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LIMITS OF WEAKLY HYPERCYCLIC AND SUPERCYCLIC OPERATORS

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Abstract. We give a spectral characterization of the norm closure of the class of all weakly hypercyclic operators on a Hilbert space. Analogous results are obtained for weakly supercyclic operators.

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1. Introduction. Throughout this paper $\mathcal{H}$ will be a complex, separable, infinite-dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$.

If $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$ we shall denote by $\mathrm{Orb}(T, x)$ the orbit of x under T which means the set

$$\{T^n x; \quad n \geq 0\}.$$

An operator T is called *hypercyclic* if there is $x \in \mathcal{H}$ such that $\mathrm{Orb}(T, x)$ is dense in $\mathcal{H}$. The set of all hypercyclic operators on $\mathcal{H}$ will be denoted by $HC(\mathcal{H})$.

An operator T is called *weakly hypercyclic* if there is $x \in \mathcal{H}$ such that $\mathrm{Orb}(T, x)$ is weakly dense in $\mathcal{H}$. We will use $WHC(\mathcal{H})$ for the set of all weakly hypercyclic operators.

It is clear that $HC(\mathcal{H}) \subset WHC(\mathcal{H})$. It was proved in [**3**] that there are weakly hypercyclic operators which are not hypercyclic.

An operator is called *supercyclic* if the set $\{\lambda \mathrm{Orb}(T, x); \lambda \in \mathbb{C}\}$ is dense in $\mathcal{H}$ for some vector x. The set of all supercyclic operators on $\mathcal{H}$ will be denoted by $SC(\mathcal{H})$.

An operator is called *weakly supercyclic* if there is a vector x such that the set $\{\lambda \mathrm{Orb}(T, x); \lambda \in \mathbb{C}\}$ is weakly dense in $\mathcal{H}$. We shall use $WSC(\mathcal{H})$ for the set of all weakly supercyclic operators.

It is clear that $SC(\mathcal{H}) \subset WSC(\mathcal{H})$. The paper [**7**] contains examples of weakly supercyclic operators which are not supercyclic.

For an operator T we shall use $\sigma(T)$ to denote the spectrum of T and $\sigma_p(T)$ for the set of all eigenvalues of T (point spectrum) while $\sigma_{p0}(T)$ is the set of all isolated eigenvalues of T of finite (geometric) multiplicity (normal eigenvalues). If $\lambda \in \sigma_{p0}(T)$ then T is similar to $A \oplus B$, where $\lambda \notin \sigma(A)$, $\sigma(B) = \{\lambda\}$ and B is an operator on a finite dimensional subspace. It is easy to see from this characterization that $\lambda \in \sigma_{p0}(T) \Longleftrightarrow \overline{\lambda} \in \sigma_{p0}(T^*)$.

We shall denote the unit circle of the complex plane by $\mathbb{T}$ and the open unit disc by $\mathbb{D}$.

The semi Fredholm domain of an operator T will be denoted by $\rho_{sF}(T)$ and, for $\lambda \in \rho_{sF}(T)$, $\mathrm{ind}(\lambda - T)$ will stand for the semi Fredholm index of $\lambda - T$. Recall that the Weyl spectrum of an operator is the set $\sigma_W(T) = \sigma(T) \setminus \{\lambda \in \rho_{sF}(T); \mathrm{ind}(\lambda - T) = 0\}$.

The spectral properties of the operators are not much different in the case of weak density. In fact, we didn't find any difference.

The classes $HC(\mathcal{H})$ and $SC(\mathcal{H})$ have been extensively studied in the last 25 years. The study of $WHC(\mathcal{H})$ and $WSC(\mathcal{H})$ is just at the beginning, from this point of view [**3**], [**4**] and [**7**] being really pioneering work. In this paper we will do for $WHC(\mathcal{H})$ and $WSC(\mathcal{H})$ what was done for $HC(\mathcal{H})$ and $SC(\mathcal{H})$ in [**6**].

2. Weakly hypercyclic operators. In this section we will list some spectral properties of weakly hypercyclic operators and we will use them to prove that the sets $HC(\mathcal{H})$ and $WHC(\mathcal{H})$ have the same interior and the same closure (in the norm topology).

