

# ESTIMATES FOR $F$ -JUMPING NUMBERS AND BOUNDS FOR HARTSHORNE–SPEISER–LYUBEZNIK NUMBERS

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**Abstract.** Given an ideal  $\mathfrak{a}$  on a smooth variety in characteristic zero, we estimate the  $F$ -jumping numbers of the reductions of  $\mathfrak{a}$  to positive characteristic in terms of the jumping numbers of  $\mathfrak{a}$  and the characteristic. We apply one of our estimates to bound the Hartshorne–Speiser–Lyubeznik invariant for the reduction to positive characteristic of a hypersurface singularity.

## §1. Introduction

Let  $\mathfrak{a}$  be a nonzero ideal on a smooth, irreducible variety  $X$  over an algebraically closed field  $k$ . A fundamental invariant of the singularities of the subscheme defined by  $\mathfrak{a}$  is the *log canonical threshold*  $\text{lct}(\mathfrak{a})$ . This can be defined in terms of either divisorial valuations or, when working over the complex numbers, integrability conditions. On the other hand, by considering models for  $X$  and  $\mathfrak{a}$  over a  $\mathbf{Z}$ -algebra of finite type  $A \subset k$ , one can take reductions  $X_s$  and  $\mathfrak{a}_s$  to positive characteristic for all closed points  $s \in \text{Spec } A$ . Using the Frobenius morphism, Takagi and Watanabe [TW] defined an analogue of the log canonical threshold in this setting, the  *$F$ -pure threshold*  $\text{fpt}(\mathfrak{a}_s)$ . A problem that has attracted a lot of interest is the relation between  $\text{lct}(\mathfrak{a})$  and  $\text{fpt}(\mathfrak{a}_s)$ .

It follows from the work of Hara and Yoshida [HY] that after possibly replacing  $A$  by a localization  $A_a$ , we may assume that  $\text{lct}(\mathfrak{a}) \geq \text{fpt}(\mathfrak{a}_s)$  for all closed points  $s \in \text{Spec } A$ . Moreover, for every  $\varepsilon > 0$ , there is an open subset  $U_\varepsilon \subseteq \text{Spec } A$  such that  $\text{fpt}(\mathfrak{a}_s) > \text{lct}(\mathfrak{a}) - \varepsilon$  for all  $s \in U_\varepsilon$ . One can see that even in very simple examples, one cannot take  $U_\varepsilon$  to be independent of  $\varepsilon$ .

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On the other hand, it is expected that there is a Zariski dense set of closed points  $s \in \text{Spec } A$  such that  $\text{lct}(\mathfrak{a}) = \text{fpt}(\mathfrak{a}_s)$  (see [MS]). As a consequence of our main results, we give an effective estimate for the difference between the log canonical threshold of  $\mathfrak{a}$  and the  $F$ -pure threshold of  $\mathfrak{a}_s$  (see Corollaries 3.5 and 4.5 below).

**THEOREM A.** *With the above notation, after possibly replacing  $A$  by a localization  $A_a$ , the following hold.*

- (i) *There is  $C > 0$  such that  $\text{lct}(\mathfrak{a}) - \text{fpt}(\mathfrak{a}_s) \leq C/(\text{char}(k(s)))$  for every closed point  $s \in \text{Spec } A$ .*
- (ii) *Assuming that  $\mathfrak{a}$  is locally principal, there is a positive integer  $N$  such that*

$$\text{lct}(\mathfrak{a}) - \text{fpt}(\mathfrak{a}_s) \geq \frac{1}{\text{char}(k(s))^N}$$

*for every closed point  $s \in \text{Spec } A$  for which  $\text{fpt}(\mathfrak{a}_s) \neq \text{lct}(\mathfrak{a})$ .*

In fact, we prove similar estimates for the higher jumping numbers, that we now describe. Recall that if  $\mathfrak{a}$  and  $X$  are as above, then one associates to  $\mathfrak{a}$  and to every  $\lambda \in \mathbf{R}_{\geq 0}$  the multiplier ideal  $\mathcal{J}(\mathfrak{a}^\lambda)$  of  $\mathcal{O}_X$ . These ideals have found a lot of applications in the study of higher-dimensional varieties, because they measure the singularities of the subscheme defined by  $\mathfrak{a}$  in a way that is relevant to vanishing theorems (see [Laz, Section 9]). The multiplier ideals can be defined using either divisorial valuations or integrability conditions (when we work over  $\mathbf{C}$ ). All multiplier ideals can be computed from a log resolution of  $\mathfrak{a}$ , and this description immediately implies that there is an unbounded sequence of positive rational numbers  $\lambda_1 < \lambda_2 < \dots$  such that

$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^{\lambda_i}) \supsetneq \mathcal{J}(\mathfrak{a}^{\lambda_{i+1}}) \quad \text{for all } i \geq 0 \text{ and all } \lambda \in [\lambda_i, \lambda_{i+1})$$

(with the convention  $\lambda_0 = 0$ ). The rational numbers  $\lambda_i$ , with  $i \geq 1$ , are the *jumping numbers* of  $\mathfrak{a}$ . The smallest such number  $\lambda_1$  can be described as the smallest  $\lambda$  such that  $\mathcal{J}(\mathfrak{a}^\lambda) \neq \mathcal{O}_X$ ; this is the log canonical threshold  $\text{lct}(\mathfrak{a})$ .

Suppose now that we choose models of  $X$  and  $\mathfrak{a}$  over  $A$  as before, and we consider the reduction  $\mathfrak{a}_s$ , where  $s \in \text{Spec } A$  is a closed point. Hara and Yoshida [HY] introduced the (generalized) *test ideals*  $\tau(\mathfrak{a}_s^\lambda)$ . While giving an analogue of multiplier ideals in the positive characteristic setting, they are defined by very different methods. (The original definition in [HY] involves a generalization of the theory of tight closure, due to Hochster and Huneke.)

One can show that in this case, too, there is an unbounded, strictly increasing sequence of positive rational numbers  $\alpha_i = \alpha_i(s)$  for  $i \geq 0$ , with  $\alpha_0 = 0$ , such that

$$\tau(\mathfrak{a}^\lambda) = \tau(\mathfrak{a}^{\alpha_i}) \supsetneq \tau(\mathfrak{a}^{\alpha_{i+1}}) \quad \text{for all } i \geq 0 \text{ and all } \lambda \in [\alpha_i, \alpha_{i+1}).$$

The rational numbers  $\alpha_i$ , with  $i \geq 1$ , are the  $F$ -jumping numbers of  $\mathfrak{a}$ . The smallest  $F$ -jumping number  $\alpha_1$  can be described as the smallest  $\lambda$  such that  $\tau(\mathfrak{a}_s^\lambda) \neq \mathcal{O}_{X_s}$ ; this is the  $F$ -pure threshold  $\text{fpt}(\mathfrak{a}_s)$ . We mention that, unlike in the case of multiplier ideals, both the rationality of the  $\alpha_i$  and the fact that they are unbounded are nontrivial (see [BMS1]).

The comparison between  $\text{lct}(\mathfrak{a})$  and  $\text{fpt}(\mathfrak{a}_s)$  comes from a relation between the multiplier ideals of  $\mathfrak{a}$  and the test ideals of  $\mathfrak{a}_s$ , proved in [HY]. This says that after possibly replacing  $A$  by a localization  $A_a$ , we may assume that

$$\tau(\mathfrak{a}_s^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)_s$$

for all closed points  $s \in \text{Spec } A$ . Furthermore, given any  $\lambda \in \mathbf{R}_{\geq 0}$ , there is an open subset  $V_\lambda \subseteq \text{Spec } A$  such that  $\tau(\mathfrak{a}_s^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)_s$  for every closed point  $s \in V_\lambda$ . This set, in general, depends on  $\lambda$ . On the other hand, it is expected that there is a Zariski dense set of closed points  $s \in \text{Spec } A$  such that  $\tau(\mathfrak{a}_s^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)_s$  for every  $\lambda$  (see [MS]). We can now state our main results concerning jumping numbers (see Theorem 3.3, Theorem 4.1, and Corollary 4.4 below).

