal{E}}(\Omega, C(X)), ||\cdot||_{\mathbb{P}})$. Pick a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}$ such that $||f_n - (f+\varepsilon)||_{\mathbb{P}} \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu + \varepsilon.$$

Therefore, by fact 3.14,

$$\lim_{n\to\infty} \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f_n, \mathfrak{s}, d) = \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -(f+\varepsilon), \mathfrak{s}, d) < 0.$$

Then, there is $N \in \mathbb{N}$ such that $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -f_N, \mathfrak{s}, d) < 0$ and

$$\int f_N \,\mathrm{d}\mu < \int f \,\mathrm{d}\mu + 2\varepsilon,$$

which implies $f_N \in \mathcal{A}_{\mathscr{L}}$. Thus, letting $\varepsilon \to 0$ we end at

$$\inf_{f\in\mathcal{A}_{\mathscr{L}}} \int f \,\mathrm{d}\mu \leq \inf_{f\in\mathcal{A}'} \int f \,\mathrm{d}\mu.$$

□

Proposition 3.19. *If* $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$*, then the function* $\overline{\mathrm{Rmdim}}_{\mathcal{E},\cdot}(\mathbf{F}, \mathfrak{s}, d) : \mathcal{M}_{\mathbb{P}}(\mathcal{E}) \to \mathbb{R}$ *is*

(1) *concave;*
(2) *upper semi-continuous.*

Proof. The concavity of $\overline{\mathrm{Rmdim}}_{\mathcal{E},\cdot}(\mathbf{F}, \mathfrak{s}, d)$ directly follows by the definition. Note that $\mathcal{M}_{\mathbb{P}}(\mathcal{E})$ is endowed with the weak*-topology, then for every fixed $f \in \mathcal{A}$ the function $F_f(\mu) = \int f \, \mathrm{d}\mu$ is continuous on $\mathcal{M}_{\mathbb{P}}(\mathcal{E})$. Hence $\overline{\mathrm{Rmdim}}_{\mathcal{E},\cdot}(\mathbf{F}, \mathfrak{s}, d)$ is upper semi-continuous since the infimum of a family of continuous functions is upper semi-continuous. □

4. Variational principle

In this section, we present a variation principle between the random upper metric mean dimension and the measure-theoretic one as the first main result of this paper. To this end, we need the following notions and lemmas:

Suppose Y is a set and $\mathcal{L}$ is a family of real-valued functions on Y. We say that $\mathcal{L}$ is a *vector lattice* if it is a linear space and $f \vee g := \max\{f, g\} \in \mathcal{L}$ for every $f, g \in \mathcal{L}$. If, in addition, $f \wedge 1 := \min\{f, 1\} \in \mathcal{L}$ for every $f \in \mathcal{L}$, then we call $\mathcal{L}$ a *Stone vector space.*

Let $\mathcal{L}$ be a vector lattice on Y. A function $L : \mathcal{L} \to \mathbb{R}$ is called a *pre-integral* if L is linear, non-negative, and $L(f_n)$ decreases to 0 for every $f_n \in \mathcal{L}$ with $f_n(x)$ decreasing to 0 for every $x \in Y$.

Lemma 4.1. (*[18]*, Theorem 4.5.2). *Let Y be a set and L be a pre-integral on a Stone vector lattice $\mathcal{L}$. Then, there exists a measure μ on $(Y, \sigma(\mathcal{L}))$ such that for all $f \in \mathcal{L}$*

$$L(f) = \int f \, \mathrm{d}\mu,$$

where $\sigma(\mathcal{L})$ is the smallest σ-algebra on Y such that all functions in $\mathcal{L}$ are measurable.

Lemma 4.2. *Let $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, then the limit points in the weak*-topology of the sequence*

$$\mu_n = \frac{1}{|F_n|} \sum_{s\in F_n} \upsilon_n s^{-1}, n \in \mathbb{N}$$

is not empty and is contained in $\mathcal{M}^G_{\mathbb{P}}(\mathcal{E})$.