PROPOSITION 2.1. *On a finite dimensional space there is no weakly hypercyclic operator.*

Proof. If there is any such operator then, because of the finite dimension, it will be hypercyclic and it is known that in finite dimension there are no hypercyclic operators. □

THEOREM 2.2. *If $T \in WHC(\mathcal{H})$ then:*

(i) *for every invariant subspace $\mathcal{M}$ of T the compression of T to the orthogonal complement of $\mathcal{M}$ is weakly hypercyclic on the space $\mathcal{M}^{\perp}$;*
(ii) $\sigma_p(T^*) = \emptyset$*;*
(iii) $\mathrm{ind}(\lambda - T) \geq 0$ *for every* $\lambda \in \rho_{sF}(T)$*;*
(iv) $\sigma_W(T) = \sigma(T)$*;*
(v) $\sigma_W(T) \cup \mathbb{T}$ *is a connected set.*

Proof. (i). This follows from the definition by considering the matrix form of T with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

(ii) and (iii). These are simple consequences of (i).

(iv). It is implied by (ii).

(v). The result follows from part (iv) and [**4**, Theorem 3]. □

THEOREM 2.3. *The closure of $WHC(\mathcal{H})$ is the class of all operators in $\mathcal{B}(\mathcal{H})$ satisfying the conditions:*

(1) $\sigma_W(T) \cup \mathbb{T}$ *is connected;*
(2) $\sigma_{p0}(T) = \emptyset$*;*
(3) $\mathrm{ind}(\lambda - T) \geq 0$ *for all* $\lambda \in \rho_{sF}(T)$.

Proof. Let $\mathcal{C}$ be the set of all operators satisfying the three conditions.

We have that $HC(\mathcal{H}) \subset WHC(\mathcal{H})$ and, by Theorem 2.2, $WHC(\mathcal{H}) \subset \mathcal{C}$.

By [**6**, Theorem 2.1], the closure of $HC(\mathcal{H}) = \mathcal{C}$, which completes the proof. □

COROLLARY 2.4. *Every weakly hypercyclic operator is limit of hypercyclic operators.*

REMARK 2.5. $\mathrm{dist}(T, WHC(\mathcal{H})) = \mathrm{dist}(T, HC(\mathcal{H}))$ and is explicitly given in [**6**, Theorem 2.1].

We shall show next that the interior of the class is empty. If K is a subset of the complex plane we shall denote by K^* the set of the conjugates of the elements of K.

PROPOSITION 2.6. *For every $n \geq 1$, the set of all operators having at least n isolated eigenvalues of finite multiplicity is dense in $\mathcal{B}(\mathcal{H})$.*

Proof. The result is really a consequence of the approximation by Apostol-Morrel simple models, see [**1**, Theorem 2.5], but we will offer here a different argument.

It is easy to see that the property under consideration is a similarity invariant.

If T has the property in the statement and $\{\lambda_j\}_{j=1}^n \subset \sigma_{p0}(T)$, then T is similar to $A \oplus \oplus_{j=1}^n B_j$, where $\sigma(A) = \sigma(T) \setminus \{\lambda_1, \lambda_2, \dots, \lambda_n$, $\sigma(B_j) = \{\lambda_j\}\}$ and each B_j is an operator on a finite dimensional subspace. Thus, if $S \in \mathcal{B}(\mathcal{H})$ and $\sigma(S) \cap \sigma(T) = \emptyset$, $\{\lambda_j\}_{j=1}^n \subset \sigma_{p0}(T \oplus S)$ and so $T \oplus S$ has at least n normal eigenvalues. Also, if α and β are complex numbers then $\{\alpha + \beta\lambda_j\}_{j=1}^n \subset \sigma_{p0}(\alpha + \beta T)$. Hence the property under consideration is a "bad property". Therefore, using Theorem 3.51 on page 91 in [**5**], we conclude that the class is dense. □

THEOREM 2.7. $\operatorname{int}(WHC)(\mathcal{H}) = \emptyset$.

Proof. By the previous proposition, every operator T is the limit of a sequence of operators T_n with $\sigma_{p0}(T_n) \neq \emptyset$.