**THEOREM B.** *With the above notation, given  $\lambda \in \mathbf{Q}_{>0}$ , after possibly replacing  $A$  by a localization  $A_a$ , the following hold.*

- (i) *There is  $C > 0$  such that for every closed point  $s \in \text{Spec } A$  with  $\text{char}(k(s)) = p_s$ , we have*

$$\mathcal{J}(\mathfrak{a}^{\lambda-C/p_s})_s = \tau(\mathfrak{a}_s^{\lambda-C/p_s}).$$

- In particular, if  $\lambda$  is a jumping number of  $\mathfrak{a}$  and if  $\lambda'$  is the largest jumping number smaller than  $\lambda$  (with the convention  $\lambda' = 0$  if  $\lambda = \text{lct}(\mathfrak{a})$ ), then we may assume that for every  $s$  as above, there is an  $F$ -jumping number  $\mu \in (\lambda', \lambda]$  for  $\mathfrak{a}_s$ , and for every such  $\mu$ , we have  $\lambda - \mu \leq C/p_s$ .*
- (ii) *Assuming that  $\mathfrak{a}$  is locally principal, there is a positive integer  $N$  such that for every  $F$ -jumping number  $\mu < \lambda$  of  $\mathfrak{a}_s$ , we have  $\lambda - \mu \geq 1/p_s^N$ .*

We deduce assertion (ii) from the description of test ideals in [BMS1] and an observation from [BMS2]. The more involved assertion (i) follows using the methods introduced by Hara and Yoshida [HY]. The statements in Theorem A then follow by applying Theorem B to  $\lambda = \text{lct}(\mathfrak{a})$ .

We apply assertion (ii) in Theorem B to Hartshorne–Speiser–Lyubeznik (HSL) numbers, as follows. Recall that given a Noetherian local ring  $(S, \mathfrak{n})$  of characteristic  $p > 0$ , a  $p$ -linear structure on an  $S$ -module  $M$  is an additive map  $\varphi: M \rightarrow M$  such that  $\varphi(az) = a^p\varphi(z)$  for all  $a \in S$  and  $z \in M$ . If  $M$  is Artinian, then by a theorem due to Hartshorne and Speiser [HS] and Lyubeznik [Lyu], the nondecreasing sequence of  $S$ -submodules

$$N_i := \{z \in M \mid \varphi^i(z) = 0\} \subseteq M$$

is eventually stationary. The HSL number of  $(M, \varphi)$  is the smallest  $\ell$  such that  $N_\ell = N_{\ell+j}$  for all  $j \geq 1$ .

We are interested in the case when  $S = R/(f)$ , for a regular local ring  $R$  of positive characteristic and a nonzero noninvertible  $f \in R$ . Let  $d = \dim(S)$ . In this case, the injective hull  $E_S$  of  $S/\mathfrak{n}$  over  $S$  can be identified with the top cohomology module  $H_{\mathfrak{n}}^d(S)$  and therefore carries a canonical  $p$ -linear structure  $\Theta$  induced by functoriality from the Frobenius action on  $S$ . If we are in a setting where the test ideals of  $R$  are defined (e.g., when  $R$  is essentially of finite type over a perfect field), then the HSL number of  $(E_S, \Theta)$  is equal to the smallest positive integer  $\ell$  such that

$$\tau(f^{1-1/p^\ell}) = \tau(f^{1-1/(p^{\ell+j})})$$

for every  $j \geq 1$ . If we are in the setting of Theorem B(ii), we obtain the following (see Theorem 5.9 below).

**THEOREM C.** *If  $X, \mathfrak{a}$ , and  $A$  are as in Theorem B, with  $\mathfrak{a}$  locally principal, and if  $Z$  is the subscheme defined by  $\mathfrak{a}$ , then there is a positive integer  $N$  such that for every closed point  $s \in \text{Spec } A$  and every point in the fiber  $Z_s$  of  $Z$  over  $s$ , the HSL number of  $(E_{\mathcal{O}_{Z_s, x}}, \Theta)$  is bounded above by  $N$ .*

We also give an example to illustrate that in the above theorem, even after possibly replacing  $A$  by a localization, we cannot take  $N = 1$ . (This gives a negative answer to a question of M. Katzman.)

The paper is structured as follows. In Section 2 we recall the definitions of multiplier ideals and test ideals, as well as the framework for reducing from characteristic zero to positive characteristic. In Section 3 we explain

how to get upper bounds for the jumping numbers of an ideal in positive characteristic. In particular, we prove Theorems A(ii) and B(ii). In Section 4 we describe how to use the methods from [HY] to get lower bounds for the  $F$ -jumping numbers of the reductions to positive characteristic of an ideal defined in characteristic zero. This gives Theorems A(i) and B(i). In Section 5 we discuss the HSL numbers and their connection with  $F$ -jumping numbers. The last section contains some examples.

## §2. Review of multiplier ideals and test ideals

In this section we recall the definitions of multiplier ideals and test ideals, and we review the results connecting these ideals via reduction mod  $p$ . For simplicity, we consider only the case of smooth ambient algebraic varieties.<sup>†</sup> We start by discussing the multiplier ideals in characteristic zero.

### 2.1. Multiplier ideals

Let  $X$  be a smooth scheme of finite type over an algebraically closed field  $k$  of characteristic zero. Suppose that  $\mathfrak{a}$  is an ideal<sup>‡</sup> of  $\mathcal{O}_X$  which is everywhere nonzero. (In other words, its restriction to every connected component of  $X$  is nonzero.) A *log resolution* of  $\mathfrak{a}$  is a projective birational morphism  $\pi: Y \rightarrow X$ , with  $Y$  smooth, such that  $\mathfrak{a} \cdot \mathcal{O}_Y$  is the ideal defining a divisor  $D$  on  $Y$  and such that  $D + K_{Y/X}$  is a simple normal crossing divisor. Here  $K_{Y/X}$  is the relative canonical divisor, an effective divisor supported on the exceptional locus of  $\pi$ , such that  $\mathcal{O}_Y(K_{Y/X}) \simeq \omega_Y \otimes \pi^*(\omega_X)^{-1}$ . (Recall that for a smooth scheme  $W$  over  $k$ , one denotes by  $\omega_W$  the line bundle of top differential forms on  $W$ .)

Given such a log resolution, we define for every  $\lambda \in \mathbf{R}_{\geq 0}$  the *multiplier ideal* of  $\mathfrak{a}$  of exponent  $\lambda$  by

$$\mathcal{J}(\mathfrak{a}^\lambda) := \pi_* \mathcal{O}_Y(K_{Y/X} - \lfloor \lambda D \rfloor).$$

Here, for a divisor with real coefficients  $F = \sum_i \alpha_i F_i$ , we put  $\lfloor F \rfloor := \sum_i \lfloor \alpha_i \rfloor F_i$ , where  $\lfloor u \rfloor$  denotes the largest integer at most  $u$ . Note that since  $K_{Y/X}$  is effective and supported on the exceptional locus, we have  $\pi_* \mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$ ; hence,  $\mathcal{J}(\mathfrak{a}^\lambda)$  is indeed an ideal of  $\mathcal{O}_X$ .

We now review some basic properties of multiplier ideals. For proofs of these facts and for a detailed introduction to this topic, see [Laz, Section 9].

<sup>†</sup>While the results in Section 4 work more generally, for the upper bounds in Section 3 we will need to restrict anyway to smooth ambient varieties.

<sup>‡</sup>All ideal sheaves are assumed to be coherent.

One can show that the definition is independent of the choice of log resolution. A few things are straightforward from the definition. If  $\lambda < \mu$ , then  $\mathcal{J}(\mathfrak{a}^\mu) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)$ . Furthermore, for every  $\lambda \in \mathbf{R}_{\geq 0}$ , there is  $\varepsilon > 0$  such that  $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^\mu)$  for every  $\mu$  with  $\lambda \leq \mu \leq \lambda + \varepsilon$ . One says that  $\lambda > 0$  is a *jumping number* of  $\mathfrak{a}$  if  $\mathcal{J}(\mathfrak{a}^\lambda)$  is strictly contained in  $\mathcal{J}(\mathfrak{a}^\mu)$  for every  $\mu < \lambda$ .

It follows from the definition that if  $D = \sum_{i=1}^r a_i D_i$ , then for every jumping number  $\lambda$  of  $\mathfrak{a}$ , there is  $i$  such that  $\lambda a_i$  is an integer. In particular, we see that the set of jumping numbers of  $\mathfrak{a}$  is a discrete set of rational numbers. As we have mentioned,  $\mathcal{J}(\mathfrak{a}^0) = \mathcal{O}_X$ . The smallest jumping number is thus the smallest  $\lambda$  such that  $\mathcal{J}(\mathfrak{a}^\lambda) \neq \mathcal{O}_X$ . This is the *log canonical threshold*  $\text{lct}(\mathfrak{a})$  of  $\mathfrak{a}$ .

A result due to Ein and Lazarsfeld allows us to reduce studying arbitrary multiplier ideals to those for which the exponent is less than the dimension of  $X$ . This is Skoda’s theorem (see [Laz, Section 11.1.A]), saying that if  $\mathfrak{a}$  is locally generated by  $r$  elements, then

$$\mathcal{J}(\mathfrak{a}^\lambda) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1}) \quad \text{for } \lambda \geq r.$$

In particular, one can take  $r = \dim(X)$ .