Proof. Note that $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, then by the compactness of $\mathcal{M}_{\mathbb{P}}(\mathcal{E})$ in the weak*-topology there are $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$ and a sequence $(n_j)_{j\geq 1}$ such that,

$$\int f \, \mathrm{d}\mu = \lim_{j\to\infty} \int f \, \mathrm{d}\mu_{n_j}$$

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for every $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$. Let $\varepsilon > 0$ and $g \in G$, then there is $k \in \mathbb{N}$ such that

$$|gF_{n_j} \Delta F_{n_j}| \le \varepsilon |F_{n_j}|,$$

for every $j \ge k$. Therefore, for any $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ and $(n_{j_l})_{l\in\mathbb{N}}$ such that $j_1 \ge k$ we have

$$\begin{aligned}
\left|\int f \circ g \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu\right| &= \lim_{l\to\infty} \left|\int f \circ g - f \,\mathrm{d}\mu_{n_{j_l}}\right| \\
&= \lim_{l\to\infty} \frac{1}{|F_{n_{j_l}}|} \left|\int \sum_{s \in F_{n_{j_l}}} (f \circ gs - f \circ s) \,\mathrm{d}\nu_{n_{j_l}}\right| \\
&= \lim_{l\to\infty} \frac{1}{|F_{n_{j_l}}|} \left|\int (\sum_{s \in gF_{n_{j_l}}} f \circ s - \sum_{s \in F_{n_{j_l}}} f \circ s) \,\mathrm{d}\nu_{n_{j_l}}\right| \\
&\le \varepsilon ||f||_{\mathbb{P}}.
\end{aligned}$$

As ε and g are arbitrarily chosen, we obtain $\mu \in \mathcal{M}^G_{\mathbb{P}}(\mathcal{E})$, which shows the statement. □

Theorem 4.3. (Variational principle). *Suppose Ω admits a compact metric, $\mathscr{F}$ is the corresponding Borel σ-algebra, and G acts ergodicly on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathcal{E} = \Omega \times X$, if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$ then*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \sup_{\mu \in \mathcal{M}^G_{\mathbb{P}}(\mathcal{E})} \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d).$$

Proof. We firstly show

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \sup_{\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})} \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d).$$

As the constant function $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)$ is an element of $\mathcal{A}$, by Proposition 3.17, $\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) \le \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)$ for every $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, which implies

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) \ge \sup_{\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})} \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d).$$

To see the opposite direction of the above inequality, let

$$\mathcal{K} = \left\{ f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)) : \sup_{(\omega,x)\in\mathcal{E}} |f(\omega, x)| < \infty \right\},$$

then $\mathcal{K}$ is a normed subspace of $(\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X)), || \cdot ||_{\mathbb{P}})$. By the proof of [25, Lemma 2.1 (i)], we know that $C(\mathcal{E})$ the set of all continuous functions on $\mathcal{E}$ is dense in $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$.

Then, the compactness of $\mathcal{E}$ implies $\mathcal{K}$ is dense in $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$. Thus, by Proposition 3.18,

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) = \inf_{f\in\mathcal{A}_{\mathcal{K}}} \int f \, \mathrm{d}\mu,$$

for every $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$. By fact 3.14, $\mathcal{A}_{\mathcal{K}}$ is a closed convex subset of $\mathcal{K}$. For simplicity, we write $\alpha := \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d), \mathfrak{s}, d)$. Note that $\alpha = 0$, so $-(\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + \varepsilon) \notin \mathcal{A}_{\mathcal{K}}$, which means $-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) \notin \mathcal{A}_{\mathcal{K}} + \varepsilon$. Let $K_1 = \{\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)\}$ and $K_2 = \mathcal{A}_{\mathcal{K}} + \varepsilon$ be two disjoint closed subsets of $\mathcal{K}$, then by Hahn–Banach separation theorem, there is a continuous linear functional $L : \mathcal{K} \to \mathbb{R}$ such that

$$\inf_{f\in K_2} L(f) + L(\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)) \geq 0. \tag{4.1}$$

