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Abstract

In this paper we give a lower bound on the waist of the unit sphere of a uniformly convex normed space by using the localization technique in codimension greater than one and a strong version of the Borsuk–Ulam theorem. The tools used in this paper follow ideas of Gromov in [*Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal. **13** (2003), 178–215] and we also include an independent proof of our main theorem which does not rely on Gromov's waist of the sphere. Our waist inequality in codimension one recovers a version of the Gromov–Milman inequality in [*Generalisation of the spherical isoperimetric inequality to uniformly convex Banach spaces*, Compositio Math. **62** (1987), 263–282].

1. Introduction

The classical isoperimetric inequality for a metric space relates the measure of compact sets to the measure of their boundaries. These inequalities are codimension-one isoperimetric inequalities (simply because the difference of the dimension of a compact set and the dimension of its boundary is equal to one).

During his research on a Morse theory for the space of cycles of a manifold, Almgren gave a sharp lower bound for the volume of a minimal k-cycle in the sphere $\mathbb{S}^n$ for every k (see [Gro83, Pit81]). This is an instance of a higher codimensional isoperimetric type inequality.

Another important example of a higher codimensional isoperimetric inequality, which in fact is a generalization of the Almgren isoperimetric inequality on the sphere, is the waist of the sphere theorem of Gromov presented in [Gro03].

In this paper we prove a higher codimensional isoperimetric inequality for the unit sphere of a uniformly convex normed space.

In [GM87], Gromov and Milman gave an *isoperimetric*-type inequality for the unit sphere of a uniformly convex normed space by using the localization technique (a nice exposition of this can be found in [Ale99]). The main result of this paper recovers a version of Gromov–Milman's inequality.

We begin by defining waist. For more details about this invariant, see [Gro03, Mem10b].

Notation 1 (Tubular neighborhoods). Let X be a metric space, Y a subset of X, and $\varepsilon > 0$. The ε-neighborhood of Y is denoted by

$$Y + \varepsilon = \{x \in X \mid d(x, Y) \leqslant \varepsilon\}.$$

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DEFINITION 1.1 (Waist of a metric-measure space, see [Gro03]). Let $X = (X, d, \mu)$ be a metric-measure (mm) space. Let Z be a topological space. Let $w(\varepsilon)$ be a positive function. We say the *waist of X relative to Z* is larger than w if for every continuous map $f : X \to Z$ there exists a $z \in Z$ such that for all $\varepsilon > 0$,

$$\mu(f^{-1}(z) + \varepsilon) \geqslant w(\varepsilon).$$

The purpose of this paper is to give a lower bound of the waist of the unit sphere of a uniformly convex normed space relative to $\mathbb{R}^k$. We are ready to state the main theorem of this paper.

THEOREM 1. *Let X be a uniformly convex normed space of finite dimension $n+1$. Let $S(X)$ be the unit sphere of X, for which the distance is induced from the norm of X. The measure defined on $S(X)$ is the conical probability measure. Then a lower bound for the waist of $S(X)$ relative to $\mathbb{R}^k$ is given by*

$$w(\varepsilon) = \frac{1}{1 + (1 - 2\delta(\varepsilon/2))^{n-k}(k+1)^{k+1}(F(k, \varepsilon/2)/G(k, \varepsilon/2))},$$

where $\delta(\varepsilon)$ is the modulus of convexity (see the next section for the definition),

$$F(k, \varepsilon) = \int_{\psi_2(\varepsilon)}^{\pi/2} \sin(x)^{k-1}\, dx$$

and

$$G(k, \varepsilon) = \int_0^{\psi_1(\varepsilon)} \sin(x)^{k-1}\, dx,$$

and where

$$\psi_1(\varepsilon) = 2 \arcsin\left(\frac{\varepsilon}{4\sqrt{k+1}}\right)$$

and

$$\psi_2(\varepsilon) = 2 \arcsin\left(\frac{\varepsilon}{2\sqrt{k+1}}\right).$$

Section 2 concerns several preliminary tools which are useful to prove this waist theorem. To aid in following this paper, an overview of the theorem's proof can be found in § 3. In § 4, we will briefly discuss the theory of convexly derived measures and use it to obtain a lower bound for the convexly derived measure of certain balls inside convex subsets of $S(X)$. In § 5, we prove our main theorem following the ideas of Gromov in [Gro03]. Section 6 will be very technical and the goal is twofold: first we introduce new techniques and ideas which may be useful for the estimation of the waist of different metric spaces (such as Riemannian or Finsler manifolds) and second we give an independent proof of the main theorem not relying on Gromov's waist of the sphere. In § 7, we give another lower bound for the waist of $S(X)$ and compare it with the result of Theorem 1. Section 8 will compare our result to the waist of the canonical sphere and in § 9 we will discuss the relation of our result with Gromov–Milman's inequality.

2. Preliminaries

Let us consider a uniformly convex normed space of dimension $(n+1)$, $X = (\mathbb{R}^{n+1}, \|\ \|)$, which we fix once and for all.

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Definition 2.1 (Modulus of convexity). The space X has modulus of convexity δ if for all $\varepsilon > 0$, for all vectors $x,\ y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geqslant \varepsilon$, we have

$$\frac{\|x+y\|}{2} \leqslant 1 - \delta(\varepsilon).$$

Example 2.2. Let E be a Euclidean space. In this case, the modulus of convexity is easily determined from the parallelogram identity. And, we have

$$\delta_E(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Remark. δ is a monotone increasing function. We use this remark later on to prove Lemma 4.4.

We denote by $B(X) := \{x \in X \mid \|x\| \leqslant 1\}$ the unit ball of X and $\partial B(X) = S(X) := \{x \in X \mid \|x\| = 1\}$ the unit sphere of X.

We define a probability measure μ on $S(X)$ and we call it the conical measure.

Definition 2.3 (Conical probability measure). For any Borel set $A \subset S(X)$, we define

$$\mu(A) := \frac{m_{n+1}\{\bigcup tA | 0 \leqslant t \leqslant 1\}}{m_{n+1}(B(X))},$$

where m_n is the n-dimensional Lebesgue measure on X.

One can easily check that the measure μ is indeed a probability measure on $S(X)$:

$$\mu(S(X)) = \frac{m_{n+1}\{tS(X), 0 \leqslant t \leqslant 1\}}{m_{n+1}B(X)} = 1.$$

Remark. For the Euclidean norm on $\mathbb{R}^{n+1}$, where the distance between two points is the Euclidean distance and where the unit sphere is the canonical n-dimensional sphere $\mathbb{S}^n$, the conical measure is the canonical Riemannian probability measure on $\mathbb{S}^n$. Generally, $S(X)$ carries a Finsler structure and there is no canonical measure which can be defined on a Finsler manifold. For a Finsler manifold, there are many different ways to define volume. For more on this subject, see the excellent survey [AT04].

The mm space on which we are settling our problem is $(S(X), \mu, d)$ with μ the conical probability measure and d the distance induced on $S(X)$ from the norm defined on X (i.e. for all $x, y \in S(X)$, $d(x, y) = \|x - y\|$).

3. Scheme of the proof of Theorem 1

We fix a continuous map $f : S(X) \to \mathbb{R}^k$. The proof of Theorem 1 goes as follows.

- Use a generalization of the Borsuk–Ulam theorem giving rise to a finite convex partition of the sphere and a fiber of f (i.e $f^{-1}(z)$ for some $z \in \mathbb{R}^k$) passing through the centers of all the pieces of the partition (the center of a convex set has to be defined).
- Narrow the pieces of the partition (by increasing their numbers) such that almost all of them are Hausdorff close to a k-dimensional convex set. Pass to a limit-infinite partition of the sphere by convex subsets of dimension less than or equal to k.
- On each piece of the partition, there exists a probability measure, convexly derived from the conical measure. This brings the n-dimensional volume estimate of the waist down to a k-dimensional measure estimate on each convex set of the partition. This method is called the localization technique. But, usually, the localization or the needle decomposition

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brings the n-dimensional measure estimate down to a one-dimensional problem. The use of a multi-dimension localization technique first appeared in [Gro03].

– On each piece of the partition, Lemma 4.6 gives an estimate of the measure of an ε-ball centered at a point where the measure of the convex set is mostly concentrated. By integrating this estimate over the space of pieces of the partition, we obtain the result of Theorem 1. There are some difficulties due to the l-dimensional convex sets of the infinite partition for all $l < k$. We prove that these 'bad' convex sets do not affect the estimation of the waist. Or, better say, the measure of these convex sets in the space of *pieces of the partition* is equal to zero.

4. Convexly derived measures on convex sets of $S(X)$

The topics studied in this section follow the ideas used in [Ale99, GM87]. For every subset $S \in S(X)$, we define the subset $\mathrm{co}(S) \in B(X)$ as

$$\mathrm{co}(S) := \left\{ \bigcup tS | 0 \leqslant t \leqslant 1 \right\}.$$

Hence, $\mathrm{co}(S)$ is the cone centered at the origin of the ball over S.

Definition 4.1 (Convexly derived measure). A convexly derived measure on $S(X)$ is a limit of a vaguely converging sequence of probability measures of the form $\mu_i = \mu | S_i / \mu(S_i)$, where S_i are open convex sets.

To understand convexly derived measures, we need the following definition.

Definition 4.2 (k-concave functions). Let K be a bounded convex subset of $\mathbb{R}^{n+1}$. A function $f : K \to \mathbb{R}_+$ is called k-concave ($k > 0$) if $f^{1/k}$ is concave.

Suppose we have a sequence of open convex sets $\{S_i\}$ of $S(X)$ which Hausdorff converges to a convex set $S' \in S(X)$, where we suppose that the dimension of S' is equal to k with $k < n$. It is clear that the sequence $\{\mathrm{co}(S_i)\}$ Hausdorff converges to the set $\mathrm{co}(S')$, where $\dim\ \mathrm{co}(S') = k + 1$. We define a probability measure μ' on $\mathrm{co}(S')$ as follows.

For every $i \in \mathbb{N}$, we define the measure $\mu'_i = m_{n+1} | S_i / m_{n+1}(S_i)$. A subsequence of this sequence of measures vaguely converges to a probability measure μ on $\mathrm{co}(S')$. We call this measure a convexly derived measure. We recall that the support of the measure μ is automatically equal to $\mathrm{co}(S')$ as the sequence converges to this set. In [Ale99], Alesker showed that the measure μ admits a continuous density function f with respect to the $(k+1)$-dimensional Lebesgue measure defined on A. Moreover, the function f is $(n-k)$-concave (the above facts follow from deep results of Borell; see [Bor75] for more details). Hence,

$$\mu = f\, dm_{k+1},$$

where m_{k+1} is the $(k+1)$-dimensional Lebesgue measure. Moreover, we have the following lemma.

Lemma 4.1. *The measure μ is $(n+1)$-homogeneous and the function f is $(n-k)$-homogeneous.*

This means $\mu(tA) = t^{n+1}\mu(A)$ for $0 \leqslant t \leqslant 1$ and $f(tx) = t^{n-k} f(x)$ for all $x \in \mathrm{co}(S')$.

Proof. The measure μ is convexly derived from the normalized $(n+1)$-dimensional Lebesgue measure. As the $(n+1)$-dimensional Lebesgue measure is $(n+1)$-homogeneous, μ is $(n+1)$-homogeneous. From the equality $\mu = f\, dm_{k+1}$, and the fact that μ is $(n+1)$-homogeneous and

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m_{k+1} is $(k+1)$-homogeneous, then clearly f is $(n-k)$-homogeneous and the proof of the lemma follows. $\square$

The convexly derived measure μ' defined on $\mathrm{co}(S')$ defines a probability measure μ on S' convexly derived from the conical measure of $S(X)$ and obtained from the sequence $\{S_i\}$, where, for every $X \subset S'$, we have

$$\mu(X) = \mu'(\mathrm{co}(X)).$$

And, on the other hand, there exists another probability measure defined on S' which is the canonical k-dimensional conical measure conically induced by m_{k+1}; we denote this measure by ν. For every Borel subset U of S',

$$\nu(U) = \frac{m_{k+1}(\mathrm{co}(U))}{m_{k+1}(\mathrm{co}(S'))}.$$

S' is a subset of the unit sphere of $\mathbb{R}^{k+1}$ equipped with a norm satisfying the same modulus of convexity.