**2.2. Test ideals**

The (generalized) test ideals have been introduced by Hara and Yoshida [HY] using a generalization of tight closure theory. Since we will work only on regular schemes, it is more convenient to use the alternative definition from [BMS1], that we now present. For proofs and more details, we refer to [BMS1].

Let  $X$  be a regular scheme of positive characteristic  $p$ . We denote by  $F: X \rightarrow X$  the absolute Frobenius morphism, which is the identity on the topological space and is given by the  $p$ th power map on the sections of  $\mathcal{O}_X$ . We assume that  $X$  is *F-finite*, that is, that  $F$  is a finite map. (Note that  $F$  is also flat since  $R$  is regular.) This is satisfied, for example, if  $X$  is a scheme of finite type over a perfect field, a local ring of such a scheme, or the completion of such a ring.

Suppose first, for simplicity, that  $X = \text{Spec } R$ , where  $R$  is a regular  $F$ -finite domain. For an ideal  $J$  in  $R$  and for  $e \geq 1$ , one denotes by  $J^{[p^e]}$  the ideal  $(h^{p^e} \mid h \in J)$ . Using the fact that  $F$  is finite and flat, one shows that, given any ideal  $\mathfrak{b}$  in  $R$ , there is a unique smallest ideal  $J$  such that  $\mathfrak{b} \subseteq J^{[p^e]}$ . We denote this  $J$  by  $\mathfrak{b}^{[1/p^e]}$ .

Suppose now that  $\mathfrak{a}$  is a nonzero ideal in  $R$  and that  $\lambda \in \mathbf{R}_{\geq 0}$ . It is not hard to see that for every  $e \geq 1$ , we have an inclusion

$$(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (\mathfrak{a}^{\lceil \lambda p^{e+1} \rceil})^{[1/p^{e+1}]}.$$

It follows from the Noetherian property that for  $e \gg 0$ , the ideal  $(\mathfrak{a}^{\lceil \lambda p^e \rceil})^{[1/p^e]}$  is independent of  $e$ . This is the (generalized) *test ideal*  $\tau(\mathfrak{a}^\lambda)$  of  $\mathfrak{a}$  of exponent  $\lambda$ . One can show that the construction of test ideals commutes with localization and completion. In particular, we can extend the above definition to the general case when  $X$  is a regular  $F$ -finite scheme of positive characteristic and  $\mathfrak{a}$  is an everywhere nonzero ideal; the test ideals  $\tau(\mathfrak{a}^\lambda)$  are coherent ideals of  $\mathcal{O}_X$ .

The formal properties that we discussed for multiplier ideals also hold in this setting. If  $\lambda < \mu$ , then  $\tau(\mathfrak{a}^\mu) \subseteq \tau(\mathfrak{a}^\lambda)$ . With a little effort (see [BMS1, Proposition 2.14]), one shows that for every  $\lambda \in \mathbf{R}_{\geq 0}$  there is  $\varepsilon > 0$  such that  $\tau(\mathfrak{a}^\lambda) = \tau(\mathfrak{a}^\mu)$  for every  $\mu$  with  $\lambda \leq \mu \leq \lambda + \varepsilon$ . A positive  $\lambda$  is an  *$F$ -jumping number* of  $\mathfrak{a}$  if  $\tau(\mathfrak{a}^\lambda) \neq \tau(\mathfrak{a}^\mu)$  for every  $\mu < \lambda$ . The set of  $F$ -jumping numbers of  $\mathfrak{a}$  is known to be a discrete set of rational numbers when  $X$  is essentially of finite type over a field (see [BMS1, Theorem 3.1]) or when  $\mathfrak{a}$  is locally principal (see [BMS2, Theorem 1.1]). Note, however, that this assertion is considerably more subtle than the corresponding one in characteristic zero. One property that is special to characteristic  $p$  says that if  $\lambda$  is an  $F$ -jumping number, then  $p\lambda$  is an  $F$ -jumping number, too. It follows from the definition that  $\tau(\mathfrak{a}^0) = \mathcal{O}_X$ ; hence, the first  $F$ -jumping number is the smallest  $\lambda$  such that  $\tau(\mathfrak{a}^\lambda) \neq \mathcal{O}_X$ . This is the  *$F$ -pure threshold*  $\text{fpt}(\mathfrak{a})$ . We note that if  $X = U_1 \cup \cdots \cup U_m$  is an open cover, then  $\lambda$  is an  $F$ -jumping number of  $\mathfrak{a}$  if and only if it is a jumping number of one of the restrictions  $\mathfrak{a}|_{U_i}$ .

There is a version of Skoda's theorem also in this setting, and this is in fact more elementary than in the case of multiplier ideals. (For a proof involving the above definition, see [BMS1, Proposition 2.25].) This says that if  $\mathfrak{a}$  is locally generated by  $r$  elements, then

$$\tau(\mathfrak{a}^\lambda) = \mathfrak{a} \cdot \tau(\mathfrak{a}^{\lambda-1}) \quad \text{for } \lambda \geq r.$$

In particular, one can always take  $r = \dim(X)$ .

We end by mentioning a formula for computing ideals of the form  $\mathfrak{b}^{[1/p^e]}$ , which we will use to compute examples in Section 6. Suppose that  $X = \text{Spec } R$ , where  $R$  is a regular domain of characteristic  $p > 0$ , such that  $R$

has a basis over  $R^{p^e}$  given by  $u_1, \dots, u_r$ . If the ideal  $\mathfrak{b}$  in  $R$  is generated by  $h_1, \dots, h_m$ , and if we write  $h_i = \sum_{j=1}^r a_{i,j}^{p^e} u_j$ , then

$$(1) \quad \mathfrak{b}^{[1/p^e]} = (a_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq r).$$

(For a proof, see [BMS1, Proposition 2.5].) We will apply this when  $R = k[[x_1, \dots, x_n]]$ , with  $k$  a perfect field, when we may consider the basis given by all monomials  $x_1^{i_1} \cdots x_n^{i_n}$  with  $0 \leq i_1, \dots, i_n \leq p^e - 1$ .

**2.3. Relating multiplier ideals and test ideals via reduction mod  $p$**

We start by briefly discussing the framework for reduction to positive characteristic. (For details, we refer to [MS, Section 2.2].) Suppose that  $Y$  is a scheme of finite type over an algebraically closed field  $k$  of characteristic zero. We can find a finitely generated  $\mathbf{Z}$ -subalgebra  $A \subseteq k$ , a scheme  $Y_A$  of finite type over  $A$  (a *model* for  $Y$ ), and we can find an isomorphism  $\varphi_A: Y_A \times_{\text{Spec } A} \text{Spec } k \rightarrow Y$ . If we choose a different  $B \subseteq k$ , a corresponding scheme  $Y_B$  over  $B$ , and an isomorphism  $\varphi_B: Y_B \times_{\text{Spec } B} \text{Spec } k \rightarrow Y$ , then we can find a finitely generated  $\mathbf{Z}$ -subalgebra  $C \subseteq k$  containing both  $A$  and  $B$  and an isomorphism  $\psi: Y_A \times_{\text{Spec } A} \text{Spec } C \rightarrow Y_B \times_{\text{Spec } B} \text{Spec } C$  such that we have  $\psi \times_{\text{Spec } C} \text{Spec } k = \varphi_B^{-1} \circ \varphi_A$ .

Given  $A$  and  $Y_A$  as above, we consider closed points  $s \in \text{Spec } A$ . Note that the residue field  $k(s)$  of  $s$  is finite. We denote by  $Y_s$  the fiber of  $Y_A$  over  $s$ .

We always choose  $A$  and  $Y_A$  as above, but all properties that we will discuss refer to closed points in some open subset of  $\text{Spec } A$ ; in particular, they are independent of the choice of  $A$  and  $Y_A$ . In light of this, we allow replacing  $A$  by some localization  $A_a$ , with  $a \in A$  nonzero, and  $Y_A$  by  $Y_A \times_{\text{Spec } A} \text{Spec } A_a$ . For example, after possibly replacing  $A$  by  $A_a$ , we may assume that  $Y_A$  is flat over  $A$ . Furthermore, if  $Y$  is smooth (and irreducible), then we may assume that for every closed point  $s \in \text{Spec } A$ , the fiber  $Y_s$  is smooth (and irreducible, of the same dimension as  $Y$ ).