Fix $f_0 \in \mathcal{K}$ with $f_0 \geq 0$. Then, by fact 3.11 and fact 3.5, for any $c > 0$,

$$\begin{aligned}\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -(cf_0 + 1 + \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)), \mathfrak{s}, d) &= \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, -cf_0, \mathfrak{s}, d) - 1 \\ &\quad - \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) \\ &\leq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) - 1 \\ &\quad - \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < 0.\end{aligned}$$

This means $cf_0 + 1 + \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) \in \mathcal{A}_{\mathcal{K}}$ and then

$$L(-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)) \leq cL(f_0) + L(1 + \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) + \varepsilon).$$

Hence, $L(f_0) \geq 0$, otherwise letting $c \to \infty$ deduces $L(-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d)) = -\infty$, which is conflict to (4.1). As f_0 is arbitrary, we have L is non-negative. Let $\{f_n\} \subset \mathcal{K}$ be a sequence pointwisely decreasing to 0, then $\{f_n(\omega, \cdot)\}_{n\in\mathbb{N}} \subset C(\mathcal{E}_\omega)$ and $f_n(\omega, x)$ decreases to 0 as $n \to \infty$ for every $\omega \in \Omega$ and $x \in X$. Hence, for every $\omega \in \Omega$, by Dini's theorem, the function $f_n(\omega) := f_n(\omega, \cdot)$ uniformly convergences to 0 as $n \to \infty$; that is, for any $\varepsilon > 0$ and $\omega \in \Omega$, there is $N(\omega, \varepsilon)$ such that $||f_n(\omega)||_\infty < \varepsilon$ for every $n \geq N(\omega, \varepsilon)$. This means $||f_n(\omega)||_\infty$ decreases to 0 as $n \to \infty$ for every $\omega \in \Omega$. By monotone convergence theorem,

$$\lim_{n\to\infty} ||f_n - 0||_{\mathbb{P}} = \lim_{n\to\infty} \int ||f_n(\omega)||_\infty \, \mathrm{d}\mathbb{P} = 0.$$

This means that $L(f_n)$ decreases to 0 as $n \to \infty$, because L is non-negative, continuous and $L(0) = 0$. As L is not constantly zero and non-negative, we can find an $\varphi \in \mathcal{K}$ such that $\varphi \geq 0$ with $L(\varphi) > 0$. We set $0 \leq h := \frac{\varphi}{M} \leq 1$, where $M := \sup_{(\omega,x)\in\mathcal{E}} \varphi(\omega, x)$. Then, $0 \leq 1 - h \leq 1$ and hence, $L(1) = L(h) + L(1 - h) = \frac{1}{M} L(\varphi) + L(1 - h) > 0$.

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This shows that $\frac{L(\cdot)}{L(1)}$ is a pre-integral on $\mathcal{K}$. Note that $\mathcal{K}$ is a Stone vector lattice, then by Lemma 4.1, there is a measure μ on $(\mathcal{E}, \sigma(\mathcal{K}))$ such that

$$\frac{L(f)}{L(1)} = \int f \, \mathrm{d}\mu \tag{4.2}$$

for every $f \in \mathcal{K}$. Note that $\sigma(\mathcal{K})$ is a sub-σ-algebra of $\mathscr{F} \times \mathscr{B}$. Let $A \in \mathscr{F}$ and B be a closed subset of X. For each $n \in \mathbb{N}$, let $\varphi_n(\omega, x) := 1_A(\omega) \cdot b_n(x) \in \mathcal{K}$, where $b_n(x) = 1 - \min\{nd(x, B), 1\}$. As

$$\limsup_{n \to \infty} \varphi_n(\omega, x) = 1_A(\omega) 1_B(x),$$

for every $(\omega, x) \in \mathcal{E}$, we have $A \times B \in \sigma(\mathcal{K})$ for every $A \in \mathscr{F}$ and closed $B \subset X$. This means that $\sigma(\mathcal{K}) = \mathscr{F} \times \mathscr{B}$ and then μ is a probability measure on $\mathcal{E}$. For each $f \in \mathcal{K}$, $n \in \mathbb{N}$ and $g \in F_n$, we have $f \circ g \in \mathcal{K}$, and then by (4.2),

$$\frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} f \circ g)}{L(1)} = \int f \, \mathrm{d} \frac{1}{|F_n|} \sum_{g \in F_n} \mu g^{-1}. \tag{4.3}$$