Since $\sigma_{p0}(T_n^*) = \sigma_{p0}(T_n)^*$, (see the argument following the definition of the normal eigenvalues in the introduction) we conclude that $\sigma_p(T_n^*) \neq \emptyset$ and so the operators T_n are not in $WHC(\mathcal{H})$. Thus T is not in the interior of the set. □

COROLLARY 2.8. *The operators which are not weakly hypercyclic are dense in* $\mathcal{B}(\mathcal{H})$.

3. Weakly supercyclic. In this section we shall determine some spectral properties of weakly supercyclic operators and we shall use them to prove that the sets $SC(\mathcal{H})$ and $WSC(\mathcal{H})$ have the same interior and the same closure (in the norm topology).

PROPOSITION 3.1. (i) *On a finite dimensional space there is no weakly supercyclic operator except for a nonzero operator on an one dimensional subspace.*

(ii) *If $T \in WSC(\mathcal{H})$ then for every invariant subspace $\mathcal{M}$ of T the compression of T to the orthogonal complement of $\mathcal{M}$ is weakly supercyclic on the space $\mathcal{M}^{\perp}$.*

(iii) *If $T \in WSC(\mathcal{H})$ then $0 \notin \sigma_p(T^*)$.*

(iv) *If $T \in WSC(\mathcal{H})$ and $\mathcal{M}$ is an invariant subspace of T then $\mathcal{M}$ has codimension either* 1 *or* ∞.

Proof. (i) If there is any such operator then, because of the finite dimension, it will be supercyclic and it is known that in finite dimension the only supercyclic operators are the nonzero operators on a one dimensional subspace.

(ii) This follows from the definition by considering the matrix form of T with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

(iii) Suppose that $0 \in \sigma_p(T^*)$. Since $\ker T^*$ is an invariant subspace of T, by part (ii) it follows that T_1, the compression of T to $\ker T^*$, is weakly supercyclic. This contradicts part (i) because $T_1 = 0$.

(iv) It is a simple consequence of (i). □

THEOREM 3.2. *If $T \in WSC(\mathcal{H})$ then:*

(i) *$\sigma_p(T^*)$ has at most one element;*

(ii) *$\operatorname{ind}(\lambda - T) \geq 0$ for every $\lambda \in \rho_{sF}(T)$;*

(iii) *if $\alpha \in \sigma_p(T^*)$ then for every $k \geq 1$, $\ker(T^* - \alpha)^k$ is a one dimensional subspace;*

(iv) *if there is any $\lambda \in \sigma(T) \setminus \sigma_W(T)$ then $\overline{\lambda}$ is the unique eigenvalue of T^* which in this case is an isolated eigenvalue of finite geometric multiplicity;*

(v) *there is $r > 0$ such that $\sigma(T) \cup r\mathbb{T}$ is connected;*

(vi) *there is $r > 0$ such that $\sigma_W(T) \cup r\mathbb{T}$ is connected.*

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Proof. (i) If α and β are two different eigenvalues of T^* and x, y are two corresponding eigenvectors then $\mathrm{Span}\{x, y\}^{\perp}$ is an invariant subspace of T of codimension 2, which contradicts part (iv) of Proposition 3.1.

(ii) If there is any component with negative index then, since the component is an open set, $\sigma_p(T^*)$ has infinitely many elements, which contradicts (i).

(iii) The proof is similar to part (i).

(iv) Follows from (iii).

(v) Suppose not. Then there is $R > 0$ such that $R\mathbb{T}$ separates the components of the spectrum of T. Since a scalar multiple of a weakly hypercyclic operator is weakly hypercyclic we can assume without loss of generality that $R = 1$.Thus T is similar to a direct sum $T_1 \oplus T_2$ where $\sigma(T_1) \subset \mathbb{D}$ and $\sigma(T_2) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, T_1 is an operator on some $\mathcal{H}_1$, T_2 is an operator on some $\mathcal{H}_2$ and of course $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Here the overline stands for the closure.

By part (iv) of Proposition 3.1, $\mathcal{H}_1$ and $\mathcal{H}_2$ have dimension either one or ∞.