Given a coherent sheaf  $\mathcal{F}$  on  $Y$ , we may choose  $A$  and a model  $Y_A$  such that there is a sheaf  $\mathcal{F}_A$  on  $Y_A$  (a *model* for  $\mathcal{F}$ ) whose pullback to  $Y$  is isomorphic to  $\mathcal{F}$ . In this case, we denote by  $\mathcal{F}_s$  the restriction of  $\mathcal{F}_A$  to  $Y_s$ . Furthermore, given a morphism of coherent sheaves  $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ , we may choose models  $Y_A, \mathcal{F}_A$ , and  $\mathcal{G}_A$  such that there is a morphism of sheaves  $\alpha_A: \mathcal{F}_A \rightarrow \mathcal{G}_A$  inducing  $\alpha$ . In particular, we may consider  $\alpha_s: \mathcal{F}_s \rightarrow \mathcal{G}_s$  for every closed point  $s \in \text{Spec } A$ . Given an exact sequence of sheaves  $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$ , we may assume, after replacing  $A$  by a suitable localization  $A_a$ ,



that the sequences  $\mathcal{F}'_s \rightarrow \mathcal{F}_s \rightarrow \mathcal{F}''_s$  are exact for all closed points  $s \in \text{Spec } A$ . In particular, if  $\mathcal{F}$  is an ideal in  $\mathcal{O}_Y$ , then we may assume that each  $\mathcal{F}_s$  is an ideal in  $\mathcal{O}_{Y_s}$ .

Given a morphism of schemes  $\pi: Y \rightarrow X$  of finite type over  $k$ , we may choose  $A$  and the models  $Y_A$  and  $X_A$  such that there is a morphism  $\pi_A: Y_A \rightarrow X_A$  of schemes over  $A$  inducing  $\pi$ . In this case, we obtain morphisms  $\pi_s: Y_s \rightarrow X_s$  for all closed points  $s \in \text{Spec } A$ . Furthermore, if  $\pi$  is projective (or birational, finite, open, or closed immersion), we may assume that each  $\pi_s$  has the same property. Given, in addition, a coherent sheaf  $\mathcal{F}$  on  $Y$ , we may assume, after restricting to a suitable open subset of  $\text{Spec } A$ , that for all  $s$  we have canonical isomorphisms

$$(2) \quad R^i \pi_* (\mathcal{F})_s \simeq R^i (\pi_s)_* (\mathcal{F}_s).$$

We now describe the setting that we will be interested in. Suppose that  $\mathfrak{a}$  is an everywhere nonzero ideal on the smooth scheme  $X$  over  $k$ . We fix a log resolution  $\pi: Y \rightarrow X$  of  $\mathfrak{a}$ , and we write  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$  and

$$D = \sum_{i=1}^N a_i E_i, \quad K_{Y/X} = \sum_{i=1}^N k_i E_i.$$

We choose  $A$  and models  $\pi_A, X_A, Y_A, D_A, (E_i)_A$ , and  $\mathfrak{a}_A$  such that for every closed point  $s \in \text{Spec } A$ , the induced map  $\pi_s: Y_s \rightarrow X_s$  is a log resolution of  $\mathfrak{a}_s$ , and we have  $\mathfrak{a}_s \cdot \mathcal{O}_{Y_s} = \mathcal{O}_{Y_s}(-D_s)$  and

$$D_s = \sum_{i=1}^N a_i (E_i)_s, \quad K_{Y_s/X_s} = \sum_{i=1}^N k_i (E_i)_s.$$

Moreover, given  $\lambda \in \mathbf{R}_{\geq 0}$ , we may consider  $\mathcal{J}(\mathfrak{a}^\lambda)_s$ , and it follows from (2) that we may assume that

$$(3) \quad \mathcal{J}(\mathfrak{a}^\lambda)_s = (\pi_s)_* \mathcal{O}_{Y_s}(K_{Y_s/X_s} - \lfloor \lambda D_s \rfloor).$$

If we consider all  $\lambda$  in some bounded interval, we have the above formula for all such  $\lambda$ , because we need only to consider finitely many ideals. (It is enough to consider only those  $\lambda$  such that  $\lambda a_i$  is an integer for some  $i$ .) If we want to consider all multiplier ideals  $\mathcal{J}(\mathfrak{a}^\lambda)$  and their reductions to prime characteristic, we simply decree, motivated by Skoda’s theorem, that  $\mathcal{J}(\mathfrak{a}^\lambda)_s = \mathfrak{a}_s \cdot \mathcal{J}(\mathfrak{a}^{\lambda-1})_s$  for  $\lambda \geq \dim(X_s)$ ; this reduces us to having to

define only  $\mathcal{J}(\mathfrak{a}^\lambda)_s$  for  $\lambda < \dim(X)$ . In what follows, we refer to all the above choices simply as a *model for the multiplier ideals of  $\mathfrak{a}$* .

We can now formulate the two main results due to Hara and Yoshida concerning the connection between the reductions of multiplier ideals and the corresponding test ideals (see [HY, Theorem 3.4]). We assume that  $X$  is a smooth scheme over  $k$  and that  $\mathfrak{a}$  is an everywhere nonzero ideal on  $X$ ; furthermore, we choose models for the multiplier ideals of  $\mathfrak{a}$  over some finitely generated  $\mathbf{Z}$ -algebra  $A \subset k$ .

**THEOREM 2.1.** *With the above notation, after possibly replacing  $A$  by a localization  $A_a$ , we have*

$$\tau(\mathfrak{a}_s^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)_s$$

for all closed points  $s \in \text{Spec } A$  and all  $\lambda \in \mathbf{R}_{\geq 0}$ .

Note that even if we start with models for  $X$  and  $\mathfrak{a}$ , in order to apply Theorem 2.1, we might need to change  $A$ . Indeed, we need to guarantee that some log resolution of  $\mathfrak{a}$  is defined over  $A$  and that (3) holds for  $\lambda < \dim(X)$ .

**THEOREM 2.2.** *With the above notation, given any  $\lambda \in \mathbf{R}_{\geq 0}$ , there is an open subset  $V_\lambda \subseteq \text{Spec } A$  such that*

$$\tau(\mathfrak{a}_s^\lambda) = \mathcal{J}(\mathfrak{a}^\lambda)_s$$

for all closed points  $s \in V_\lambda$ .

It is definitely not the case that  $V_\lambda$  can be taken independently of  $\lambda$ . However, it is expected that there is a dense set of closed points in  $s \in \text{Spec } A$  such that the equality in Theorem 2.2 holds for all  $\lambda \in \mathbf{R}_{\geq 0}$  (see [MS]).

The proof of Theorem 2.1 is elementary. For a proof in our simplified setting, and with our definitions, see [BHMM, Proposition 4.3]. The proof of Theorem 2.2 is deeper and makes use of the action of the Frobenius on the de Rham complex. We will make use of the main ingredient in this proof in Section 4, in order to give lower bounds for the  $F$ -jumping numbers of the reductions of  $\mathfrak{a}$  to positive characteristic.

In particular, Theorems 2.1 and 2.2 give the following relation between the log canonical threshold of  $\mathfrak{a}$  and the  $F$ -pure thresholds of the reductions  $\mathfrak{a}_s$  to positive characteristic. If  $A$  is chosen as in Theorem 2.2, then the theorem implies that  $\text{lct}(\mathfrak{a}) \geq \text{fpt}(\mathfrak{a}_s)$  for all closed points  $s \in \text{Spec } A$ . On the other hand, Theorem 2.2 implies that for every  $\varepsilon > 0$ , there is an open

subset  $U_\varepsilon \subseteq \text{Spec } A$  such that  $\text{lct}(\mathfrak{a}) - \text{fpt}(\mathfrak{a}_s) < \varepsilon$  for every closed point  $s \in U_\varepsilon$ .

The above results raise the following problem that we will consider in Sections 3 and 4. Suppose that  $\lambda$  is a jumping number of  $\mathfrak{a}$ , and denote by  $\lambda'$  the largest jumping number  $< \lambda$ . (When  $\lambda = \text{lct}(\mathfrak{a})$ , we put  $\lambda' = 0$ .) After replacing  $A$  by some localization  $A_a$ , we may assume by Theorem 2.2 that for every closed point  $s \in \text{Spec } A$  we have

$$\mathcal{J}(\mathfrak{a}^{\lambda'})_s = \tau(\mathfrak{a}_s^{\lambda'}) \quad \text{and} \quad \mathcal{J}(\mathfrak{a}^\lambda)_s = \tau(\mathfrak{a}_s^\lambda).$$

It follows that in this case there is an  $F$ -jumping number of  $\mathfrak{a}_s$  in the interval  $(\lambda', \lambda]$ . The following problem addresses the question of how close this jumping number is from  $\lambda$ .

**PROBLEM 2.3.** With the above notation, show that there are  $C \in \mathbf{R}_{>0}$  and  $N \in \mathbf{Z}_{>0}$  such that after possibly replacing  $A$  by a localization, for every closed point in  $\text{Spec } A$  and every  $F$ -jumping number  $\mu$  of  $\mathfrak{a}_s$  in the open interval  $(\lambda', \lambda)$ , we have

$$(4) \quad \frac{1}{\text{char}(k(s))^N} \leq \lambda - \mu \leq \frac{C}{\text{char}(k(s))}.$$

In Section 3 we prove the existence of  $N$  in Problem 2.3 when  $\mathfrak{a}$  is a locally principal ideal, while in Section 4 we show how to find  $C$  for arbitrary  $\mathfrak{a}$ .