As $\mathcal{E}$ is compact, by [50, Theorem 6.5], the set of all probability measures on $\mathcal{E}$ denoted by $\mathcal{M}(\mathcal{E})$ is compact in the weak*-topology which is the topology such that the map $\mu \mapsto \int f \, \mathrm{d}\mu$ is continuous on $\mathcal{M}(\mathcal{E})$ for every $f \in C(\mathcal{E})$. Without loss of generality, we may assume that

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} \mu g^{-1} = \nu \in \mathcal{M}(\mathcal{E}).$$

Hence, by (4.3), for each compact subset $A \subset \Omega$, we have

$$\limsup_{n \to \infty} \frac{L(\frac{1}{|F_n|} \sum_{g \in F_n} 1_{A \times X} \circ g)}{L(1)} \leq \nu(A \times X).$$

For each $\mathbb{P}$-integrable function $h : \Omega \to \mathbb{R}$, we can regard h as a function $h^*(\cdot, \cdot) \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ by setting $h^*(\omega, x) = h(\omega)$ for every $(\omega, x) \in \mathcal{E}$. Note that $1^*_K = 1_{K \times X}$. As $\mathbb{P}$ is G-ergodic, by mean ergodic theorem [22, Theorem 4.23],

$$\begin{aligned}\lim_{n \to \infty} ||(\frac{1}{|F_n|} \sum_{g \in Fn} 1_K \circ g)^* - \mathbb{P}(K)||_{\mathbb{P}} &= \lim_{n \to \infty} \int |\frac{1}{|F_n|} \sum_{g \in Fn} 1_K(g\omega) - \mathbb{P}(K)| \, \mathrm{d}\mathbb{P} \\ &= \lim_{n \to \infty} ||\frac{1}{|F_n|} \sum_{g \in Fn} 1_K \circ g - \mathbb{P}(K)||_{L^1} = 0,\end{aligned}$$

where $||\cdot||_{L^1}$ is the L^1-norm on the integrable function space $L^1(\Omega,\mathscr{F},\mathbb{P})$. As L is continuous, we obtain

$$\lim_{n\to\infty}\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}1_K^*\circ g)}{L(1)}=\lim_{n\to\infty}\frac{L((\frac{1}{|F_n|}\sum_{g\in F_n}1_K\circ g)^*)}{L(1)}=\mathbb{P}(K).$$

This means that $\mathbb{P}(K)\le\nu(K\times X)$ for all compact subsets $K\subset\Omega$. Analogously, we can derive $\mathbb{P}(O)\ge\nu(O\times X)$ for every open subset $O\subset\Omega$ with the similar argument. As the measures $\mathbb{P}$ and ν are regular, we have $\mathbb{P}(A)=\nu(A\times X)$ for every $A\in\mathscr{F}$. Thus,

$$\nu\circ\mathrm{proj}_\Omega^{-1}=\mathbb{P},$$

where $\mathrm{proj}_\Omega:\Omega\times X\to\Omega$ is the natural projection. By (4.3), for any $f\in C(\mathcal{E})$, we have

$$\lim_{n\to\infty}\frac{L(\frac{1}{|F_n|}\sum_{g\in F_n}f\circ g)}{L(1)}=\int f\,\mathrm{d}\nu. \tag{4.4}$$

This shows

$$\frac{L(-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d))}{L(1)}=\int-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d)\,\mathrm{d}\nu=-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d). \tag{4.5}$$

By Proposition 3.18, there is the equality that

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\nu}(\mathbf{F},\mathfrak{s},d)=\inf_{f\in\mathcal{A}_{C(\mathcal{E})}}\int f\,\mathrm{d}\nu, \tag{4.6}$$

where $\mathcal{A}_{C(\mathcal{E})}:=\{f\in C(\mathcal{E}):\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},-f,\mathfrak{s},d)\le 0\}$. By Fact 3.15, for each $f\in\mathcal{A}_{C(\mathcal{E})}$ and $g\in G$, we have $f\circ g\in\mathcal{A}_{\mathcal{K}}$, which yields by (4.4) and the linearity of L that

$$\int f\,\mathrm{d}\nu=\lim_{n\to\infty}\frac{1}{|F_n|}\sum_{g\in F_n}\frac{L(f\circ g)}{L(1)}\ge\inf_{f\in\mathcal{A}_{\mathcal{K}}}\frac{L(f)}{L(1)}. \tag{4.7}$$