Then we have

$$\mu(U) = \mu'(\mathrm{co}(U)) = \int_{\mathrm{co}(U)} f \, dm_{k+1} = \int_U f \, d\nu.$$

Hence, in conclusion, we have

$$d\mu = f \, d\nu,$$

where we take the restriction of f on the set U.

The function f is $(n-k)$-concave on $\mathrm{co}(A)$ but the restriction of this function on the spherical part of the border of $\mathrm{co}(A)$ is not any more $(n-k)$-concave.

However, the restriction function still has nice concavity properties, as we will explain now.

DEFINITION 4.3. An *arc* $\sigma \subset S(X)$ is a subarc of the intersection of a 2-plane passing through the origin of the ball with $S(X)$.

We know that for all $x, y \in S_\pi$,

$$f^{1/(n-k)}\left(\frac{x+y}{2}\right) \geqslant \frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2}.$$

But, the point $(x+y)/2$ is no more on $S(X)$, so we set $z = ((x+y)/2)/\|(x+y)/2\| \in S(X)$. By the definition of the modulus of convexity, we have

$$\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta(\|x-y\|). \tag{1}$$

So, we can conclude the following lemma.

LEMMA 4.2. *Let f denote the density of a convexly derived measure on $S(X)$. Let x, $y \in S_\pi$ and let $z = ((x+y)/2)/\|(x+y)/2\| \in S_\pi$. Then*

$$\frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2} \leqslant (1 - \delta(\|x-y\|) f^{1/(n-k)}(z).$$

Proof. As $(x+y)/2 = \|(x+y)/2\| z$ and as the function f is $(n-k)$-homogeneous,

$$f^{1/(n-k)}\left(\frac{x+y}{2}\right) = \left\|\frac{x+y}{2}\right\| f^{1/(n-k)}(z)$$

and, by (1), the proof of the lemma follows. $\square$

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DEFINITION 4.4. Let f be a function defined on an arc of $S(X)$. We say f is *weakly* $(n-k)$-*concave* if for all $x, y \in \sigma$, $z = ((x+y)/2)/\|(x+y)/2\|$,

$$\frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2} \leqslant (1 - \delta(\|x-y\|)f^{1/(n-k)}(z).$$

LEMMA 4.3. *A nonzero weakly $(n-k)$-concave function defined on an arc of $S(X)$ has at most one maximum point and has no local minima.*

Proof. If there were two distinct maxima x and y, we would get $f^{1/(n-k)}(x) \leqslant (1 - \delta(\|x - y\|)f^{1/(n-k)}(x)$, a contradiction. Suppose f has a local minimum at point m. Take nearby points x' and y' such that $m = (x'+y')/2$. Then $x = x'/\|x'\|$ and $y = y'/\|y'\|$ belong to the arc, and $m = ((x+y)/2)/\|(x+y)/2\| = m$. This leads again to a contradiction. The proof of the lemma follows. □

Let f be the density of a convexly derived measure supported on a k-dimensional convex subset S of $S(X)$. By Lemma 4.3, we can conclude that at most one point $z \in S$ exists at which f achieves its maximum. Indeed, suppose f achieves its maximum in at least two points x_1 and x_2. Since there exists an arc passing through x_1 and x_2 and contained in S, this would contradict Lemma 4.3.

Let z be the point of S where f achieves its maximum. We want to give a (uniform) lower bound for $\mu(B(z, \varepsilon))$, where $B(z, \varepsilon)$ is the k-dimensional ball in S of norm radius ε,

$$B(z, \varepsilon) := \{x \in S_\pi \mid \|x - z\| \leqslant \varepsilon\}.$$

Therefore, from now on, the mm space we are working on will be $(S, \mu, \|\ \|)$.

We define two subsets on S: $A := B(z, \varepsilon)$, $B := S \setminus B(z, 2\varepsilon) = B(z, 2\varepsilon)^c$ and we are interested in estimating the ratio

$$\frac{\mu_\pi(B)}{\mu_\pi(A)}.$$

We need the following lemma.

LEMMA 4.4. *Let f be the density of a convexly derived measure supported on a k-dimensional convex subset S of $S(X)$. Assume f achieves its maximum at z. Let $x \in B(z, 2\varepsilon)^c = S \setminus B(z, 2\varepsilon)$ and consider the arc $\sigma = [z, x]$ in $S(X)$. Then*

$$f(x) \leqslant (1 - 2\delta(\varepsilon))^{n-k} \operatorname{Min}_{\sigma \cap B(z,\varepsilon)} f.$$

Proof. (Compare [Ale99]) Pick $y \in [x, z] \cap B(z, \varepsilon)$. By weak concavity, we know that f is monotone nondecreasing along $[x, z]$, so

$$f(x) \leqslant f(y) \leqslant f(z).$$

So, the maximum of f on the subarc $[x, y]$ is achieved at y. By Lemma 4.2,

$$\frac{f^{1/(n-k)}(x) + f^{1/(n-k)}(y)}{2} \leqslant (1 - \delta(\|x-y\|) \operatorname*{Max}_{w \in [x,y]} f^{1/(n-k)}(w),$$

which implies

$$f(x) \leqslant (1 - 2\delta(\|x - y\|))^{n-k} f(y).$$

By the triangle inequality, $\|x-y\| \geqslant \varepsilon$ and we remember that the modulus of convexity is nondecreasing, so

$$\delta(\|x-y\|) \geqslant \delta(\varepsilon).$$

Hence,

$$(1-2\delta(\|x-y\|))^{n-k} \leqslant (1-2\delta(\varepsilon))^{n-k}.$$

And, at last, we have

$$f(x) \leqslant (1-2\delta(\|x-y\|))^{n-k} f(y) \leqslant (1-2\delta(\varepsilon))^{n-k} f(y).$$

The proof of the lemma follows. □

We are ready now to integrate both sides of the inequality of Lemma 4.4 and give an upper bound for $\mu(B)/\mu(A)$.

LEMMA 4.5. *Let $\varepsilon > 0$ be given. Let $S \subset S(X)$ be a k-dimensional convex set. Let a convexly derived measure μ be defined on S. Let z be the maximum point for the density function of the measure μ. Let $A := B(z,\varepsilon)$, $B := S \smallsetminus B(z, 2\varepsilon)$. Then*

$$\frac{\mu(B)}{\mu(A)} \leqslant (1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}\frac{F(k,\varepsilon)}{G(k,\varepsilon)},$$

where

$$F(k,\varepsilon) = \int_{\psi_2(\varepsilon)}^{\pi/2} \sin(x)^{k-1}\, dx$$

and

$$G(k,\varepsilon) = \int_{0}^{\psi_1(\varepsilon)} \sin(x)^{k-1}\, dx,$$

and where

$$\psi_1(\varepsilon) = 2\arcsin\left(\frac{\varepsilon}{4\sqrt{k+1}}\right)$$

and

$$\psi_2(\varepsilon) = 2\arcsin\left(\frac{\varepsilon}{2\sqrt{k+1}}\right).$$

Proof. Let σ be an arc of $S(X)$ emanating from z. Denote

$$m = \operatorname{Min} \underset{\sigma\cap B(z,\varepsilon)}{f}.$$

Then

$$x \in \sigma \cap B(z, 2\varepsilon)^c \Rightarrow f(x) \leqslant (1-2\delta(\varepsilon))^{n-k} m$$

and

$$y \in \sigma \cap B(z,\varepsilon) \Rightarrow f(y) \geqslant m.$$

Assume first that the norm $\|\cdot\|$ is Euclidean. We need to convert Euclidean distances into Riemannian distances along the unit sphere, i.e. angles. If x and y are unit vectors making an angle ϕ, then $|x-y| = 2\sin(\phi/2)$. Therefore, $|x-y| = \epsilon$ corresponds to an angle ϕ_1 and $|x-y| = 2\epsilon$ corresponds to an angle ϕ_2. Therefore, for a fixed θ, $t \leqslant \phi_1 \Rightarrow f(t,\theta) \geqslant m(\theta)$ and $t \geqslant \phi_2 \Rightarrow f(t,\theta) \leqslant (1-2\delta(\varepsilon))^{n-k} m(\theta)$. Using polar coordinates (t,θ) on the unit sphere,

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we compute

$$\frac{\mu(B)}{\mu(A)} \leqslant \frac{\int_{\phi_2}^{\pi} \int_{\mathbb{S}^{k-1}} f(t,\theta)\ \sin(t)^{k-1}\,dt\,d\theta}{\int_0^{\phi_1} \int_{\mathbb{S}^{k-1}} f(t,\theta)\ \sin(t)^{k-1}\,dt\,d\theta}$$
$$\leqslant \max_{\theta \in \mathbb{S}^{k-1}} \frac{\int_{\phi_2}^{\pi} f(t,\theta)\ \sin(t)^{k-1}\,dt}{\int_0^{\phi_1} f(t,\theta)\ \sin(t)^{k-1}\,dt}.$$

For each θ,

$$\frac{\int_{\phi_2}^{\pi/2} f(t,\theta)\ \sin(t)^{k-1}\,dt}{\int_0^{\phi_1} f(t,\theta)\ \sin(t)^{k-1}\,dt} \leqslant \frac{\int_{\phi_2}^{\pi}(1-2\delta(\varepsilon))^{n-k} m(\theta) \sin(t)^{k-1}\,dt}{\int_0^{\phi_1} m(\theta) \sin(t)^{k-1}\,dt}$$
$$= \frac{\int_{\phi_2}^{\pi} \sin(t)^{k-1}\,dt}{\int_0^{\phi_1} \sin(t)^{k-1}\,dt}(1-2\delta(\varepsilon))^{n-k}.$$

To handle general norms, we use the fact that the Banach–Mazur distance between any $(k+1)$-dimensional normed space and Euclidean space is at most $\sqrt{k+1}$. On the affine extension of $\mathrm{co}(S)$ there exists a Euclidean structure $|\cdot|$ such that for every $x \in \mathit{Aff}(\mathrm{co}(S))$, we have

$$\frac{1}{\sqrt{k+1}}|x| \leqslant \|x\| \leqslant |x|.$$

Or, equivalently, we have

$$B \subset K \subset \sqrt{k+1}B,$$

where B is the Euclidean ball of dimension $k+1$ and K is the uniformly convex ball defined by $S(X)$.