### §3. Upper bounds for $F$ -jumping numbers

The key ingredient for giving an upper bound for  $F$ -jumping numbers is provided by the following result from [BMS2]. We include the proof for the sake of completeness.

**LEMMA 3.1** ([BMS2, Proposition 4.3]). *Let  $X$  be a regular  $F$ -finite scheme of characteristic  $p > 0$ , and let  $\mathfrak{a}$  be an everywhere nonzero locally principal ideal in  $\mathcal{O}_X$ . Given  $\lambda = r/(p^e - 1)$  for positive integers  $r$  and  $e$ , let us put  $\lambda_m = (1 - (1/p^{me}))\lambda$  for  $m \geq 0$ . If there is an  $F$ -jumping number of  $\mathfrak{a}$  in  $(\lambda_m, \lambda_{m+1}]$  with  $m \geq 1$ , then there is also an  $F$ -jumping number of  $\mathfrak{a}$  in  $(\lambda_{m-1}, \lambda_m]$ . In particular, in this case there are at least  $m + 1$   $F$ -jumping numbers of  $\mathfrak{a}$  in  $(0, \lambda)$ .*

*Proof.* Note that Skoda’s theorem for a locally principal ideal implies that  $\mu > 1$  is an  $F$ -jumping number if and only if  $\mu - 1$  is an  $F$ -jumping number. Furthermore, as we have mentioned, if  $\mu$  is an  $F$ -jumping number for  $\mathfrak{a}$ , then

also  $p\mu$  is an  $F$ -jumping number. It is easy to see that if  $\mu \in (\lambda_m, \lambda_{m+1}]$ , then  $p^e\mu - r \in (\lambda_{m-1}, \lambda_m]$ , and as we have seen,  $p^e\mu - r$  is an  $F$ -jumping number of  $\mathfrak{a}$ . □

**COROLLARY 3.2.** *With  $X$  and  $\mathfrak{a}$  as in Lemma 3.1, if there are at most  $d$   $F$ -jumping numbers of  $\mathfrak{a}$  that are less than  $\lambda = r/(p^e - 1)$ , then there are no  $F$ -jumping numbers of  $\mathfrak{a}$  in the open interval  $(\lambda_d, \lambda)$ .*

We can now state and prove the main result of this section. Suppose that  $A$  is a finitely generated  $\mathbf{Z}$ -algebra and that  $X$  is a smooth scheme of finite type over  $\text{Spec } A$ , of relative dimension  $n$ . Let  $\mathfrak{a}$  be a locally principal ideal on  $X$ , whose restriction to every fiber is everywhere nonzero.

**THEOREM 3.3.** *With the above notation, if  $\lambda \in \mathbf{Q}_{>0}$ , then there are positive integers  $N$  and  $p_0 = p_0(\lambda)$  such that, for every closed point  $s \in \text{Spec } A$  with  $\text{char}(k(s)) > p_0$  and every  $F$ -jumping number  $\mu$  of  $\mathfrak{a}_s$  that is less than  $\lambda$ , we have*

$$\lambda - \mu \geq \frac{1}{\text{char}(k(s))^N}.$$

*In particular, this holds for every closed point  $s \in \text{Spec } A_a$ , for some nonzero  $a \in A$ .*

*Proof.* After taking a suitable affine open cover of  $X$ , we may assume that  $X = \text{Spec}(R)$  is affine and that  $\mathfrak{a} = (f)$  is a principal ideal. Let us write  $\lambda = a/b$ , with  $a, b \in \mathbf{Z}_{>0}$ . We first require that  $p_0$  is such that all divisors of  $b$  are at most  $p_0$ .

**CLAIM.** *There is a positive integer  $d$  such that for every closed point  $s \in \text{Spec } A$ , there are at most  $d$   $F$ -jumping numbers of  $f_s$  that are less than  $\lambda$ .*

Indeed, since  $X$  is of finite type over  $A$ , we can write  $R \simeq A[x_1, \dots, x_m]/J$  for some ideal  $J$ , and let  $g \in A[x_1, \dots, x_m]$  be a polynomial whose class corresponds to  $f$ . It follows from [BMS1, Proposition 3.6] that  $\tau(f_s^\mu) = \tau((J_s + (g_s))^{\mu+m-n}) \cdot R_s$ . On the other hand, let  $M$  be such that  $J + (g) \subseteq A[x_1, \dots, x_m]$  is generated in degree at most  $M$ . It follows from [BMS1, Proposition 3.2] that  $\tau((J_s + (g_s))^{\mu+m-n})$  is generated in degree at most  $\lfloor M(\mu + m - n) \rfloor$  for every  $\mu$ . In particular, the number of  $F$ -jumping numbers of  $f_s$  that are less than  $\lambda$  is bounded above by the number of  $F$ -jumping numbers of  $J_s + (g_s)$  that are less than  $\lambda + m - n$ , which in turn is bounded above by the dimension of the vector space of polynomials in  $k(s)[x_1, \dots, x_m]$  of degree at most  $\lfloor M(\lambda + m - n) \rfloor$ . Note that this dimension is independent of the closed point  $s \in \text{Spec } A$ ; hence, we obtain  $d$  as in the claim.

Let us fix  $d$  as in the claim, and let  $s \in \text{Spec } A$  be a closed point. Let  $p = \text{char}(k(s))$ , and let  $e \geq 1$  be the order of  $p$  in the group of units of  $\mathbf{Z}/b\mathbf{Z}$ . (Recall that  $p$  does not divide  $b$ .) In this case we can write  $\lambda = r/(p^e - 1)$ , and it follows from Corollary 3.2 that if  $\mu < \lambda$  is an  $F$ -jumping number of  $\mathfrak{a}_s$ , then

$$\lambda - \mu \geq \lambda - \lambda_d = \frac{\lambda}{p^{de}}.$$

Note that  $e \leq b$ , and if we require also that  $p_0 \geq \lambda^{-1}$ , we see that  $N = db + 1$  satisfies the first assertion in the theorem.

The second assertion is clear, too: if we consider  $u: \text{Spec } A \rightarrow \text{Spec } \mathbf{Z}$ , it is enough to take a nonzero  $a \in A$  such that  $u(\text{Spec } A_a)$  does not contain any prime  $p\mathbf{Z}$ , with  $0 < p \leq p_0$ .  $\square$

**COROLLARY 3.4.** *Given  $\lambda \in \mathbf{Q}_{>0}$  and  $n, M \in \mathbf{Z}_{>0}$ , there are positive integers  $N = N(n, M, \lambda)$  and  $p_0 = p_0(\lambda)$  such that for every  $F$ -finite field  $k$  of characteristic  $p \geq p_0$  and every  $f \in k[x_1, \dots, x_n]$  with  $\deg(f) \leq M$ , we have*

$$\lambda - \mu \geq \frac{1}{p^N}$$

for every  $F$ -jumping number  $\mu < \lambda$  of  $f$ .

*Proof.* We could either apply Theorem 3.3 for the universal polynomial of degree at most  $M$  in  $n$  variables (with  $\text{Spec } A$  being the parameter space for such polynomials), or simply apply the argument in the proof of the theorem, noting that in this case we obtain directly the bound for the number of  $F$ -jumping numbers of  $f$  that are less than  $\lambda$ , in terms of  $n$ ,  $M$ , and  $\lambda$ .  $\square$

Suppose now that  $X$  is a smooth scheme over an algebraically closed field  $k$  and that  $\mathfrak{a}$  is an everywhere nonzero, locally principal ideal on  $X$ . Let us consider models for  $X$  and  $\mathfrak{a}$  over a finitely generated  $\mathbf{Z}$ -algebra  $A \subset k$ . By applying Theorem 3.3 to  $\lambda = \text{lct}(\mathfrak{a})$ , we obtain the following.

**COROLLARY 3.5.** *With the above notation, there is  $N$  such that, after possibly replacing  $A$  by a localization  $A_a$ , we would then have that  $\text{lct}(\mathfrak{a}) - \text{fpt}(\mathfrak{a}_s) \geq 1/(\text{char}(k(s))^N)$  for every closed point  $s \in \text{Spec } A$  such that  $\text{lct}(\mathfrak{a}) \neq \text{fpt}(\mathfrak{a}_s)$ .*

#### §4. Lower bounds for $F$ -jumping numbers

In this section we assume that  $X$  is a smooth scheme of finite type over an algebraically closed field  $k$  of characteristic zero. We consider an everywhere nonzero ideal sheaf  $\mathfrak{a}$  in  $\mathcal{O}_X$ , and we fix a model over a finitely generated

**Z**-algebra  $A \subset k$  for  $X$ ,  $\mathfrak{a}$ , and the multiplier ideals of  $\mathfrak{a}$ . Our goal in this section is to prove the following theorem.