Thus, by (4.1), (4.5), (4.6) and (4.7) and the linearity and non-negativity of L, we obtain

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\nu}(\mathbf{F},\mathfrak{s},d)-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d)+2\varepsilon\ge\inf_{f\in K_2}\frac{L(f)}{L(1)}+\frac{L(-\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d))}{L(1)}+\varepsilon>0.$$

Letting $\varepsilon\to 0$ and μ range over $\mathcal{M}_{\mathbb{P}}(\mathcal{E})$ yields

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d)\le\sup_{\mu\in\mathcal{M}_{\mathbb{P}}(\mathcal{E})}\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F},\mathfrak{s},d).$$

To prove $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d)=\sup_{\mu\in\mathcal{M}^G_{\mathbb{P}}(\mathcal{E})}\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F},\mathfrak{s},d)$, it suffices to show that there exists $\mu\in\mathcal{M}^G_{\mathbb{P}}(\mathcal{E})$ such that $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F},\mathfrak{s},d)\le\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F},\mathfrak{s},d)$. By Fact 3.15,

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for every $f \in \mathcal{A}$ and $g \in G$, we have $f \circ g \in \mathcal{A}$, and thus by Proposition 3.17, for each $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$, there is the inequality that

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) \leq \int f \, \mathrm{d}\mu g^{-1}.$$

This yields, by letting f range over $\mathcal{A}$,

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) \leq \mathrm{Rmdim}_{\mathcal{E},\mu g^{-1}}(\mathbf{F}, \mathfrak{s}, d), \tag{4.8}$$

for every $g \in G$. For each $n \in \mathbb{N}$, pick $\nu_n \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$ such that

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) - \frac{1}{|F_n|} \leq \mathrm{Rmdim}_{\mathcal{E},\nu_n}(\mathbf{F}, \mathfrak{s}, d).$$

Let $\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} \nu_n g^{-1}$. By Lemma 4.2, there is a limit point $\mu \in \mathcal{M}_{\mathbb{P}}^G(\mathcal{E})$ of $(\mu_n)_{n \in \mathbb{N}}$ in weak*-topology. Without loss of generality, we may assume $\lim_{n \to \infty} \mu_n = \mu$. For every $n \in \mathbb{N}$ and $g \in F_n$ by (4.8), we have

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) - \frac{1}{|F_n|} < \mathrm{Rmdim}_{\mathcal{E},\nu_n}(\mathbf{F}, \mathfrak{s}, d) \leq \mathrm{Rmdim}_{\mathcal{E},\nu_n g^{-1}}(\mathbf{F}, \mathfrak{s}, d),$$

then summing $g \in F_n$ and dividing $|F_n|$ yields by (1) of Proposition 3.19 that

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) - \frac{1}{|F_n|} < \frac{1}{|F_n|} \sum_{g \in F_n} \mathrm{Rmdim}_{\mathcal{E},\nu_n g^{-1}}(\mathbf{F}, \mathfrak{s}, d) \leq \mathrm{Rmdim}_{\mathcal{E},\mu_n}(\mathbf{F}, \mathfrak{s}, d).$$

Letting $n \to \infty$ results in by (2) of Proposition 3.19 that $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) \leq \mathrm{Rmdim}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d)$, which shows the statement. □

Remark 4.4. The first key technique in the proof of our variational principle is a convex analysis approach set up by Biś–Carvalho–Mendes–Varandas in [3] or Cioletti–Silva–Stadlbauer in [8], where they considered an upper semi-continuous affine entropy-like map, established an abstract variational principle for both countably and finitely additive probability measures and proved that equilibrium states always exist. As our mean dimensions can be regraded as is an entropy-like maps, we borrow part of their idea to establish our variational principle.