We denote by pr the radial projection of the uniformly convex sphere ∂K to the Euclidean sphere ∂B. Recall that ν is the conical measure on ∂K and we denote by dv_k the conical measure on ∂B, i.e. the Riemannian probability measure. Then the density $h = pr_*\,d\nu/dv_k$ satisfies

$$\frac{1}{\sqrt{k+1}^{k+1}} \leqslant h \leqslant \sqrt{k+1}^{k+1}.$$

Let $x,\ y \in \partial K,\ x' = \mathrm{pr}(x)$ and $y' = \mathrm{pr}(y)$. Since radial projection to the sphere decreases Euclidean distance outside the Euclidean ball,

$$|x'-y'| \leqslant |x-y| \leqslant \sqrt{k+1}\|x-y\|.$$

For a general norm, radial projection to the unit sphere is 2-Lipschitz. Indeed, let x'', y'' be points such that $1 \leqslant \|x''\| \leqslant \|y''\|$. Rescaling both by $\|x''\|$ decreases $\|x''-y''\|$, so we can assume that $\|x''\| = 1$. Then $\|y''\| \leqslant 1 + \|x''-y''\|$ and

$$\left\| x'' - \frac{y''}{\|y''\|} \right\| = \left\| \frac{x''}{\|y''\|} - \frac{y''}{\|y''\|} + x\left(1 - \frac{1}{\|y''\|}\right) \right\|$$
$$\leqslant \|x''-y''\| + \|y''\| - 1 \leqslant 2\|x''-y''\|.$$

If $x'' = \sqrt{k+1}x'$ and $y'' = \sqrt{k+1}y'$, then

$$\|x-y\| \leqslant 2\|x''-y''\| = 2\sqrt{k+1}\|x'-y'\| \leqslant 2\sqrt{k+1}|x'-y'|.$$

We radially project the set S to a set S' on the sphere. S' is k dimensional and is a convex set as radial projection preserves convexity. We denote the projection of the point z on the sphere by $z' = \mathrm{pr}(z)$. In polar coordinates (t,θ) centered at z', fix θ. Let $\psi_1(\theta)$ (respectively $\psi_2(\theta)$)

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denote the angle t such that $y = pr^{-1}(t, \theta) \in \partial K$ satisfies $\|y - z\| = \varepsilon$ (respectively $= 2\varepsilon$). The above distance estimates yield

$$2 \sin \frac{\psi_1(\theta)}{2} \geqslant \frac{\varepsilon}{2\sqrt{k+1}}$$

and

$$2 \sin \frac{\psi_2(\theta)}{2} \geqslant \frac{\varepsilon}{\sqrt{k+1}}.$$

Then

$$\begin{aligned}\frac{\mu(B)}{\mu(A)} &\leqslant \frac{\int_{\mathbb{S}^{k-1}} \int_{\psi_2(\theta)}^{\pi} h(t,\theta) f(t,\theta)\ \sin(t)^{k-1}\, dt\, d\theta}{\int_{\mathbb{S}^{k-1}} \int_0^{\psi_1(\theta)} h(t,\theta) f(t,\theta)\ \sin(t)^{k-1}\, dt\, d\theta} \\ &\leqslant \max_{\theta \in \mathbb{S}^{k-1}} \frac{\int_{\psi_2(\theta)}^{\pi} h(t,\theta) f(t,\theta)\ \sin(t)^{k-1}\, dt}{\int_0^{\psi_1(\theta)} h(t,\theta) f(t,\theta)\ \sin(t)^{k-1}\, dt}.\end{aligned}$$

For each θ,

$$\begin{aligned}\frac{\int_{\psi_2(\theta)}^{\pi} h(t,\theta) f(t,\theta)\ \sin(t)^{k-1}\, dt}{\int_0^{\psi_1(\theta)} h(t,\theta) f(t,\theta)\ \sin(t)^{k-1}\, dt} &\leqslant \frac{\int_{\psi_2}^{\pi} (1-2\delta(\varepsilon))^{n-k} m(\theta) h(t,\theta) \sin(t)^{k-1}\, dt}{\int_0^{\psi_1} m(\theta) h(t,\theta) \sin(t)^{k-1}\, dt} \\ &= (1-2\delta(\varepsilon))^{n-k} \frac{\int_{\psi_2}^{\pi} h(t,\theta) \sin(t)^{k-1}\, dt}{\int_0^{\psi_1} h(t,\theta) \sin(t)^{k-1}\, dt} \\ &\leqslant (1-2\delta(\varepsilon))^{n-k} (k+1)^{k+1} \frac{\int_{\psi_2}^{\pi} \sin(t)^{k-1}\, dt}{\int_0^{\psi_1} \sin(t)^{k-1}\, dt}.\end{aligned}$$

Replacing ψ_1 and ψ_2 with the above lower bounds yields

$$\begin{aligned}\frac{\mu(B)}{\mu(A)} &\leqslant (1-2\delta(\varepsilon))^{n-k} (k+1)^{k+1} \frac{\int_{\psi_2}^{\pi} \sin(t)^{k-1}\, dt}{\int_0^{\psi_1} \sin(t)^{k-1}\, dt} \\ &\leqslant (1-2\delta(\varepsilon))^{n-k} (k+1)^{k+1} \frac{F(k,\varepsilon)}{G(k,\varepsilon)}.\end{aligned}$$

The proof of the lemma follows. □

LEMMA 4.6. *Let S be a convex set of dimension k in $S(x)$. Let a convexly derived measure μ be defined on S. Let z be the maximum point of the density of the measure μ. For every $\varepsilon > 0$, the following estimation holds:*

$$\mu(B(z,\varepsilon)) \geqslant \frac{1}{1 + (1-2\delta(\varepsilon/2))^{n-k}(k+1)^{k+1}(F(k,\varepsilon/2)/G(k,\varepsilon/2))},$$

where the functions F and G are defined as before.

Proof. We use the result of Lemma 4.5, which tells us that

$$\frac{\mu(B)}{\mu(A)} \leqslant (1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1} \frac{F(k,\varepsilon)}{G(k,\varepsilon)}.$$

We recall that μ is a probability measure and we have

$$\frac{\mu(B(z,2\varepsilon))}{\mu(B(z,2\varepsilon))^c} \geqslant \frac{\mu(B(z,\varepsilon))}{\mu(B(z,2\varepsilon))^c} \geqslant \frac{1}{(1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}(F(k,\varepsilon)/G(k,\varepsilon))}.$$

Hence,

$$\mu(B(z,2\varepsilon)) = \frac{\mu(B(z,2\varepsilon))}{\mu(B(z,2\varepsilon)) + \mu(B(z,2\varepsilon))^c} \geqslant \frac{1}{1+(1-2\delta(\varepsilon))^{n-k}(k+1)^{k+1}(F(k,\varepsilon)/G(k,\varepsilon))}.$$

The proof of the lemma follows. □

5. Proof of Theorem 1 following Gromov

In this section, we follow the ideas used in [Gro03, Mem10b]. Let $f : S(X) \to \mathbb{R}^k$ be as in Theorem 1. We want to partition the sphere $S(X)$ by at most k-dimensional convex sets. The continuous map f defines a continuous map $Pr(f)$ on the sphere $\mathbb{S}^n$ which is the radial projection of f on $\mathbb{S}^n$. We use the following theorem announced by Gromov in [Gro03]. He remarked that this theorem is not entirely proved in [Gro03] and unfortunately we are not able to give a proof for this theorem either. However, if we believe Gromov, then the proof of our Theorem 1 becomes much easier. On the other hand, we will give another method, which will be independent of the following theorem, to finalize the results of this paper.

THEOREM 2 (Gromov). *Let $f : \mathbb{S}^n \to \mathbb{R}^k$ be a continuous map. There exist an infinite partition of the sphere by at most k-dimensional convex sets, denoted by Π_∞, and a point $z \in \mathbb{R}^k$ such that for every $S \in \Pi_\infty$, $f^{-1}(z)$ passes through the maximum point of the density of the convexly derived measure defined on S.*

Using Gromov's theorem 2, we announce the following corollary.

COROLLARY 5.1. *Let $f : S(X) \to \mathbb{R}^k$ be as in Theorem 1. There exist an infinite partition of $S(X)$ by at most k-dimensional convex sets, denoted by Π_∞, and a point $z \in \mathbb{R}^k$ such that for every $S \in \Pi_\infty$, $f^{-1}(z)$ passes through the maximum point of the density of the (unique) convexly derived measure defined on S.*

Proof. We apply Theorem 2 for the continuous map $Pr(f)$. We know that there exists an infinite partition of the sphere, Π_∞, by at most k-dimensional convex sets. By radially projecting each piece of the partition on $S(X)$, we obtain an infinite partition of $S(X)$ by at most k-dimensional convex sets. Let $S \subset \mathbb{S}^n$ and $S \in \Pi_\infty$ and let $S' = \mathrm{pr}(S)$. Denote by z (respectively z') the maximum point of the density of the convexly derived measure defined on S (respectively S'). It remains to prove that $z' = \mathrm{pr}(z)$. Indeed, as we are taking the radial projection, the density of the convexly derived measure on each S' is just the radial projection of the density of the measure defined on S. We recall that the radial projection of the normalized Riemannian measure of S is the conical measure defined on S' up to a constant, but this is irrelevant for our purpose. We are now ready to give a proof of Theorem 1. □

5.1 Proof of Theorem 1 following Theorem 2

We apply the previous corollary. There exist an infinite partition of $S(X)$ by at most k-dimensional convex sets and a fiber $f^{-1}(z)$ passing through all the maximum points of the densities of the convexly derived measure defined on all pieces of the partition. where x_π is the maximum point of the density of the (unique) convexly derived measure μ_π defined on S_π. Hence, on every S_π, we have

$$\mu_\pi((f^{-1}(z)+\varepsilon) \cap S_\pi) \geqslant \mu_\pi(B(x_\pi,\varepsilon)) \geqslant w(\varepsilon).$$

And, at the end,

$$\begin{aligned}\mu(f^{-1}(z)+\varepsilon) &= \int_{\Pi_\infty} \mu_\pi((f^{-1}(z)+\varepsilon)\cap S_\pi)\,d\pi \\ &= \int_{\dim S_\pi = k} \mu_\pi((f^{-1}(z)+\varepsilon)\cap S_\pi)\,d\pi + \int_{\dim S_\pi < k} \mu_\pi((f^{-1}(z)+\varepsilon)\cap S_\pi)\,d\pi.\end{aligned}$$

The measure of the measurable partition is equal to one. In [Mem10b], we proved that the measure of the set of pieces of partition which has dimension $<k$ on the sphere is equal to zero; radially projecting this on $S(X)$ implies that the measure of the set of pieces of partition of $S(X)$ which has dimension $<k$ is also equal to zero; hence, we have

$$\mu(f^{-1}(z)+\varepsilon) \geqslant w(\varepsilon).$$

Hence, the proof of the theorem follows. □

6. Alternative proof of Theorem 1

This section will be long and very technical. As the author is unable to prove Theorem 2, he found, by the enormous help of Pierre Pansu, the following arguments replacing Theorem 2. We begin by giving the following useful definition.

DEFINITION 6.1. Let S be an open convex subset of $S(X)$. S is called a (k,ε)-pancake if there exists a convex set S_π of dimension k such that every point of S is at distance at most ε from S_π.

We remark again that the distance on $S(X)$ is the restriction of the norm being defined on $\mathbb{R}^{n+1}$ on $S(X)$.

The two following theorems are strong generalizations of the classical Borsuk–Ulam theorem in algebraic topology and the construction of finite and infinite partitions of $S(X)$ is provided by them.

THEOREM 3 (Gromov–Borsuk–Ulam, finite case). *Let $f:\mathbb{S}^n\to\mathbb{R}^k$ $(k\leqslant n)$ be a continuous map from the n-sphere to Euclidean space of dimension k. For every $i\in\mathbb{N}$, there exists a partition of the sphere $\mathbb{S}^n$ into 2^i open convex sets $\{S_i\}$ of equal volumes $(=\mathrm{Vol}(S^n)/2^i)$ and such that all the center points $c.(S_i)$ of the elements of partition have the same image in $\mathbb{R}^k$.*

THEOREM 4 (Gromov–Borsuk–Ulam, almost-infinite case). *Let $f:\mathbb{S}^n\to\mathbb{R}^k$ be a continuous map. For all $\varepsilon>0$, there exists an integer i_0 such that for all $i\geqslant i_0$, there exists a finite partition of $\mathbb{S}^n$ into 2^i open convex subsets such that:*

(I) *every convex subset of the partition is a (k,ε)-pancake;*

(II) *the centers of all convex subsets of the partition have the same image in $\mathbb{R}^k$;*

(III) *all convex subsets of the partition have the same volume.*

The proof of Theorem 3 is long and uses algebraic topology arguments. We will not give the proof of these theorems here and refer the reader to [Mem10b].