**THEOREM 4.1.** *With the above notation, for every  $\lambda > 0$  there is  $C \in \mathbf{R}_{>0}$  such that after possibly replacing  $A$  by a localization  $A_a$ , for every closed point  $s \in \text{Spec } A$  we have*

$$(5) \quad \mathcal{J}(\mathfrak{a}^{\lambda-C/p_s})_s = \tau(\mathfrak{a}_s^{\lambda-C/p_s}),$$

where  $p_s = \text{char}(k(s))$ .

**REMARK 4.2.** Note the interesting inclusion in (5) of “ $\subseteq$ ”, since the reverse one can be guaranteed using Theorem 2.1.

**REMARK 4.3.** With the notation in the above theorem, we may assume that for every closed point  $s \in \text{Spec } A$ , we have  $\lambda - (C/p_s) \geq \lambda'$ , where  $\lambda'$  is the largest jumping number of  $\mathfrak{a}$  that is less than  $\lambda$  (with the convention that  $\lambda' = 0$  if there is no such jumping number). In this case, the ideal on the left-hand side of (5) is  $\mathcal{J}(\mathfrak{a}^{\lambda'})_s$ .

**COROLLARY 4.4.** *With the notation in Remark 4.3, if  $\lambda$  is a jumping number of  $\mathfrak{a}$ , then there is  $C > 0$  such that after possibly replacing  $A$  by a localization  $A_a$ , the following holds: for every closed point  $s \in \text{Spec } A$ , there is an  $F$ -jumping number  $\mu \in (\lambda', \lambda]$  for  $\mathfrak{a}_s$ , and for every such  $\mu$  we have*

$$\lambda - \mu \leq \frac{C}{p_s}.$$

*Proof.* It follows from Theorem 2.1 that we may assume that  $\tau(\mathfrak{a}_s^\alpha) \subseteq \mathcal{J}(\mathfrak{a}^\alpha)_s$  for every  $\alpha \in \mathbf{R}_{>0}$ . Suppose now that the conclusion of Theorem 4.1 holds. In this case, for every closed point  $s \in \text{Spec } A$  and every  $\alpha$  with  $\lambda' \leq \alpha \leq \lambda - (C/p_s)$ , we have

$$(6) \quad \mathcal{J}(\mathfrak{a}^{\lambda-C/p_s})_s = \tau(\mathfrak{a}_s^{\lambda-C/p_s}) \subseteq \tau(\mathfrak{a}_s^\alpha) \subseteq \tau(\mathfrak{a}_s^{\lambda'}) \subseteq \mathcal{J}(\mathfrak{a}^{\lambda'})_s.$$

Since  $\mathcal{J}(\mathfrak{a}^{\lambda-C/p_s})_s = \mathcal{J}(\mathfrak{a}^{\lambda'})_s$ , we conclude that all inclusions in (6) are equalities. In particular, there is no  $F$ -jumping number for  $\mathfrak{a}_s$  in the interval  $(\lambda', \lambda - (C/p_s)]$ .

On the other hand, since  $\lambda$  is a jumping number for  $\mathfrak{a}$ , we have

$$\tau(\mathfrak{a}_s^\lambda) \subseteq \mathcal{J}(\mathfrak{a}^\lambda)_s \subsetneq \mathcal{J}(\mathfrak{a}^{\lambda-C/p_s})_s = \tau(\mathfrak{a}_s^{\lambda-C/p_s}).$$

Therefore, there is an  $F$ -jumping number of  $\mathfrak{a}_s$  in the interval  $(\lambda - (C/p_s), \lambda]$ , and we thus obtain both assertions in the corollary. □

**COROLLARY 4.5.** *With the notation in Theorem 4.1, there is  $C > 0$  such that after possibly replacing  $A$  by a localization  $A_a$ , we have  $\text{lct}(\mathfrak{a}) - \text{fpt}(\mathfrak{a}_s) < (C/p_s)$  for every closed point  $s \in \text{Spec } A$ .*

*Proof.* Applying Theorem 4.1 with  $\lambda = \text{lct}(\mathfrak{a})$ , we see that we may assume that  $\tau(\mathfrak{a}_s^{\lambda - (C/p_s)}) = \mathcal{O}_{X_s}$ ; hence,  $\text{fpt}(\mathfrak{a}_s) > \lambda - (C/p_s)$ .  $\square$

**REMARK 4.6.** The assertion in Theorem 4.1 is interesting only when  $\lambda$  is a jumping number of  $\mathfrak{a}$ . Indeed, otherwise we can find  $\varepsilon > 0$  such that  $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(\mathfrak{a}^{\lambda - \varepsilon})$ . By applying Theorem 2.2, we see that we may assume that for all closed points  $s \in \text{Spec } A$  we have

$$\tau(\mathfrak{a}_s^{\lambda - \varepsilon}) = \mathcal{J}(\mathfrak{a}^{\lambda - \varepsilon})_s = \mathcal{J}(\mathfrak{a}^\lambda)_s = \tau(\mathfrak{a}_s^\lambda).$$

If we consider a localization  $A_a$  of  $A$  such that for every closed point  $s \in \text{Spec } A$  we have  $\text{char}(k(s)) > 1/\varepsilon$ , it is clear that the conclusion of the theorem holds by taking  $C = 1$ .

**REMARK 4.7.** In order to prove Theorem 4.1, we may assume that  $X$  is affine and irreducible and that  $\mathfrak{a}$  is a principal ideal. Indeed, after taking an open affine cover, we reduce to the case when  $X$  is affine and irreducible. Consider now generators  $g_1, \dots, g_m$  for the ideal  $\mathfrak{a}$ . If  $M > \lambda$  is an integer and if  $h_1, \dots, h_M$  are general linear combinations of the  $g_i$  with coefficients in  $k$ , then

$$\mathcal{J}(\mathfrak{a}^\alpha) = \mathcal{J}(h^{\alpha/M})$$

for every  $\alpha < M$ , where  $h = h_1 \cdots h_M$  (see [Laz, Proposition 9.2.28]). If the theorem holds for  $h$  and  $\lambda/M$ , then we can find  $C' > 0$  such that after replacing  $A$  by a localization,

$$\mathcal{J}(h^{(\lambda/M) - (C'/p_s)})_s = \tau(h_s^{(\lambda/M) - (C'/p_s)})$$

for every closed point  $s \in \text{Spec } A$ . Using the fact that  $h \in \mathfrak{a}^M$ , we now obtain

$$\mathcal{J}(\mathfrak{a}^{\lambda - C'M/p_s})_s = \mathcal{J}(h^{(\lambda/M) - (C'/p_s)})_s = \tau(h_s^{(\lambda/M) - (C'/p_s)}) \subseteq \tau(\mathfrak{a}_s^{\lambda - C'M/p_s});$$

hence, we obtain the inclusion “ $\subseteq$ ” in Theorem 4.1 if we take  $C = C'M$ , while the reverse inclusion is trivial (see Remark 4.2).

Before giving the proof of Theorem 4.1, we recall the criterion from [HY] that guarantees the equality of multiplier ideals and test ideals in a fixed positive characteristic. This is the heart of the proof of Theorem 2.2. We start by describing the setting.

Suppose that  $X'$  is a smooth, irreducible,  $n$ -dimensional affine scheme over a perfect field  $L$  of characteristic  $p > 0$ . We assume that we have a log resolution  $\pi': Y' \rightarrow X'$  of a nonzero principal ideal  $\mathfrak{a}'$  on  $X'$ . Let  $Z'$  be the divisor on  $Y'$  such that  $\mathfrak{a}' \cdot \mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-Z')$ . By assumption, there is a simple normal crossing divisor  $E' = E'_1 + \dots + E'_N$  such that both  $K_{Y'/X'}$  and  $Z'$  are supported on  $E'$ . Under these assumptions, we put  $\mathcal{J}(\mathfrak{a}'^\alpha) = \pi_* \mathcal{O}_{Y'}(K_{Y'/X'} - \lfloor \alpha Z' \rfloor)$ . It is shown in [HY] that in this case we have  $\tau(\mathfrak{a}'^\alpha) \subseteq \mathcal{J}(\mathfrak{a}'^\alpha)$  for every  $\alpha$ . (This is the result that implies Theorem 2.1.)