Let us consider first the case when one of the dimensions is one, let say the dimension of $\mathcal{H}_2$. Then, by part (iv), $T = T_1 \oplus \alpha I$, where $|\alpha| > 1$. Thus, using [**7**, Theorem 2.2], we conclude that $\alpha^{-1}T_1$ is weakly hypercyclic. This contradicts part (v) of Theorem 2.2 which states that every component of the spectrum of $\alpha^{-1}T_1$ intersects the unit circle because the spectrum of the operator is in fact included in $\mathbb{D}$.

Let $0 < r < 1$ be the spectral radius of T_1 and let $r < q < 1$. Since $\lim_n \|T_1^n\|^{\frac{1}{n}} = r$, there is n_1 such that $\|T_1^n\|^{\frac{1}{n}} < q$ for every $n \geq n_1$. Thus $\|T_1^n\| < q^n$ for every $n \geq n_1$.

Let $x_1 \oplus x_2$ be a weakly supercyclic vector for $T_1 \oplus T_2$. Modulo a scaling, we can assume without loss of generality that $\|x_2\| = 1$.

Let $q < p < 1$. Since

$$\lim_{n\to\infty} \left(\frac{p}{q}\right)^n \frac{p}{\|x_1\|} = \infty$$

there is n_2 such that

$$\left(\frac{p}{q}\right)^n \frac{p}{\|x_1\|} > 1$$

for all $n \geq n_2$.

Since $\sigma(T_2) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$, there is $s > 1$ and n_3 such that $\|T_2^n x\| > s^n\|x\|$ for all $n \geq n_3$.

Let u_1 be a unit vector such that $u_1 \perp x_1$. Let u_2 be a unit vector such that $u_2 \perp \{x_1, Tx_1, u_1\}$. Let u_3 be a unit vector such that $u_3 \perp \{x_1, Tx_1, T^2x_1, u_1, u_2\}$. Continuing in the same way we construct a sequence u_n, of unit vectors, such that $u_n \perp \{x_1, Tx_1, \dots T^{n-1}x_1, u_1, \dots u_{n-1}\}$ for every n.

Let

$$u = \sum_{n=1}^{\infty} p^n u_n.$$

Then, for every n, $\langle u, u_n\rangle = p^n$.

Since $T_1 \oplus T_2$ is weakly supercyclic, there is a net of complex numbers λ_α and a net of natural numbers k_α such that $\lambda_\alpha(T_1 \oplus T_2)^{k_\alpha}(x_1 \oplus x_2)$ converges weakly to $u \oplus 0$. Therefore $\lambda_\alpha T_1^{k_\alpha} x_1$ converges weakly to u while $\lambda_\alpha T_2^{k_\alpha} x_2$ converges weakly to 0.

Let $m \geq \max\{n_1, n_2, n_3\}$, fixed. Then

$$\lim_{\alpha} \langle \lambda_\alpha T_1^{k_\alpha} x_1, u_m \rangle = \langle u, u_m \rangle = p^m.$$

Since $p^{m+1} < p^m$, there is α_0 such that

$$\left| \langle \lambda_\alpha T_1^{k_\alpha} x_1, u_n \rangle \right| > p^{m+1}$$

for all $\alpha \geq \alpha_0$.

Suppose that there is $\alpha \geq \alpha_0$ such that $k_\alpha < m$. Then, by construction, $\langle \lambda_\alpha T_1^{k_\alpha} x_1, u_m \rangle = 0$, which is a contradiction. Thus for every $\alpha \geq \alpha_0$, $k_\alpha \geq m$.