We need Theorems 3 and 4 on $S(X)$, but we cannot proceed directly; we again pass via the round sphere and, by radially projecting the results of these two theorems on $S(X)$, we obtain the desired partitions on $S(X)$.

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6.1 Approximation of general norms by smooth norms

For a technical reason imposed by Lemma 6.5, we need to approximate general norms by smooth norms. Indeed, as we will see in the next subsection, we cannot allow the convexly derived measures charging any mass for the boundary of balls. In this subsection, we show by approximation that we can in fact exclude this technical problem.

Lemma 6.1. *Let X denote a finite-dimensional space equipped with a C^2-smooth norm. Let $S(X)$ denote its unit sphere. Fix an auxiliary Euclidean structure. There exists K such that for every 2-plane Π passing through the origin, $S(X) \cap P$ is a disjoint union of curves whose curvatures κ satisfy $|\kappa| \leqslant K$ at all points.*

Proof. Since the norm is homogeneous of degree one, its derivative along a line passing through the origin does not vanish. It follows that at every point $x \in S(X)$, the restriction of the differential to P does not vanish identically, i.e. P is transverse to the tangent hyperplane $T_x S(X)$. This shows that $S(X) \cap P$ is a C^2-smooth one-dimensional submanifold, i.e. a finite disjoint union of curves. Furthermore, the curvature $\kappa(x, P)$ of $S(X) \cap P$ at x is a continuous function of $(x, P) \in I = \{(x, P) \mid x \in \partial B(0, 1), x \in P\}$. Since I is compact, κ is bounded. □

Notation 2. The *Hessian* of a C^2-smooth function $f : \mathbb{R}^d \to \mathbb{R}$ at x is the quadratic form

$$\mathrm{Hess}_x(v) = \frac{\partial^2}{\partial t^2} f(x + tv)_{|t=0}.$$

We say a C^2-smooth norm on a finite-dimensional vector space is *strongly convex* if, at every nonzero point, the Hessian of $x \mapsto \|x\|^2$ is positive definite.

Proposition 5. *Let X denote a finite-dimensional space equipped with a C^2-smooth strongly convex norm. Let $S(X)$ denote its unit sphere. There exists $r_0 > 0$ such that, for every $r < r_0$, for every 2-plane P passing through the origin, for every $x \in S(X)$, $S(X) \cap P \cap \partial B(x, r)$ is a finite set.*

Proof. The map $x \mapsto \mathrm{Hess}_x \|\cdot\|^2$ is homogeneous of degree zero. Fix an auxiliary Euclidean inner product on X. By compactness of the unit sphere, there exists a positive constant c such that for all $x \neq 0$ and all v,

$$(\mathrm{Hess}_x \|\cdot\|^2)(v, v) \geqslant c\, v \cdot v. \tag{2}$$

Also, the differential $x \mapsto D_x\|\cdot\|^2$ is homogeneous of degree one. Therefore, there exists a positive constant C such that for all $x \neq 0$ and all v,

$$|(D_x\|\cdot\|^2)(v)| \leqslant C\, \|x\| \sqrt{v \cdot v}. \tag{3}$$

Fix $x \in X$. Let P be a 2-plane. Let f denote the restriction of $z \mapsto \|z - x\|^2$ to P. It satisfies the previous two inequalities. Let $s \mapsto \gamma(s)$ be a C^2-smooth curve in P parameterized by arc length, $z = \gamma(0)$, $\tau = \gamma'(0)$. Then

$$\gamma(s) = z + s\tau + \frac{s^2}{2}\gamma''(0) + o(s^2),$$

since, for all small v,

$$f(z + v) = f(z) + D_z f(v) + \frac{1}{2}\,\mathrm{Hess}_x\, f(v, v) + o(v \cdot v),$$

$$f(\gamma(s)) = f(z) + D_z f\left(s\tau + \frac{s^2}{2}\gamma''(0)\right) + \frac{1}{2}\,\mathrm{Hess}_z\, f(\tau, \tau) + o(s^2).$$

Now assume that $f(\gamma(s_j)) = f(z)$ for a sequence s_j that tends to 0. Then, comparing asymptotic expansions gives

$$D_z f(\tau) = 0, \quad D_z f(\gamma''(0)) + \mathrm{Hess}_z\, f(\tau, \tau).$$

Since $\tau \cdot \tau = 1$, inequalities (2) and (3) give

$$c \leqslant -D_z f(\gamma''(0)) \leqslant C\, \|z - x\| \sqrt{\gamma''(0) \cdot \gamma''(0)}.$$

This shows that the curvature κ of the plane curve at γ at z satisfies

$$\kappa(z) \geqslant \frac{c}{C\|z - x\|}.$$

Therefore, if z is an accumulation point of $\gamma \cap P \cap \partial B(x, r)$, the curvature of γ at z is $\geqslant c/Cr$. With Lemma 6.1, we conclude that if $r < r_0 := c/CK$, for all P, $S(X) \cap P \cap \partial B(x, r)$ has only isolated points and thus is finite. □

Lemma 6.2. *Let X_1 be a finite-dimensional normed space. Let $S(X_1)$ denote its unit sphere. For every $\lambda > 1$, there exists a C^2-smooth strongly convex norm on X_1, with unit sphere $S(X_2)$, such that the radial projection $S(X_1) \to S(X_2)$ is λ-bi-Lipschitz.*

Proof. Fix an auxiliary Euclidean inner product on X_1. Fix a smooth compactly supported nonnegative function $\psi : X \to \mathbb{R}_+$ such that $\int \psi = 1$. The convolution

$$f(x) = \int_{X_1} \|y\|_1 \psi(x - y)\, dy = \int_{X_1} \|x - y\|_1 \psi(y)\, dy$$

is smooth and convex. For all $x \in X_1$,

$$|f(x) - \|x\|_1| \leqslant \int_{X_1} \|y\|_1 \psi(y)\, dy$$

is uniformly bounded. Therefore, when one restricts f to a large Euclidean sphere and extends it to become positively homogeneous of degree one, one gets a smooth norm $\|\cdot\|'$ uniformly close to $\|\cdot\|_1$. By convexity, the Hessian of $\|\cdot\|'^2$ is nonnegative. For $\delta > 0$, let

$$\|v\|_\delta = \sqrt{\|v\|'^2 + \delta\, v \cdot v}.$$

This is a smooth norm, and $\mathrm{Hess}(\|v\|_\delta^2) \geqslant \delta\, v \cdot v$ is positive definite. For δ small enough, this norm is close to $\|\cdot\|_1$ and therefore radial projection between unit spheres is λ-bi-Lipschitz. □

Lemma 6.2 allows us to reduce the proof of Theorem 1 to the special case of C^2-smooth strongly convex norms, for which we know, from Proposition 5, that convexly derived measures do not give any mass to small enough spheres. Until the end of § 6.2, we suppose the norm to be of class C^2 and strongly convex.

6.2 Infinite partitions

The proof follows [Mem10b], where the case of the round sphere $\mathbb{S}^n$ was treated. But, we need these results for the unit spheres of uniformly convex normed spaces. This merely requires a few minor changes, but we include complete proofs for completeness sake.

Definition 6.2 (Space of convexly derived measures). Let $\mathcal{MC}^n$ denote the set of probability measures on $S(X)$ of the form $\mu_S = \mu_{|S}/\mu(S)$, where $S \subset S(X)$ is open and convex and where μ is the conical probability measure defined on $S(X)$. The space $\mathcal{MC}$ of *convexly derived probability measures* on $S(X)$ is the vague closure of $\mathcal{MC}^n$.

It is a compact metrizable topological space.

Lemma 6.3. *For all open convex sets $S \subset \mathbb{S}^n$ and all $x \in S$,*

$$\frac{\operatorname{vol}(S \cap B(x,r))}{\operatorname{vol}(S)} \geqslant \frac{\operatorname{vol}(B(x,r))}{\operatorname{vol}(\mathbb{S}^n)}.$$

Proof. Apply Bishop–Gromov's inequality in Riemannian geometry. In this special case ($\mathbb{S}^n$ has constant curvature 1), it states that the ratio

$$\frac{\operatorname{vol}(S \cap B(x,r))}{\operatorname{vol}(B(x,r))}$$

is a nonincreasing function of r. It follows that

$$\frac{\operatorname{vol}(S \cap B(x,r))}{\operatorname{vol}(B(x,r))} \geqslant \frac{\operatorname{vol}(S)}{\operatorname{vol}(\mathbb{S}^n)}. \qquad \square$$

Corollary 6.4. *For all open convex sets $S \subset S(X)$ and all $x \in S$,*

$$\frac{\mu(S \cap B(x,r))}{\mu(S)} \geqslant \frac{\operatorname{vol}(\cdot, \phi(r))}{\operatorname{vol}(\mathbb{S}^n)},$$

where $\operatorname{vol}(\cdot, \phi(r))$ is the volume of a ball of radius $\phi(r)$ on $\mathbb{S}^n$ and where

$$2\sin\left(\frac{\phi(r)}{2}\right) = \frac{r}{2\sqrt{n+1}}.$$

Proof. By radially projecting $S(X)$ to $\mathbb{S}^n$, the convex set S maps to a convex set S' on the round sphere. By our previous observations, the image of the ball $B(x,r)$ contains a spherical ball of radius $\phi(r)$, where

$$2\sin\left(\frac{\phi(r)}{2}\right) = \frac{r}{2\sqrt{n+1}}.$$

Hence,

$$\begin{aligned}
\frac{\mu(S \cap B(x,r))}{\mu(S)} &\geqslant \frac{\mu'(S' \cap B(x',r'))}{\mu'(S')} \\
&\geqslant \frac{\mu'(S' \cap B(x',\phi(r)))}{\mu'(S')} \\
&\geqslant \frac{(n+1)^{n+1}\operatorname{vol}(B(x',\phi(r))}{(n+1)^{n+1}\operatorname{vol}(\mathbb{S}^n)} \\
&= \frac{\operatorname{vol}(B(x',\phi(r)))}{\operatorname{vol}(\mathbb{S}^n)}. \qquad \square
\end{aligned}$$

This inequality extends to all convexly derived measures, thanks to the following lemma.

Lemma 6.5 (See [HL99]). *Let μ_i be a sequence of positive Radon measures on a locally compact space X which vaguely converges to a positive Radon measure μ. Then, for every relatively compact subset $A \subset X$ such that $\mu(\partial A) = 0$,*

$$\lim_{i\to\infty} \mu_i(A) = \mu(A).$$

Corollary 6.6. *For all measures $\nu \in \mathcal{MC}$ on $S(X)$, all $x \in \operatorname{support}(\nu)$ and small enough r,*

$$\nu(S \cap B(x,r)) \geqslant \text{const.}\, r^n.$$

Proof. Let $\nu = \lim \mu_{S_j}$. Up to extracting a subsequence, one can assume that S_j Hausdorff converges to a compact convex set S. Then $\operatorname{support}(\nu) \subset S$. Indeed, if $x \notin S$, there exists $r > 0$

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such that $S \cap B(x,r) = \emptyset$. Let f be a continuous function on $S(X)$, supported in $B(x,r/2)$. Then, for j large enough, $S_j \cap B(x,r/2) = \emptyset$, $\int f\, d\nu_{S_j} = 0$, so $\int f\, d\nu = 0$, showing that $x \notin \mathrm{support}(\nu)$.