The second key technique is borrowed from [55] by Yang et al. We use an approach of Stone vector lattice to overcome the problem: according to Biś's method in [3], the Riesz's representation is a critical tool to prove variational principle; however, the Riesz's representation theorem holds only for all continuous functions. Unfortunately, $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ composing $g \in G$ may be no longer continuous, which is an obstruction to construct a probability measure on $\Omega \times X$ having marginal $\mathbb{P}$. Moreover, although $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ is also a Stone vector lattice, we still need construct the auxiliary set $\mathcal{K}$, on which $\frac{L(\cdot)}{L(1)}$ is a pre-integral, so that we can apply Lemma 4.1.

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5. Equilibrium states

In this section, we introduce a notion of equilibrium state for amenable random upper metric mean dimension with potentials and discuss its properties.

Definition 5.1. *Let* $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$*, if a measure* $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$ *satisfies*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) = \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) + \int f \, \mathrm{d}\mu,$$

then μ *is said to be an equilibrium state of* f *with respect to the Følner sequence* $\mathfrak{s}$ *and metric* d*. Denote* $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ *the set of all equilibrium states of* f *with respect to* $\mathfrak{s}$ *and* d*.*

Note that if $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) = \infty$ then by the convention that any infimum over empty set equals ∞ we have $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) = \mathcal{M}_{\mathbb{P}}(\mathcal{E})$.

Definition 5.2. *Let* μ *be a finite sign measure on* $(\Omega \times X, \mathscr{F} \times \mathscr{B})$ *with marginal measure* $\mathbb{P}$*. We say* μ *is a tangent functional at* $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ *with respect to* $\mathfrak{s}$ *and* d *if*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f + \varphi, \mathfrak{s}, d) - \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \varphi, \mathfrak{s}, d) \geq \int \varphi \, \mathrm{d}\mu$$

for every $\varphi \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$*. Denote* $\mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ *the set of all tangent functional at* f *with respect to* $\mathfrak{s}$ *and* d*.*

Theorem 5.3. *Suppose* $\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \mathfrak{s}, d) < \infty$ *then*

(1) $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ *is non-empty, convex and compact for every* $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$*;*
(2) *For every* $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$*,*

$$\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) = \mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) = \bigcap_{n \geq 1} \overline{M_n},$$

where $M_n = \{\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}) : \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) + \int f \, \mathrm{d}\mu > \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \frac{1}{n}\}$ *for every* $n \in \mathbb{N}$*;*
(3) *there exists a dense subset* $\mathscr{L} \subset \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ *such that* $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ *is a singleton for every* $f \in \mathscr{L}$*.*

Proof. (1) By Proposition 3.19, the function $\overline{\mathrm{Rmdim}}_{\mathcal{E},\cdot}(\mathbf{F}, \mathfrak{s}, d) : \mathcal{M}_{\mathbb{P}}(\mathcal{E}) \to \mathbb{R}$ is upper semicontinuous, which implies $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) \neq \emptyset$.

Let $\mu, \nu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ and $t \in [0,1]$, then

$$\begin{aligned}\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) &= t\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) + (1-t)\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \\ &= t\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) + (1-t)\overline{\mathrm{Rmdim}}_{\mathcal{E},\nu}(\mathbf{F}, \mathfrak{s}, d) \\ &+ t\int f\,\mathrm{d}\mu + (1-t)\int f\,\mathrm{d}\nu \\ &\le \overline{\mathrm{Rmdim}}_{\mathcal{E},t\mu+(1-t)\nu}(\mathbf{F}, \mathfrak{s}, d) + \int f\,\mathrm{d}(t\mu + (1-t)\nu) \\ &\le \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d),\end{aligned}$$

which shows $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ is convex.

Let $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$ and $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ such that $\lim_{n\to\infty}\mu_n = \mu$. By (2) of Proposition 3.19,

$$\begin{aligned}\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) &= \limsup_{n\to\infty}\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu_n}(\mathbf{F}, \mathfrak{s}, d) + \int f\,\mathrm{d}\mu_n \\ &\le \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d).\end{aligned}$$

This yields $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ is closed and compact.