Suppose now that  $\alpha \in \mathbf{R}_{\geq 0}$  is fixed and that we choose  $\mu > \alpha$  such that  $\mathcal{J}(\mathfrak{a}'^\alpha) = \mathcal{J}(\mathfrak{a}'^\mu)$ . (Note that if  $Z' = \sum_i a_i Z'_i$ , then it is enough to take  $\mu < (\lfloor \alpha a_i \rfloor + 1 - \alpha a_i) / a_i$  for all  $i$  with  $a_i > 0$ .) Suppose also that we have a  $\mathbf{Q}$ -divisor  $D'$  on  $Y'$  supported on  $E'$  such that  $D'$  is ample over  $X'$  and  $-D'$  is effective. We put  $G' = \mu(D' - Z')$  and assume, in addition, that  $\lceil G' \rceil = \lceil -\mu Z' \rceil$ . The following is the main criterion for the equality of  $\tau(\mathfrak{a}'^\alpha)$  and  $\mathcal{J}(\mathfrak{a}'^\alpha)$ . We denote by  $\Omega_{Y'}^i(\log E')$  the sheaf of  $i$ -differential forms on  $Y'$  with log poles along  $E'$ . If  $F' = \sum_{i=1}^N \alpha_i E'_i$  is a divisor with real coefficients, then we put  $\lceil F' \rceil = \sum_{i=1}^N \lceil \alpha_i \rceil E'_i$ , where  $\lceil u \rceil$  denotes the smallest integer at least  $u$ .

PROPOSITION 4.8. *With the above notation, if the conditions*

- (A)  $H^i(Y', \Omega_{Y'}^{n-i}(\log E')(-E' + \lceil p^\ell G' \rceil)) = 0$  for all  $i \geq 1$  and  $\ell \geq 1$ ;
- (B)  $H^{i+1}(Y', \Omega_{Y'}^{n-i}(\log E')(-E' + \lceil p^\ell G' \rceil)) = 0$  for all  $i \geq 1$  and  $\ell \geq 0$

hold, then  $\tau(\mathfrak{a}'^\alpha) = \mathcal{J}(\mathfrak{a}'^\alpha)$ .

We refer to [HY] for a proof of this result. For a somewhat simplified version of the argument, using our definition of test ideals, see the presentation in [BHMM, Section 4]. We can now prove the main result of this section.

*Proof of Theorem 4.1.* It is easy to see that the assertion in the theorem is independent of the models we have chosen. The point is that if  $A \subset B$  is an inclusion of finitely generated  $\mathbf{Z}$ -algebras, and  $s \in \text{Spec } A$  is the image of the closed point  $t \in \text{Spec } B$ , then we have a finite field extension  $k(s) \subseteq k(t)$ ; if we have models of  $X$  and  $\mathfrak{a}$  over both  $A$  and  $B$  such that  $X_B \simeq X_A \times_{\text{Spec } A} \text{Spec } B$ , then it follows from the description of test ideals in [BMS1, Proposition 2.5] that  $\tau(\mathfrak{a}_t^\mu) = \tau(\mathfrak{a}_s^\mu) \otimes_{k(s)} k(t)$ .

In particular, we may change the log resolution, and we may assume that all divisors that we define in characteristic zero have models over  $A$ . We



also note that by Remark 4.6, we may assume that  $\lambda$  is a jumping number of  $\mathfrak{a}$ ; hence, it is rational. Finally, by Remark 4.7, we may and will assume that  $X$  is affine and irreducible and that  $\mathfrak{a}$  is principal.

By assumption, we have a log resolution  $\pi: Y \rightarrow X$  of  $\mathfrak{a}$  that admits a model over  $A$  such that construction of multiplier ideals commutes with taking the fiber over the closed points in  $\text{Spec } A$ . Since we may choose a convenient log resolution, we may assume that  $\pi$  is an isomorphism over  $X \setminus V(\mathfrak{a})$ . Let  $Z$  be the divisor on  $Y$  such that  $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ . By assumption,  $Z$  is supported on a simple normal crossing divisor  $E = E_1 + \dots + E_N$  on  $Y$  such that the support of  $K_{Y/X}$  is also contained in  $E$ .

We fix an integral divisor  $H$  on  $Y$  that is ample over  $X$  and such that  $-H$  is effective and supported on  $E$ . (For example, if we express  $Y$  as the blowup of  $X$  along some closed subscheme  $T$ , then we may choose  $H = -\pi^{-1}(T)$ .) Let us write

$$Z = \sum_{i=1}^N a_i E_i \quad \text{and} \quad -H = \sum_{i=1}^N h_i E_i,$$

so by assumption,  $a_i > 0$  and  $h_i \geq 0$  for all  $i$ .

LEMMA 4.9. *With the above notation, if  $\mathcal{F}$  is a coherent sheaf on  $Y$ , there is  $m_0$  such that after possibly replacing  $A$  by a localization  $A_a$ , we have*

$$H^i(Y_s, \mathcal{F}_s \otimes \mathcal{O}_{Y_s}(mH_s)) = 0$$

for all  $i \geq 1$ , all  $m \geq m_0$ , and all closed points  $s \in \text{Spec } A$ .

*Proof.* We may assume that  $\mathcal{O}_Y(H)$  is very ample over  $X$ ; indeed, there is  $d \geq 1$  such that  $\mathcal{O}_Y(dH)$  is very ample over  $X$ , and it is enough to prove the assertion in the lemma for  $\mathcal{O}_Y(dH)$  and each of the sheaves  $\mathcal{F} \otimes \mathcal{O}_Y(jH)$ , with  $0 \leq j \leq d - 1$ . We now assume that  $\mathcal{O}_Y(H)$  is very ample over  $X$ . By asymptotic Serre vanishing, there is  $m_0$  such that  $H^i(Y, \mathcal{F} \otimes \mathcal{O}_Y(mH)) = 0$  for every  $i \geq 1$  and every  $m \geq m_0$ . After possibly replacing  $A$  by some localization  $A_a$ , we may assume that  $H^i(Y_s, \mathcal{F}_s \otimes \mathcal{O}_{Y_s}(mH_s)) = 0$  for every  $i \geq 1$ , every  $m$  with  $m_0 \leq m \leq m_0 + n - 1$ , and every closed point  $s \in \text{Spec } A$ . For every such  $s$ , the sheaf  $\mathcal{F}_s$  on  $Y_s$  is  $(m_0 + n)$ -regular with respect to  $\mathcal{O}_{Y_s}(H_s)$  in the sense of Castelnuovo–Mumford regularity. (We refer to [Laz, Section 1.8] for the basic facts on Castelnuovo–Mumford regularity.) In this case,  $\mathcal{F}_s$  is  $m$ -regular for every  $m \geq m_0 + n$ , and we obtain all the vanishings in the statement of the lemma by the definition of Castelnuovo–Mumford regularity. □

COROLLARY 4.10. *With the above notation, if  $\mathcal{F}$  is a coherent sheaf on  $Y$ , if  $d$  is a positive integer, and if  $\gamma$  is a positive rational number, then there is  $m_1$  such that*

$$H^i(Y_s, \mathcal{F}_s(\lceil m\gamma H_s - qZ_s \rceil)) = 0$$

for all  $m \geq m_1/\gamma$ , all  $q \in \mathbf{Q}$  with  $qd \in \mathbf{Z}$ , all  $i \geq 1$ , and all closed points  $s \in \text{Spec } A$ .

*Proof.* By assumption, both  $m\gamma$  and  $q$  have bounded denominators; hence,  $m\gamma H - qZ$  can be written as  $\lfloor m\gamma \rfloor H - \lfloor q \rfloor Z + T$ , where, when we vary  $m$  and  $q$ , the  $\mathbf{Q}$ -divisor  $T$  can take only finitely many values  $T_1, \dots, T_r$ . Furthermore, note that since  $\mathfrak{a}$  is assumed to be principal,  $Z$  is the pullback of a divisor from  $X$ . Since  $X$  is affine, it follows that  $H^i(Y_s, \mathcal{F}_s(\lceil m\gamma H_s - qZ_s \rceil)) = 0$  if  $H^i(Y_s, \mathcal{F}_s(\lceil T \rceil + \lfloor m\gamma \rfloor H)) = 0$ . Hence, we obtain our assertion by applying Lemma 4.9 to each of the sheaves  $\mathcal{F}(\lceil T_i \rceil)$ , for  $1 \leq i \leq r$ .  $\square$

We now return to the proof of Theorem 4.1. We first apply Corollary 4.10 to choose  $m_1$  such that after possibly replacing  $A$  by a localization, we have

$$(7) \quad H^i(Y_s, \Omega_{Y_s}^j(\log E_s)(-E_s + \lceil m\gamma H_s - qZ_s \rceil)) = 0$$

for all closed points  $s \in \text{Spec } A$ , all  $i \geq 1$ ,  $j \geq 0$ ,  $m \geq m_1/\gamma$ , and  $q \in \mathbf{Q}$  with  $qd \in \mathbf{Z}$ , where  $\gamma = 1/3 \cdot \prod_{h_i > 0} (1/h_i)$  and  $d \in 2\mathbf{Z}$  is such that  $\lambda d \in \mathbf{Z}_{>0}$ .