If ν is a Dirac measure, then the inequality trivially holds. Otherwise, let $x \in \mathrm{support}(\nu)$. There exist $x_j \in \mathrm{support}(\mu_j)$ such that the x_j tend to x. Since ν gives no measure to boundaries of small metric balls (by Proposition 5, since we assume that the norm is C^2 and strongly convex), Lemma 6.5 applies, and the inequality of Corollary 6.6 passes to the limit. □

Lemma 6.7. *Let* $\mathrm{Comp}(S(X))$ *denote the space of compact subsets of* $S(X)$ *equipped with Hausdorff distance. The map* $\mathrm{support} : \mathcal{MC} \to \mathrm{Comp}(S(X))$ *which maps a measure to its support is continuous.*

Proof. Let $\mu_j \in \mathcal{MC}$ converge to ν. One can assume that $S_j = \mathrm{support}(\mu_j)$ converge to a compact set S. We saw in the proof of Corollary 6.6 that $\mathrm{support}(\nu) \subset S$. To prove the opposite inclusion, let us define, for $r > 0$ and $x \in S(X)$,

$$f_{r,x}(y) = \begin{cases} 1 & \text{if } d(y,x) < \dfrac{r}{2}, \\ 2 - 2\dfrac{d(y,x)}{r} & \text{if } \dfrac{r}{2} \leqslant d(y,x) < r, \\ 0 & \text{otherwise,} \end{cases}$$

where d is the distance induced by the norm of $\mathbb{R}^{n+1}$. Let $x \in S$. Let $x_j \in S_j$ converge to x. According to Lemma 6.6, if $d(x_j, x) < r/4$,

$$\int f_{x,r}(y)\, d\mu_j(y) \geqslant \mathrm{const.}\, r^n,$$

i.e. $\int f_{x,r}\, d\mu_j$ does not tend to 0. It follows that $\int f_{x,r}\, d\nu > 0$, and x belongs to $\mathrm{support}(\nu)$. This shows that support is a continuous map on $\mathcal{MC}$. □

The support of a convexly derived probability measure is a closed convex set; it has a dimension.

Notation 3. $\mathcal{MC}^k$ denotes the set of convexly derived probability measures whose support has dimension k, $\mathcal{MC}^{\leqslant k} = \bigcup_{\ell=0}^{k} \mathcal{MC}^k$, $\mathcal{MC}^+ = \mathcal{MC} \backslash \mathcal{MC}^0$. For $\rho > 0$, $\mathcal{MC}_\rho$ denotes the set of convexly derived probability measures whose support has diameter $\geqslant \rho$.

Lemma 6.8. *As* r *tends to* 0, $\nu(B(x,r))$ *tends to* 0 *uniformly on* $\mathcal{MC}_\rho \times S(X)$.

Proof. We first prove the lemma in $\mathbb{R}^n$; the spherical case follows by projectively mapping hemispheres of $S(X)$ to $\mathbb{R}^n$. We can assume that ρ is very small as well. Let μ be a convexly derived measure supported by a k-dimensional convex set S, let $x \in \mathbb{R}^n$ and let $B = S \cap B(x,r)$. Since S has diameter at least ρ, there is a point y at distance at least $\rho/2$ of x. Up to a translation, we can assume that y is the origin of $\mathbb{R}^k$. Let ϕ be the density of μ. Then $\phi^{1/(n-k)}$ is concave. Thus, for $x' \in B$ and $\lambda \in]0,1[$,

$$\phi(\lambda x) \geqslant \lambda^{n-k} \phi(x).$$

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Changing variables gives

$$\begin{aligned}\mu(\lambda B) &= \int_{\lambda B} \phi(z)\, dz \\ &= \lambda^k \int_B \phi(\lambda z)\, dz \\ &\geqslant \lambda^n \int_B \phi(z)\, dz \\ &= \lambda^n \mu(B).\end{aligned}$$

If N is an integer such that $N \leqslant \rho/4r$, then one can choose N values of λ between 1/2 and 1 leading to disjoint subsets λB of S, and this yields

$$1 = \mu(S) \geqslant N(\tfrac{1}{2})^n \mu(B),$$

i.e.

$$\mu(B) \leqslant 2^n/N \simeq \text{const.}\, r/\rho.$$

Now let $S \subset S(X)$ be the support of a convexly derived measure $\nu \in \mathcal{MC}_\rho$ and let $B = B(x,r) \cap S$. We projectively map B to $\mathbb{R}^n$ and we choose the center of this projection to be the point x. Hence, it follows again that

$$\nu(B) \leqslant Cr/\rho. \qquad \square$$

LEMMA 6.9. *Let $\rho > 0$. Let $\mathcal{K}$ be a compact set of probability measures on $S(X)$ with the following property: for every $\nu \in \mathcal{K}$, all x and all $r < \rho$, $\nu(\partial B(x,r)) = 0$. Then the function $(\nu, x, r) \mapsto \nu(B(x,r))$ is uniformly continuous on $\mathcal{K} \times S(X) \times (0,\rho)$. It follows that it is continuous on $\mathcal{MC}^+ \times S(X) \times [0,\rho)$.*

Proof. Let $(\nu_i, x_i, r_i) \to (\nu, x, r)$. Let $\{x_i'\}$ (respectively x') be the sequence of points (respectively the point) on the $\mathbb{S}^n$ image of radial projection of the sequence $\{x_i\}$ (respectively x). Let $\phi_i \in Iso(\mathbb{R}^{n+1})$ be such that $\lim_{i\to\infty} \phi_i = Id$ and, for every i, $\phi_i(x_i') = x'$. Such a sequence of isometry acts on $S(X)$ by taking the action on $\mathbb{S}^n$ and projecting to $S(X)$. For every $\delta > 0$, for big enough i we have

$$B(x, r-\delta) \subset \phi_i(B(x_i, r_i)) \subset B(x, r+\delta).$$

This implies

$$\nu_i(\phi_i^{-1}(B(x, r-\delta))) < \nu_i(B(x_i, r_i)) < \nu_i(\phi_i^{-1}(B(x, r+\delta))).$$

Hence,

$$\begin{aligned}\limsup \nu_i(B(x_i, r_i)) &< \lim_{i\to\infty} \phi_{i*}\nu_i(B(x, r+\delta)) = \nu(B(x, r+\delta)), \\ \liminf \nu_i(B(x_i, r_i)) &> \lim_{i\to\infty} \phi_{i*}\nu_i(B(x, r-\delta)) = \nu(B(x, r-\delta)).\end{aligned}$$

Let $\delta \to 0$. As we supposed the norm being smooth, we know that $\nu(\partial B(x,r)) = 0$. We can apply Lemma 6.5 and deduce that $\lim_{\delta\to 0} \nu(B(x, r+\delta)) = \nu(B(x,r))$. We can apply Lemma 6.8 and the continuity on $\mathcal{MC}^+ \times S(X) \times [0,\rho)$ is deduced. $\square$

DEFINITION 6.3 (Limits of finite convex partitions). Let Π be a finite convex partition of $S(X)$. We view it as an atomic probability measure $m(\Pi)$ on $\mathcal{MC}$ as follows: for each piece S of Π, let $\mu_S = \mu_{|S}/\mu(S)$ be the normalized volume of S. Then set

$$m(\Pi) = \sum_{\text{pieces } S} \mu(S)\delta_{\mu_S}.$$

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We define the *space of* (infinite) *convex partitions* $\mathcal{CP}$ as the vague closure of the image of the map m in the space $\mathcal{P}(\mathcal{MC})$ of probability measures on the space of convexly derived measures. The subset $\mathcal{CP}^{\leqslant k}$ of convex partitions of dimension $\leqslant k$ consists of elements of $\mathcal{CP}$ which are supported on the subset $\mathcal{MC}^{\leqslant k}$ of convexly derived measures with support of dimension at most k.

Note that $\mathcal{CP}$ is compact and $\mathcal{CP}^{\leqslant k}$ is closed in it. Measures in the support of a convex partition can be thought of as the pieces of the partition.

Lemma 6.10 (Disintegration formula). *Let $A \subset S(X)$ be a set such that the intersection of ∂A with every ℓ-dimensional subsphere has vanishing ℓ-dimensional measure, for all ℓ, $0<\ell<n$. Let $\Pi \in \mathcal{CP}$. Assume that $\Pi(\mathcal{MC}^0)=0$. Then*

$$\mu(A)=\int_{\mathcal{MC}} \nu(A)\, d\,\Pi(\nu).$$

Proof. For finite partitions Π_i, equality holds. According to Lemma 6.5, the function $\nu \mapsto \nu(A)\chi(\nu)$ is continuous on $\mathcal{MC}^+$. Therefore, the identity still holds for vague limits of finite partitions. This completes the proof of the lemma. □

6.3 Choice of a center map

In the previous sections, we did not make any particular assumption about the center map. In fact, the only property of this map which was used was the continuity. In this section, we construct a family of center maps, which will lead us to the proof of Theorem 1.

Definition 6.4 (Approximate centers of convexly derived measures). Let $\nu \in \mathcal{MC}$ and let $r>0$. Consider the function $S(X)\to\mathbb{R}$, $x \mapsto v_{r,\nu}(x)=\nu(B(x,r))$. Let $M_r(\nu)$ be the set of points where $v_{r,\nu}$ achieves its maximum on support(ν).

If the support of ν is ℓ dimensional, $\ell<n$, we denote by $M_0(\nu)$ the unique point where the density of ν achieves its maximum.

The next lemma states a semi-continuity property of M_r.

Notation 4. When A_i, $i\in\mathbb{N}$, are subsets of a topological space, we shall denote by

$$\lim_{i\to\infty} A_i=\bigcap_i \overline{\bigcup_{j\geqslant i} A_j}$$

the set of all possible limits of subsequences $x_{i(j)} \in A_{i(j)}$.

Lemma 6.11. *Let ν_i be convexly derived measures which converge to $\nu \in \mathcal{MC}$. Then, for all $r>0$,*

$$\lim_{i\to\infty} M_r(\nu_i) \subset M_r(\nu).$$

If follows that

$$\lim_{i\to\infty} \operatorname{conv.hull}(M_r(\nu_i)) \subset \operatorname{conv.hull}(M_r(\nu)).$$

Proof. Let ν_i tend to ν. Then the support of ν_i Hausdorff converges to the support of ν. If $\nu\in\mathcal{MC}^0$ equals the Dirac measure at x, then $M_r(\nu_i)$ automatically converges to $\{x\}=M_r(\nu)$. Otherwise, $\nu\in\mathcal{MC}^+$. Let $x\in\lim_{i\to\infty} M_r(\nu_i)$, i.e. $x=\lim_{i\to\infty} x_i$ for some $x_i\in M_r(\nu_i)$. Pick $y\in$ support(ν). Pick a sequence $y_i\in$ support(ν_i) converging to y. According to Lemma 6.9,

$$v_{r,\nu}(x)=\lim_{i\to\infty} v_{r,\nu_i}(x_i), \quad v_{r,\nu}(y)=\lim_{i\to\infty} v_{r,\nu_i}(y_i).$$

Since $v_{r,\nu_i}(x_i)\geqslant v_{r,\nu_i}(y_i)$, we get $v_{r,\nu}(x)\geqslant v_{r,\nu}(y)$, showing that $x\in M_r(\nu)$.

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We claim that for arbitrary compact sets $A_i \in S(X)$,

$$\lim_{i\to\infty} \text{conv. hull}(A_i) \subset \text{conv. hull}(\lim_{i\to\infty} A_i).$$

Indeed, taking cones, it is sufficient to check this in $\mathbb{R}^{n+1}$. If $x \in \lim_{i\to\infty} \text{conv. hull}(A_i)$, $x = \lim x_i$ with $x_i \in \text{conv. hull}(A_i)$, then there exist $n+1$ numbers $t_{i,j} \in [0,1]$ and points $a_{i,j} \in A_i$ such that $\sum_j t_{i,j} = 1$, $x_i = \sum_j t_{i,j} a_{i,j}$. One can assume that all sequences $i \mapsto t_{i,j}$, $a_{i,j}$ converge to t_j, a_j. Then $t_j \in [0,1]$, $\sum_j t_j = 1$, $a_j \in A = \lim_{i\to\infty} A_i$ and $x = \sum_j t_j a_j \in \text{conv. hull}(A)$. This completes the proof of Lemma 6.11. □

The above semi-continuity property is sufficient to apply Michael's theory of continuous selections [Mic59].