(2) Note that $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) = \bigcap_{n\in\mathbb{N}} M_n$. Let $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$, then

$$\begin{aligned}&\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f+\varphi, \mathfrak{s}, d) - \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \\ &\ge \overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) + \int f+\varphi\,\mathrm{d}\mu - (\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) + \int f\,\mathrm{d}\mu) \\ &= \int \varphi\,\mathrm{d}\mu.\end{aligned}$$

Thus, $\mu \in \mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ and $\mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) \subset \mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$.

Let $\mu \in \mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$, then for any $\varepsilon > 0$ and $\varphi \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ with $\varphi > 0$,

$$\begin{aligned}\int \varphi\,\mathrm{d}\mu + \varepsilon &= -\int -(\varphi+\varepsilon)\,\mathrm{d}\mu \\ &\ge -\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f-(\varphi+\varepsilon), \mathfrak{s}, d) + \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \\ &\ge -\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f - \inf_{(\omega,x)\in\mathcal{E}}(\varphi(\omega,x)+\varepsilon)), \mathfrak{s}, d) + \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) \\ &= \inf_{(\omega,x)\in\mathcal{E}}(\varphi(\omega,x)+\varepsilon).\end{aligned}$$

This means that μ is a non-negative measure on $\mathcal{E}$. By fact 3.11, if $n \ge 1$, then we have

$$\int n\,\mathrm{d}\mu \le \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f+n, \mathfrak{s}, d) - \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) = n,$$

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which shows $\mu(\mathcal{E}) \leq 1$. Likewise, if $n \leq -1$, then we can derive $\mu(\mathcal{E}) \geq 1$. Hence, $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E})$. Fix $\varphi \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$. As $\mu \in \mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$, we have

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f+\varphi, \mathfrak{s}, d) - \int f+\varphi \,\mathrm{d}\mu \geq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f \,\mathrm{d}\mu.$$

Then, by substituting φ with $\psi = \varphi - f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ in above inequality, we obtain

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, \varphi, \mathfrak{s}, d) - \int \varphi \,\mathrm{d}\mu \geq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f \,\mathrm{d}\mu,$$

which means

$$\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d) \geq \overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, f, \mathfrak{s}, d) - \int f \,\mathrm{d}\mu.$$

Therefore, $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$ and then $\mathcal{T}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d) \subset \mathcal{M}_{\mathbb{P}}(\mathcal{E}, f, \mathfrak{s}, d)$.

(3) As $\mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$ is a Banach space, then by [15, Theorem 8, V.9.8] and Fact 3.6, we derive the statement. □

Corollary 5.4. *Let $f \in \mathbf{L}^1_{\mathcal{E}}(\Omega, C(X))$, $b \in \mathbb{R}$ and $\mu \in \mathcal{M}_{\mathbb{P}}(\mathcal{E}, bf, \mathfrak{s}, d)$ such that $\int f \,\mathrm{d}\mu \neq 0$, then*

$$\overline{\mathrm{Rmdim}}_{\mathcal{E}}(\mathbf{F}, bf, \mathfrak{s}, d) = 0 \iff b = -\frac{\overline{\mathrm{Rmdim}}_{\mathcal{E},\mu}(\mathbf{F}, \mathfrak{s}, d)}{\int f \,\mathrm{d}\mu}.$$

6. Open questions

There is still a problem untouched in this paper, i.e., what is the difference between a random action and the non-random action of a group with respect to its metric mean dimension? However, interesting examples of metric mean dimension for non-autonomous dynamical systems are considered in [44, Examples 3.8, 5.7 and 5.8]. Their results suggest that there are important differences between of a non-autonomous dynamical system and an autonomous dynamical system with respect to its metric mean dimension. This motivates us that there could be many other differences between a random action and the non-random action of a group with respect to its metric mean dimension. We leave it as an open question and further work.

Acknowledgements. The authors are grateful to the referee for a careful reading of the paper and a multitude of corrections and helpful suggestions.

Funding Statement. This work is supported by the National Natural Science Foundation of China (NSFC) (No: 12201372).

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