Now, for $\alpha \geq \alpha_0$,

$$\begin{aligned} \left| \langle \lambda_\alpha T_1^{k_\alpha} x_1, u_m \rangle \right| &\leq |\lambda_\alpha| \cdot \left\| T_1^{k_\alpha} x_1 \right\| \cdot \|u_m\| = |\lambda_\alpha| \cdot \left\| T_1^{k_\alpha} x_1 \right\| \leq |\lambda_\alpha| \cdot \left\| T_1^{k_\alpha} \right\| \cdot \|x_1\| \\ &\leq |\lambda_\alpha| q^{k_\alpha} \|x_1\| \leq |\lambda_\alpha| q^m \|x_1\|. \end{aligned}$$

Hence, for every $\alpha \geq \alpha_0$,

$$p^{m+1} < |\lambda_\alpha| q^m \|x_1\|,$$

which implies that

$$|\lambda_\alpha| > \left(\frac{p}{q} \right)^m \frac{p}{\|x_1\|} > 1.$$

Let y be a vector in $\mathcal{H}_2$ and $\varepsilon > 0$. Since $\lambda_\alpha T_2^{k_\alpha} x_2$ converges weakly to 0, there is α_1 such that

$$\left| \langle \lambda_\alpha T_2^{k_\alpha} x_2, y \rangle \right| < \varepsilon$$

for every $\alpha \geq \alpha_1$.

Let α_ε be such that $\alpha_\varepsilon \geq \alpha_0$ and $\alpha_\varepsilon \geq \alpha_1$ and let $\alpha \geq \alpha_\varepsilon$. Then, since

$$|\lambda_\alpha| \left| \langle T_2^{k_\alpha} x_2, y \rangle \right| < \varepsilon$$

we obtain

$$\langle T_2^{k_\alpha} x_2, y \rangle \Big| < \frac{\varepsilon}{|\lambda_\alpha||} < \varepsilon.$$

From this we conclude that 0 is in the weak closure of the set $\{T_2^{k_\alpha} x_2; \alpha \geq \alpha_0\}$. The set is included in $\{T_2^n x_2; n \geq m\}$ and so 0 is in the weak closure of this last set.

Now, for $n \geq m$,

$$\left\| T_2^m x_2 \right\| \geq c^n \|x_2\| = c^n.$$

According to [**4**, Lemma 1], 0 is not in the weak closure of $\{T_2^n x_2; n \geq m\}$, which provides us with a contradiction.

(vi) The proof follows from part (iv) and part (v). □

We can proceed now to characterize the closure of $WSC(\mathcal{H})$.

THEOREM 3.3. *The closure of* $WSC(\mathcal{H})$ *is the class of all operators in* $\mathcal{B}(\mathcal{H})$ *satisfying the conditions:*

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(1) $\sigma(T) \cup r\mathbb{T}$ *is connected for some* $r > 0$*;*
(2) $\sigma_W(T) \cup r\mathbb{T}$ *is connected for some* $r > 0$*;*
(3) $\sigma_{p0}(T)$ *has at most one element;*
(4) $\operatorname{ind}(\lambda - T) \geq 0$ *for all* $\lambda \in \rho_{sF}(T)$.

Proof. Let $\mathcal{C}$ be the set of all operators satisfying the four conditions.
We have that $SC(\mathcal{H}) \subset WSC(\mathcal{H})$ and, by Theorem 3.2, $WSC(\mathcal{H}) \subset \mathcal{C}$.
By [**6**, Theorem 3.3], the closure of $SC(\mathcal{H}) = \mathcal{C}$, which completes the proof. □

COROLLARY 3.4. *Every weakly supercyclic operator is limit of supercyclic operators.*

REMARK 3.5. $\operatorname{dist}(T, WSC(\mathcal{H})) = \operatorname{dist}(T, SC(\mathcal{H}))$ and is explicitly given in [**6**, Theorem 3.3].

THEOREM 3.6. $\operatorname{int} WSC(\mathcal{H}) = \emptyset$.

Proof. Using again Proposition 2.6, we infer that every operator T is the limit of a sequence of operators T_n with at least two points in $\sigma_{p0}(T_n)$.

Since $\sigma_{p0}(T_n^*) = \sigma_{p0}(T_n)^*$ we conclude that $\sigma_p(T_n^*)$ has at least two elements which implies that the operators T_n are not in $WSC(\mathcal{H})$. Thus T is not in the interior of the set. □

COROLLARY 3.7. *The operators which are not weakly supercyclic are dense in* $\mathcal{B}(\mathcal{H})$.

After this paper was submitted we learned of a different proof of part (v) of Theorem 3.2 obtained in [**2**].

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