Theorem 6 (Michael's continuous selection theorem). *Let X be paracompact, Y a Banach Space and $\mathfrak{S}$ the space of closed convex nonempty subsets of Y. Then every lower semi-continuous map $\phi : X \to \mathfrak{S}$ admits a continuous selection.*

Let $\delta > 0$ and small be fixed. We use Theorem 6 for $X = \mathcal{MC}$ and $Y = \mathbb{R}^{n+1}$ and the map $\phi = \text{conv. hull} M_r(S) + \delta$, where the convex hull is taken with respect to the geometry of $\mathbb{R}^{n+1}$ and $M_r(S) + \delta$ is the *delta*-neighborhood of $M_r(S)$ in $\mathbb{R}^{n+1}$. In this case, Theorem 6 provides a continuous selection for the map ϕ. To have a continuous selection on $S(X)$, it will be sufficient to take $\text{conv. hull}\phi(S) \cap S(X)$, where this time the convex hull is taken with respect to the geometry of $S(X)$.

Definition 6.5 (Centers of open convex sets). Let $r > 0$. According to Theorem 6, we can choose a continuous map $C_r : \mathcal{MC}^n \to S(X)$ such that, for every $S \in \mathcal{MC}^n$, $C_r(S)$ belongs to $\text{conv. hull}(M_r(S))$.

6.4 Construction of partitions adapted to a continuous map

Definition 6.6 (Partitions adapted to a continuous map). Let $f : S(X) \to \mathbb{R}^k$ be a continuous map. Let $r \geqslant 0$. We say a convex partition $\Pi \in \mathcal{CP}$ is *r-adapted to f* if there exists $z \in \mathbb{R}^k$ such that $f^{-1}(z)$ intersects the convex hull of $M_r(\nu)$ for all measures ν in the support of Π. Let

$$\mathcal{F}_r = \left\{ \Pi \in \mathcal{CP} \,\middle|\, \bigcap_{\nu \in \text{support}(\Pi)} f(\text{conv. hull}(M_r(\nu))) \neq \emptyset \right\}$$

denote the set of partitions which are r-adapted to f.

Proposition 7. *For all $r > 0$, $\mathcal{F}_r$ is closed in $\mathcal{CP}$.*

Proof. If $\lim_{i\to\infty} \Pi_i = \Pi$, $\text{support}(\Pi) \subset \lim_{i\to\infty} \text{support}(\Pi_i)$, i.e. every piece ν of Π is the limit of a sequence of pieces ν_i of Π_i. By assumption, there is a $z_i \in \mathbb{R}^k$ which belongs to all $f(\text{conv. hull}(M_r(\nu)))$, $\nu \in \text{support}(\Pi_i)$. One can assume z_i converges to z. Then z belongs to all $f(\text{conv. hull}(M_r(\nu)))$, $\nu \in \text{support}(\Pi)$. Indeed, in general, if g is a continuous map and A_i are subsets of a compact space, $g(\lim_{i\to\infty} A_i) = \lim_{i\to\infty} g(A_i)$. So, if $\nu = \lim \nu_i$, $\nu_i \in \text{support}(\Pi_i)$,

$$\begin{aligned} z = \lim_{i\to\infty} z_i &\in \lim_{i\to\infty} f(\text{conv. hull}(M_r(\nu_i))) \\ &\subset f(\lim_{i\to\infty} \text{conv. hull}(M_r(\nu_i))) \\ &\subset f(\text{conv. hull}(M_r(\nu))), \end{aligned}$$

thanks to Lemma 6.11. □

Corollary 6.12. *Let $f : S(X) \to \mathbb{R}^k$ be a continuous map. For all $r > 0$, $\mathcal{F}_r \cap \mathcal{CP}^{\leqslant k}$ is nonempty.*

Proof. Theorem 3 states that for every $r > 0$, $\mathcal{F}_r$ contains uniform atomic measures with arbitrarily many pieces. Theorem 4 produces elements of $\mathcal{F}_r$ whose support is contained in arbitrarily thin neighborhoods of the compact subset $\mathcal{MC}^{\leqslant k}$. With Proposition 7, this gives elements in $\mathcal{F}_r \cap \mathcal{CP}^{\leqslant k}$. □

6.5 Convergence of $M_r(\nu)$ as r tends to 0

Lemma 6.13. *Let $\ell < n$. For every ℓ-dimensional convexly derived measure ν,*

$$\lim_{r \to 0} d_H(M_r(\nu), M_0(\nu)) = 0.$$

Proof. We prove the lemma by contradiction. Otherwise, we get a $\delta > 0$ and a sequence of radii r_i tending to 0 such that $d_H(M_{r_i}(\nu), M_0(\nu)) \geqslant \delta$. Pick a point $x_i \in S$ where $v_{r_i,\nu}$ achieves its maximum and such that $d(x_i, M_0(\nu)) \geqslant \delta$. Up to extracting a subsequence, we can assume that x_i converges to $x \in S$. Then $v_{r_i,\nu}(x_i)/\alpha_k r_i^k$ converges to $\phi_\nu(x)$. For every $y \in S$, $v_{r_i,\nu}(y) \leqslant v_{r_i,\nu}(x)$ and $v_{r_i,\nu}(y)/\alpha_k r_i^k$ converges to $\phi_\nu(y)$. Therefore, $\phi_\nu(y) \leqslant \phi_\nu(x)$. This shows that $\{x\} = M_0(\nu)$, a contradiction. □

A stronger statement (Corollary 6.17) will be given after the following technical lemmas.

Lemma 6.14. *Let ν be a convexly derived measure on $S(X)$ whose support is a k-dimensional convex set S. Write $d\nu = \phi \, d\mu_k$. Then*

$$\max_S \phi \leqslant \frac{2^{n+1}}{\mu_k(S)}.$$

Proof. Replace S with $C = \mathrm{co}(S) \subset \mathbb{R}^{n+1}$, and ϕ by its $n-k$-homogeneous extension. Then $\phi^{1/(n-k)}$ is concave. Assume ϕ achieves its maximum at $x \in C$. Translate C so that $x = 0$. On $\frac{1}{2}C$, $\phi^{1/(n-k)} \geqslant \frac{1}{2}\phi^{1/(n-k)}(x)$; thus,

$$\begin{aligned} 1 = \nu(S) &\geqslant \int_{\frac{1}{2}C} \phi \, d\,\mathrm{vol}_{k+1} \\ &\geqslant \frac{1}{2^{n-k}} \phi(x) \, \mathrm{vol}_{k+1}\left(\frac{1}{2}C\right) \\ &= \frac{1}{2^{n+1}} \phi(x) \, \mathrm{vol}_{k+1}(C) \\ &= \frac{1}{2^{n+1}} \phi(x) \mu_k(S). \end{aligned}$$ □

Lemma 6.15. *Let S, S_i be full compact convex subsets of $\mathbb{R}^n$ such that S_i Hausdorff converges to S. Let $\phi_i : S_i \to [0,1]$ be concave functions. Then there exist a concave function $\phi : S \to [0,1]$ and a subsequence with the following properties.*

- *On every compact subset of the interior of S, ϕ_i converges uniformly to ϕ.*
- *For all $x \in \partial S$ and all sequences $x_i \in S_i$ converging to x,*

$$\limsup_{i \to \infty} \phi_i(x_i) \leqslant \phi(x).$$

Proof. In general, bounded concave functions f on compact convex sets Σ are locally Lipschitz,

$$\text{for } x \in \Sigma \quad \text{with } d(x, \partial\Sigma) = r \quad \text{and all } y \in \Sigma, \quad |f(x) - f(y)| \leqslant \frac{1}{r} d(x, y).$$

Indeed, let $[x', y']$ be the intersection of Σ with the line through x and y, with x', x, y' and y sitting along the line in this order. Let ℓ be the affine function on $[x', y']$ such that $\ell(x') = f(x')$ and $\ell(x) = f(x)$. Then $f(y) \leqslant \ell(y)$; thus, $f(y) - f(x) \leqslant (1/d(x', x))|f(x) - f(x')|d(x, y) \leqslant (1/r)d(x, y)$. Also, let ℓ' be the affine function on $[x', y']$ such that $\ell'(x) = f(x)$ and $\ell'(y') = f(y')$. Then $f(y) \geqslant \ell'(y)$; thus, $f(y) - f(x) \geqslant -(1/d(x, y'))|f(x) - f(y')|d(x, y) \geqslant -(1/r)d(x, y)$.

This shows that on every compact subset of the interior of S, the sequence f_j is equicontinuous, so a subsequence can be found which converges uniformly on all such compact sets to a continuous function ϕ. Of course, ϕ is concave and bounded, so it extends continuously to ∂S. Let $x \in \partial S$ and $x_i \in S_i$ converge to x. Pick an interior point x_0 of S and a second interior point $x' \neq x_0$ such that x_0 lies on the segment $[x', x]$. Pick x'_i on the line passing through x_0 and x_i and converging to x'. The Lipschitz estimate for ϕ_i reads

$$\phi_i(x_i) - \phi_i(x_0) \leqslant \frac{d(x_0, x_i)}{d(x_0, x'_i)} |\phi_i(x'_i) - \phi_i(x_0)|.$$

Letting i tend to infinity yields

$$\limsup \phi_i(x_i) \leqslant \phi(x_0) + \frac{d(x_0, x)}{d(x_0, x')} |\phi(x') - \phi(x_0)|.$$

Letting x_0 and x' tend to x (while keeping x', x_0 and x aligned and $d(x_0, x)/d(x_0, x')$ bounded) gives $\limsup \phi_i(x_i) \leqslant \phi(x)$. □

Lemma 6.16. *For each $k < n$, the restriction of $(\nu, r) \mapsto d_H(M_r(\nu), M_0(\nu))$ to $\mathbb{R}_+ \times \mathcal{MC}^k$ tends to 0 along $\{0\} \times \mathcal{MC}^k$, i.e. for all $\nu \in \mathcal{MC}^k$,*

$$\lim_{r \to 0,\, \nu' \to \nu,\, \nu' \in \mathcal{MC}^k} d_H(M_r(\nu), M_0(\nu)) = 0.$$

Proof. Let $\nu \in \mathcal{MC}^k$. Let ν_i be a sequence of k-dimensional convexly derived measures which converges to ν and r_i be positive numbers tending to 0. For every i, we project the support of ν_i into the k-sphere which contains the support of ν (if intrinsically this poses a problem, one can always think of the cones over the support of these measures and do all projections in $\mathbb{R}^{n+1}$). In other words, one can assume that all ν_i have support S_i in the same k-sphere. Of course, S_i Hausdorff converges to the support S of ν. Let ϕ_i denote the density of ν_i with respect to k-dimensional conical measure. Since $\mu_k(S_i)$ does not tend to 0, the ϕ_i are uniformly bounded, by Lemma 6.14. Furthermore, on any compact convex subset K of the relative interior of S, the ϕ_i are equicontinuous (this follows by the cone construction from Lemma 6.15). Therefore, one can assume that the ϕ_i converge uniformly on compact subsets of the relative interior of S. Since, for all $r' > 0$, v_{r',ν_i} converges to $v_{r',\nu}$, the limit must be equal to the density ϕ of ν. From Lemma 6.15, one can assert that at boundary points $x \in \partial S$, for every sequence $x_i \in S_i$ converging to x, $\limsup \phi_i(x_i) \leqslant \phi(x)$.

We repeat the argument of Lemma 6.13. If $M_{r_i}(\nu_i)$ does not converge to $M_0(\nu)$, some sequence $x_i \in M_{r_i}(\nu_i)$ satisfies $d(x_i, M_0(\nu)) \geqslant \delta$ for some $\delta > 0$. Up to extracting a subsequence, we can assume that x_i converges to $x \in S$. If $x \notin \partial S$, then $v_{r_i,\nu}(x_i)/\alpha_k r_i^k$ converges to $\phi(x)$. If $x \in \partial S$, $\limsup v_{r_i,\nu}(x_i)/\alpha_k r_i^k \leqslant \phi(x)$. For every $y \in S \backslash \partial S$, $v_{r_i,\nu}(y) \leqslant v_{r_i,\nu}(x)$ and $v_{r_i,\nu}(y)/\alpha_k r_i^k$ converges to $\phi(y)$. Therefore, $\phi(y) \leqslant \phi(x)$. Since $S \backslash \partial S$ is dense in S, this holds for all $y \in S$; thus, ϕ achieves its maximum at x, i.e. $\{x\} = M_0(\nu)$, a contradiction. □

Corollary 6.17. *On any compact subset of $\mathcal{MC}^k$, the functions*

$$\nu \mapsto d_H(M_r(\nu), M_0(\nu))$$

converge uniformly to 0 *as* r *tends to* 0.

Proposition 8. *Assume* $f: S(X) \to \mathbb{R}^k$ *is a generic smooth map. Let* r_i *tend to* 0 *and let* $\Pi_i \in \mathcal{CP}^{\leqslant k} \cap \mathcal{F}_{r_i}$ *be convex partitions of dimension* $\leqslant k$, r_i*-adapted to* f. *Then, for all* $\varepsilon > 0$,

$$\max_{z \in \mathbb{R}^k} \mu(f^{-1}(z) + \varepsilon) \geqslant w(\varepsilon) \limsup_{i \to \infty} \Pi_i(\mathcal{MC}^k),$$

where

$$w(\varepsilon) = \frac{1}{1 + (1 - 2\delta(\varepsilon/2))^{n-k}(k+1)^{k+1}(F(k, \varepsilon/2)/G(k, \varepsilon/2))}$$

and where the functions $F(\cdot, \cdot)$ *and* $G(\cdot, \cdot)$ *were defined previously.*

Proof. By assumption, for each i, there exists $z_i \in \mathbb{R}^k$ such that for all $\mu \in \mathrm{support}(\Pi_i)$, there exists $x_{i,\nu} \in \mathrm{conv.\,hull}(M_{r_i}(\nu))$ such that $f(x_{i,\nu}) = z_i$. Let $\mathcal{K} \subset \mathcal{MC}^k$ be a compact set. According to Corollary 6.17 and Lemma 6.9, for all $\varepsilon > 0$,

$$\delta_i := \sup_{\nu \in \mathcal{K}} |\nu(B(x_{i,\nu}, \varepsilon)) - \nu(B(M_0(\nu), \varepsilon))|$$

tends to 0. Considerations in previous sections show that for every k-dimensional convexly derived measure ν,

$$\nu(B(M_0(\nu), \varepsilon)) \geqslant w(\varepsilon).$$

For a generic smooth map f, the intersection of $f^{-1}(z_i) + \varepsilon$ with k-dimensional convex sets has vanishing k-dimensional measure, so the disintegration formula applies, and

$$\begin{aligned}
\mu(f^{-1}(z_i) + \varepsilon) &\geqslant \int_{\mathcal{MC}^+} \nu(f^{-1}(z_i) + \varepsilon)\, d\,\Pi_i(\nu) \\
&\geqslant \int_{\mathcal{K}} \nu(B(x_{i,\nu}, \varepsilon))\, d\,\Pi_i(\nu) \\
&\geqslant \Pi_i(\mathcal{K}) w(\varepsilon) - \delta_i.
\end{aligned}$$

Taking the supremum over all compact subsets of $\mathcal{MC}^k$ and then a limit as i tends to infinity yields the announced inequality. □

6.6 End of the proof of Theorem 1

It remains to show that convex partitions in $\mathcal{CP}^{\leqslant k} \cap \mathcal{F}_r$, r small, put most of their weight on k-dimensional pieces. This will be proven indirectly. Pieces of dimension $<k$ may exist, but they provide a lower bound on $\mu(f^{-1}(z) + r)$ which is so large that they must have small weight. We shall need a weak concavity property of $v_{\mu,r}$, which in turn relies on the corresponding Euclidean statement.

Lemma 6.18. *Let* $S \subset \mathbb{R}^n$ *be an open convex set and* ϕ *an* m*-concave function defined on* S. *Let* $\mu = \phi d\,\mathrm{vol}_n$. *Then the map* $x \mapsto \mu(B(x, r) \cap S)$ *is* $(m+n)$*-concave on* S.

Proof. We use the following estimate (generalized Prekopa–Leindler inequality), which can be found in [LB08]. For $\alpha \in [-\infty, +\infty]$ and $\theta \in [0, 1]$, the α-mean of two nonnegative numbers a and b with weight θ is

$$M_\alpha^{(\theta)}(a, b) = (\theta a^\alpha + (1-\theta) b^\alpha)^{1/\alpha}.$$

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Let $-1/n \leqslant \alpha \leqslant +\infty$, $\theta \in [0,1]$, u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ such that for all x, $y \in \mathbb{R}^n$,

$$w(\theta x + (1-\theta)y) \geqslant M_\alpha^{(\theta)}(u(x), v(y)).$$

Let $\beta = \alpha/(1+\alpha n)$. Then

$$\int w \geqslant M_\beta^{(\theta)}\left(\int u, \int v\right).$$

We apply this to restrictions of ϕ to balls, $u = 1_{B(x,r)}\phi$, $v = 1_{B(y,r)}\phi$, $w = 1_{B(\theta x+(1-\theta)y,r)}\phi$. By m-convexity of ϕ, the assumptions of the generalized Prekopa–Leindler inequality are satisfied with $\alpha = 1/m$. Then, for $\beta = 1/(m+n)$,

$$\mu(B(\theta x + (1-\theta)y), r) \geqslant M_\beta^{(\theta)}(\mu(B(x,r)), \mu(B(y,r))),$$

which means

$$\mu(B(\theta x + (1-\theta)y), r)^{1/(m+n)} \geqslant \theta\mu(B(x,r))^{1/(m+n)} + (1-\theta)\mu(B(y,r))^{1/(m+n)}. \qquad \square$$

Lemma 6.19. *The functions $v_{\nu,r}$ are weakly concave on $S(X)$. In other words, there exists a constant $c = c(n) > 0$ such that for every convexly derived measure ν and every sufficiently small $r > 0$, if $K \subset \mathrm{support}(\nu)$, then*

$$\min_{\mathrm{conv}(K)} v_{\nu,r/c} \geqslant c \min_K v_{\nu,r}.$$

Proof. Since a half-sphere is projectively equivalent with Euclidean space, it suffices to prove weak concavity when K consists of two points.

Let ν be a k-dimensional convexly derived measure on $S(X)$. Denote its density by ϕ, a weak $(n-k)$-concave function on the support S of ν. Let Φ denote the $(n-k)$-homogeneous extension of ϕ to the cone on S. This is $(n-k)$-concave. Fix a point $x_0 \in S(X)$ and let $\mathbb{R}^n$ denote the tangent space (cone) of $S(X)$ at x_0. Denote by ϕ' the restriction of Φ to $\mathbb{R}^n$, and ν' the measure with density ϕ'. Lemma 6.18 implies that $x' \mapsto \mu(B(x', r))$ is $(2n-k)$-concave. This implies that for every x', $y' \in \mathbb{R}^n$ and z' belonging to the middle third of the line segment $[x', y']$,

$$\nu'(B(z', r)) \geqslant \frac{1}{3^{2n-k}} \max\{\nu'(B(x', r)), \nu'(B(y', r))\}.$$

The radial projection from a neighborhood $V \subset S(X)$ of x_0 to $\mathbb{R}^n$ is nearly isometric and nearly maps ϕ' to ϕ. Thus, there exists a constant $c_1 > 0$ such that if x, $y \in V$ and z belong to the middle third of the segment $[x, y]$,

$$\nu\left(B\left(z, \frac{r}{c_1}\right)\right) \geqslant c_1 \max\{\nu(B(x,r)), \nu(B(y,r))\}.$$

Covering long segments $[x, y]$ with N neighborhoods like V (N can be bounded independently of n) provides a constant $c > 0$ such that for all $z \in [x, y]$ which is not too close to the end points,

$$\nu\left(B\left(z, \frac{r}{c_1^N}\right)\right) \geqslant c_1^N \max\{\nu(B(x,r)), \nu(B(y,r))\}.$$

In particular, for $c = c_1^N$,

$$\nu\left(B\left(z, \frac{r}{c}\right)\right) \geqslant c \min\{\nu(B(x,r)), \nu(B(y,r))\}. \qquad \square$$

PROPOSITION 9. *There exists a constant $c = c(n) > 0$ such that if $f : S(X) \to \mathbb{R}^k$ is smooth and generic and Π belongs to $\mathcal{F}_r \cap \mathcal{CP}^{\leqslant k}$ for some small enough $r > 0$, then*

$$\max_{z\in\mathbb{R}^k} \mu\left(f^{-1}(z) + \frac{r}{c}\right) \geqslant c \sum_{\ell=0}^{k} w_l(r)\Pi(\mathcal{MC}^{\ell}),$$

where $w_l(r)$ is equal to $w(r)$ in codimension l.

Proof. By assumption, there exists $z \in \mathbb{R}^k$ such that for every measure ν in the support of Π, there exists $x \in \text{conv. hull}(M_r(\nu))$ such that $f(x) = z$. If the support of ν is ℓ dimensional, Lemma 6.11 and our previous computations give

$$\begin{aligned}\nu\left(f^{-1}(z) + \frac{r}{c}\right) &\geqslant \nu\left(B\left(x, \frac{r}{c}\right)\right)\\ &= v_{\mu,r/c}(x)\\ &\geqslant c \min_{M_r(\nu)} v_{\nu,r}\\ &= c \max_{\text{support}(\nu)} v_{\nu,r}\\ &\geqslant c\, v_{\nu,r}(M_0(\nu))\\ &= c\,\nu(B(M_0(\nu), r))\\ &\geqslant c\, w_l(\rho).\end{aligned}$$

Again, for generic smooth f, one can integrate this with respect to Π.

$$\begin{aligned}\mu(f^{-1}(z) + r) &= \int_{\mathcal{MC}} \nu(f^{-1}(z) + r)\, d\,\Pi(\nu)\\ &\geqslant c \sum_{\ell=0}^{k} w_l(\rho)\Pi(\mathcal{MC}^{\ell}).\end{aligned}$$ □

LEMMA 6.20. *For every $l < k$, we have*

$$\lim_{r\to 0} w_l(r)/w_k(r) = \infty.$$

Proof. Simple observation shows that for every $m \in \mathbb{N}$, $\lim_{r\to 0} G(m, r) \to 0$ and $\lim_{r\to 0} F(m, r) = 1$. Simple calculation leads to

$$w_l(r)/w_k(r) = \frac{1 + (1 - 2\delta(r/2))^{n-k}(F(k, r/2)/G(k, r/2))(k+1)^{k+1}}{1 + (1 - 2\delta(r/2))^{n-l}(F(l, r/2)/G(l, r/2))(l+1)^{l+1}} \backsim_{r\to 0} C\frac{G(l, r)}{G(k, r)}$$

and, by the well-known asymptotic behavior of the function $G(m, r)$, we have

$$\frac{G(l, r)}{G(k, r)} \backsim_{r\to 0} r^{l-k}.$$

Hence, the proof of the lemma follows. □

Proof of Theorem 1.

PROPOSITION 10. *Let $\varepsilon > 0$. Let $f : S(X) \to \mathbb{R}^k$ be a continuous map. Then*

$$\max_{z\in\mathbb{R}^k} \mu(f^{-1}(z) + \varepsilon) \geqslant w(\varepsilon).$$

Proof. Assume first that f is smooth and generic. Then there exists a constant W such that for all sufficiently small r,

$$\max_{z\in\mathbb{R}^k} \mu(f^{-1}(z)+r) \leqslant Wr^k.$$

For every $r>0$, there exists a convex partition $\Pi_r \in \mathcal{CP}^{\leqslant k} \cap \mathcal{F}_r$ which is r-adapted to f (Corollary 6.12). Proposition 9 yields

$$\sum_{\ell=0}^{k} w_l(r)\Pi_r(\mathcal{MC}^\ell) \leqslant \frac{1}{c} \max_{z\in\mathbb{R}^k} \mu\left(f^{-1}(z)+\frac{r}{c}\right) \leqslant \frac{W}{c}\left(\frac{r}{c}\right)^k.$$

As r tends to 0, this implies that for all $\ell<k$ (including $\ell=0$), $\Pi_r(\mathcal{MC}^\ell)$ tends to 0, and thus $\Pi_r(\mathcal{MC}^k)$ tends to 1. Letting r tend to 0 in Proposition 8 then shows that

$$\max_{z\in\mathbb{R}^k} \mu(f^{-1}(z)+\varepsilon) \geqslant w(\varepsilon). \qquad \square$$

Every continuous map $f: S(X)\to\mathbb{R}^k$ is a uniform limit of smooth generic maps. Hausdorff semi-continuity of $X\mapsto\mu(X+\varepsilon)$ then extends the result to all continuous maps. Indeed, let the continuous map $f: S(X)\to\mathbb{R}^k$ of Theorem 1 be fixed. Let $g_j : S(X)\to\mathbb{R}^k$ be a sequence of C^∞ maps such that $\delta_j = \|g_j - f\|_{C^0}$ tends to 0. For every j, there exists a $z_j\in\mathbb{R}^k$ such that

$$\mu(g_j^{-1}(z_j)+\varepsilon) \geqslant w(\varepsilon).$$

We know that for every j, $g_j^{-1}(z_j) \subseteq f^{-1}(B(z_j,\delta_j))$. Then

$$\mu(f^{-1}(B(z_j,\delta_j))+\varepsilon) \geqslant \mu(g_j^{-1}(z_j)+\varepsilon) \geqslant w(\varepsilon).$$

Up to extracting a subsequence, we can assume that $\{z_j\}$ converges to a point z. There exists a decreasing sequence $\varepsilon_j\to 0$ such that for every j, $|z-z_j|\leqslant\varepsilon_j$. Then

$$f^{-1}(B(z_j,\delta_j))+\varepsilon \subseteq f^{-1}(B(z,\delta_j+\varepsilon_j))+\varepsilon;$$

thus, for all j,

$$\mu(f^{-1}(B(z,\delta_j+\varepsilon_j)+\varepsilon) \geqslant w(\varepsilon)$$

and, by the Fatou lemma,

$$\mu\left(\bigcap_j f^{-1}(B(z,\delta_j+\varepsilon_j))+\varepsilon\right) \geqslant w(\varepsilon).$$

If, for all j, $x\in f^{-1}(B(z,\delta_j+\varepsilon_j))+\varepsilon$, then there exists y_j such that $d(x,y_j)\leqslant\varepsilon$ and $f(y_j)\in B(z,\delta_j+\varepsilon_j)$. We choose a subsequence y_k which converges to y. By construction, $d(x,y)\leqslant\varepsilon$, $f(y)=z$ and thus $x\in f^{-1}(z)+\varepsilon$. Hence,

$$\bigcap_j f^{-1}(B(z,\delta_j+\varepsilon_j))+\varepsilon) \subset f^{-1}(z)+\varepsilon$$

and

$$\mu(f^{-1}(z)+\varepsilon) \geqslant w(\varepsilon).$$

7. Why all these complications?

Remember the following theorem.

Theorem 11 (Gromov [Gro03]). *Let $f:\mathbb{S}^n\to\mathbb{R}^k$ be a continuous map from the canonical unit n-sphere to a Euclidean space of dimension k, where $k\leqslant n$. There exists a point $z\in\mathbb{R}^k$ such* *that the n-spherical volume of the ε-tubular neighborhood of $f^{-1}(z)$, denoted by $f^{-1}(z)+\varepsilon$, satisfies, for every $\varepsilon > 0$,*

$$\mathrm{vol}_n(f^{-1}(z)+\varepsilon) \geqslant \mathrm{vol}_n(S^{n-k}+\varepsilon).$$

Here $\mathbb{S}^{n-k}$ is the $(n-k)$-equatorial sphere of $\mathbb{S}^n$.

Several times during the last sections, we used the radial projection between the canonical sphere and the unit sphere $S(X)$. One could ask why bother with all we did and not just radially project the result of Theorem 11 on $S(X)$? Indeed, this gives another lower bound for the waist of $S(X)$, as we will show in the next proposition.

Proposition 12. *Let $S(X)$ be the unit sphere of an $(n+1)$-dimensional normed space X, for which the distance is induced from the norm of X. The measure defined on $S(X)$ is the conical probability measure. A lower bound for the waist of $S(X)$ relative to $\mathbb{R}^k$ is given by*

$$w_2(\varepsilon) = (n+1)^{-n-1}\frac{\mathrm{vol}(\mathbb{S}^{n-k}+\varepsilon/(n+1))}{\mathrm{vol}(\mathbb{S}^n)}.$$

Proof of the proposition. Let pr be the radial projection of $\mathbb{S}^n$ to $S(X)$. We apply Theorem 11 to the map $g = pr^{-1}\circ f$. Hence, there exists a fiber X such that for every $\varepsilon > 0$,

$$\mathrm{vol}(X+\varepsilon) \geqslant \mathrm{vol}(\mathbb{S}^{n-k}+\varepsilon).$$

We radially project $X+\varepsilon$ to $S(X)$. We have

$$\mathrm{pr}(X+\varepsilon) \subset \mathrm{pr}(X) + (n+1)\varepsilon.$$

Hence,

$$\begin{aligned}\mu(\mathrm{pr}(X)+\varepsilon) &\geqslant \mu\left(pr\left(X+\frac{\varepsilon}{n+1}\right)\right)\\ &\geqslant (n+1)^{-n-1}\frac{\mathrm{vol}(X+\varepsilon/(n+1))}{\mathrm{vol}(\mathbb{S}^n)}\\ &\geqslant (n+1)^{-n-1}\frac{\mathrm{vol}(\mathbb{S}^{n-k}+\varepsilon/(n+1))}{\mathrm{vol}(\mathbb{S}^n)}.\end{aligned}$$

The proposition is proved. □

We see that a brutal application of Gromov's theorem gives a lower bound for the waist of the unit sphere of a uniformly convex normed space, $S(X)$. But, comparing $w_1(\varepsilon)$ and $w_2(\varepsilon)$, we can see that the lower bound $w_1(\varepsilon)$ has a much better dependence on the variable n, even if the dependence on the variable k is very bad.

For example, if k is fixed and n tends to infinity, $w_2(\varepsilon)$ tends (exponentially fast) to 0, while, for this case, the lower bound $w_1(\varepsilon)$ tends to 1. One can hope to have a better dependence on the variable k by knowing the best degree of dilation of the radial projection of $\mathbb{S}^n \to S(X)$. Here we gave a trivial bound for the degree of dilation, not taking into account uniform convexity.

8. Comparison between the waist of $S(X)$ with the waist of the round sphere

One major benefit of having a metric invariant is the ability to compare it between different metric spaces. For instance, Theorem 11 gives the sharp estimation of the waist of the round (canonical) sphere for every n and k. Our main Theorem 1 gives an estimation of the waist of $S(X)$ which is not sharp. It may seems strange that using the *sharp* estimate of the waist of the round sphere to obtain an estimate of the waist of $S(X)$ brings a sharp value to a nonsharp one.

But, this is not surprising, since to obtain our value we had to integrate on a round sphere and then use the radial projection. This is why we obtain a result which is far from being optimal. We believe for a sharp value of the waist of $S(X)$, one needs to use other methods than the one used in this paper. Denote the optimal value of the waist of $S(X)$ (for every $\varepsilon > 0$) by $w_{op}(\varepsilon)$. Then clearly

$$w_{op}(\varepsilon) \geqslant w(\varepsilon).$$

Unfortunately, the annoying factor $(k+1)^{(k+1)}$ in the expression $w(\varepsilon)$ makes the comparison of $w(\varepsilon)$ and $w_{\mathbb{S}^n}(\varepsilon) = \mathrm{vol}(\mathbb{S}^{n-k} + \varepsilon)/\,\mathrm{vol}(\mathbb{S}^n)$ uninteresting. But, we believe in the following conjecture.

CONJECTURE 8.1. *For every $\varepsilon > 0$, k and n,*

$$w_{op}(\varepsilon) \geqslant w_{\mathbb{S}^n}(\varepsilon) = \frac{\mathrm{vol}(\mathbb{S}^{n-k} + \varepsilon)}{\mathrm{vol}(\mathbb{S}^n)}.$$

9. Comparison with Gromov–Milman's inequality

We want to compare the result of Theorem 1 for $k = 1$ with Gromov–Milman's isoperimetric-type inequality, which we recall here. This inequality was proved first by Gromov–Millman in [GM87]. The proof was completed later on by Alesker in [Ale99] (S. Sodin had the kindness of referring Alesker's paper to the author). There is a very short and easy proof given by Arias-de-Reyna, Ball and Villa in [ABV98].

THEOREM 13. *Let $S(X)$ be a uniformly convex unit sphere with modulus δ. For every Borel set $A \subset S(X)$ such that $\mu(A) \geqslant \frac{1}{2}$ and for every $\varepsilon > 0$, we have*

$$\mu(A + \varepsilon) \geqslant 1 - e^{-a(\varepsilon)n},$$

where $a(\varepsilon) = \delta(\varepsilon/8 - \theta_n)$ and where θ_n is such that $\delta(\theta_n) = 1 - (1/2)^{1/(n-1)}$.

Our Theorem 1, in the case $k = 1$, recovers a version of the above inequality.

We need the following proposition, which relates isoperimetry and 1-waist.

PROPOSITION 14. 1-*waist $\Rightarrow$ isoperimetry: for every open subset $A \subset S(X)$ and for all $\varepsilon > 0$, we have*

$$\max\{\mu(A + \varepsilon), \mu(A^c + \varepsilon)\} \geqslant w(\varepsilon).$$

For the proof, see [Mem10a], where we proved this proposition in a more general context.

Proposition 14 is far from optimal for small ε and fixed n. To see this, compare $w(\varepsilon)$ of our main theorem with the right-hand side of the inequality of Theorem 13 when $\varepsilon \to 0$. One can see that when $\varepsilon \to 0$, $w(\varepsilon) \to 0$ but $\lim_{\varepsilon \to 0}(1 - e^{-a(\varepsilon)n}) = 1 - e^{n\delta(\theta_n)} > 1/2$ (since $1 - (1/2)^{1/(n-1)} \approx \log(2)/(n-1)$). On the other hand, let ε be fixed and let $n \to \infty$. The complicated expression of our main theorem in this particular case simplifies as

$$w(\varepsilon) = \frac{1}{1 + ((2\pi - 4\psi_2(\varepsilon))/\psi_1(\varepsilon))(1 - 2\delta(\varepsilon/2))^{n-1}}.$$

In this regime, our main Theorem 1 combined with Proposition 14 yields

$$\max\{\mu(A + \varepsilon), \mu(A^c + \varepsilon)\} \geqslant 1 - e^{-b(\varepsilon)n - c(\varepsilon)},$$

where $b(\varepsilon) = 2\delta(\varepsilon/2)$ and $c(\varepsilon)$ has an ugly expression. Since $b(\varepsilon) = 2\delta(\varepsilon/2) > \delta((\varepsilon/8) - \theta_n) = a(\varepsilon)$, our Theorem 1 gives a better estimate.

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