Let  $C \in \mathbf{Z}_{>0}$  be such that  $(C/3) \cdot \min\{a_i/h_i \mid h_i > 0\} \geq m_1$ . After possibly replacing  $A$  by a localization  $A_a$ , we may assume that for every closed point  $s \in \text{Spec } A$ , the characteristic  $p_s$  of  $k(s)$  is large enough. In particular, we may assume that  $\mathcal{J}(\mathfrak{a}^{\lambda - (C/p_s)}) = \mathcal{J}(\mathfrak{a}^{\lambda - (C/2p_s)})$  and that

$$(8) \quad \left\lfloor \left( \lambda - \frac{C}{2p_s} \right) a_i \right\rfloor = \lceil \lambda a_i \rceil - 1 \quad \text{for all } i \text{ with } 1 \leq i \leq N.$$

Suppose now that  $s \in \text{Spec } A$  is a closed point. We use primes to denote the corresponding varieties and divisors obtained after restricting the models over  $\text{Spec } k(s)$ , and we write  $p = p_s$ . In order to show that the condition in the theorem is satisfied, it is enough to show that we may apply Proposition 4.8 to  $\alpha = \lambda - (C/p)$ ,  $\mu = \lambda - (C/2p)$ , and  $G' = \mu(\eta H' - Z')$ , where  $\eta \in \mathbf{Q}_{>0}$  is given by

$$(9) \quad \eta = \frac{C}{3p\mu} \min \left\{ \frac{a_i}{h_i} \mid h_i > 0 \right\}.$$

In order to show that  $\lceil G' \rceil = \lceil -\mu Z' \rceil$ , it is enough to show that

$$(10) \quad -\mu\eta h_i - \mu a_i > \lceil -\mu a_i \rceil - 1 \quad \text{for all } i \leq N.$$

This is clear if  $h_i = 0$ ; hence, let us assume that  $h_i > 0$ . Using (8) and (9), we obtain

$$\mu\eta h_i \leq \frac{Ca_i}{3p} < \frac{Ca_i}{2p} \leq \lceil \lambda a_i \rceil - \left(\lambda - \frac{C}{2p}\right)a_i = \lfloor \mu a_i \rfloor - \mu a_i + 1,$$

which gives (10).

Arguing as in the proof of [HY, Theorem 3.4], it is easy to guarantee the vanishing in Proposition 4.8(B) for  $\ell = 0$ . Indeed, after possibly replacing  $A$  by a localization, we may assume that

$$\dim_{k(s)} H^{i+1}(Y', \Omega_{Y'}^{n-i}(\log E')(-E' + \lceil G' \rceil))$$

is independent of the closed point  $s \in \text{Spec } A$ . On the other hand, there is such a closed point  $t \in \text{Spec } A$  with the properties that  $\text{char}(k(t)) > \dim(X)$  and that  $Y_t$  and  $E_t$  admit liftings to the second ring of Witt vectors  $W_2(k(t))$  of  $k(t)$ . In this case, a version of the Akizuki–Nakano vanishing theorem (see [Hara, Corollary 3.8]) gives

$$H^{i+1}(Y_t, \Omega_{Y_t}^{n-i}(\log E_t)(-E_t + \lceil G_t \rceil)) = 0,$$

where  $G = (\lambda - (C/2p))(\eta H - Z)$ .

On the other hand, the other vanishings in Proposition 4.8 items (A) and (B) hold by our choice of  $m_1$ . Indeed, for  $\ell \geq 1$  we have  $p^\ell G' = p^\ell \mu \eta H' - p^\ell \mu Z'$ , and  $p^\ell \mu \eta \geq p \mu \eta \geq m_1$  by the definition of  $C$ . Moreover,  $p^\ell \mu d = p^{\ell-1}(p\lambda d - (Cd/2)) \in \mathbf{Z}$ , and  $p^\ell \mu \eta \in \mathbf{Z}\gamma$ . Therefore, we may apply Proposition 4.8 to obtain the assertion in the theorem. □

### §5. Bounds for Hartshorne–Speiser–Lyubeznik numbers

In this section we apply the results in Section 3 to get bounds for the HSL numbers. We start by recalling the definition of these numbers.

Let  $(S, \mathfrak{n}, k)$  be a  $d$ -dimensional Noetherian local ring of characteristic  $p > 0$ . Given any  $S$ -module  $M$ , a  $p$ -linear structure on  $M$  is an additive map  $\varphi: M \rightarrow M$  such that  $\varphi(az) = a^p \varphi(z)$  for all  $a \in S$  and  $z \in M$ . For example, the Frobenius endomorphism on  $S$  gives a  $p$ -linear structure on  $S$ , and it also induces by functoriality a  $p$ -linear structure  $\Theta$  on the top local cohomology module  $H_{\mathfrak{n}}^d(S)$ .

We recall the following theorem, originally proved by Hartshorne and Speiser [HS, Proposition 1.11] and later generalized by Lyubeznik [Lyu, Proposition 4.4] (see also [Sha] for a simplified proof).

**THEOREM 5.1.** *Let  $(S, \mathfrak{n}, k)$  be a Noetherian local ring of characteristic  $p > 0$ , and let  $M$  be an Artinian  $S$ -module with a  $p$ -linear structure  $\varphi: M \rightarrow M$ . If each element  $z \in M$  is nilpotent under  $\varphi$ , then  $M$  is nilpotent under  $\varphi$ ; that is, there exists a positive integer  $\ell$  such that  $\varphi^\ell(M) = 0$ .*

The above theorem has the following immediate consequence. Let  $\varphi$  be a  $p$ -linear structure on an  $S$ -module  $M$ , and set

$$N_i = \{z \in M \mid \varphi^i(z) = 0\}.$$

When  $M$  is an Artinian  $S$ -module, it follows from Theorem 5.1 that the ascending chain of submodules

$$\cdots \subseteq N_i \subseteq N_{i+1} \subseteq \cdots$$

eventually stabilizes; that is, there is an integer  $\ell$  such that  $N_\ell = N_{\ell+j}$  for all  $j \geq 1$ . This motivates the following definition.

**DEFINITION 5.2** (Hartshorne–Speiser–Lyubeznik number). With the above notation, if  $M$  is an Artinian  $S$ -module and if  $\varphi$  is a  $p$ -linear structure on  $M$ , then the HSL number of  $(M, \varphi)$  is the smallest positive integer  $\ell$  such that  $N_\ell = N_{\ell+j}$  for all  $j \geq 1$ .

**REMARK 5.3.** For every complete Noetherian local ring  $T$ , Matlis duality gives an inclusion-reversing one-to-one correspondence between the ideals of  $T$  and the submodules of the injective hull  $E_T$  of the residue field of  $T$ . This correspondence is given by

$$I \rightarrow \text{Ann}_{E_T} I = \{u \in E_T \mid Iu = 0\}, \quad \text{with inverse } M \rightarrow \text{Ann}_T M.$$

Therefore, the stabilization of an ascending chain of submodules  $\cdots \subseteq N_i \subseteq N_{i+1} \subseteq \cdots$  of  $E_T$  is equivalent to the stabilization of  $\cdots \supseteq I_i \supseteq I_{i+1} \supseteq \cdots$ , where  $I_i = \text{Ann}_T N_i$ .

The injective hull  $E_S$  of the residue field of  $S$  is of particular interest due to the following remark.

REMARK 5.4. Suppose that  $S$  is a complete local Noetherian ring of characteristic  $p > 0$ ; hence, by Cohen's theorem,  $S$  is a homomorphic image of a complete regular local ring  $R$ ; that is,  $S \simeq R/I$ . In this case, there is a one-to-one correspondence between  $(I^{[p]}: I)/I^{[p]}$  and the set of  $p$ -linear structures  $\varphi$  on  $E_S$ , as follows.

By Cohen's theorem, we may assume that  $R = k[[x_1, \dots, x_n]]$ , for a field  $k$ . The injective hull  $E_R$  is isomorphic to  $H_{\mathfrak{m}}^n(R)$ , where  $\mathfrak{m} = (x_1, \dots, x_n)$ . The natural  $p$ -linear structure  $F$  on  $E_R$  is given by

$$\left[ \frac{r}{x_1^{j_1} \cdots x_n^{j_n}} \right] \rightarrow \left[ \frac{r^p}{x_1^{j_1 p} \cdots x_n^{j_n p}} \right]$$

for each cohomology class  $[r/(x_1^{j_1} \cdots x_n^{j_n})] \in H_{\mathfrak{m}}^n(R) \simeq R_{x_1 \cdots x_n} / \sum_{i=1}^n R_{x_1 \cdots \widehat{x}_i \cdots x_n}$ . Given an element  $u \in (I^{[p]}: I)$ , we claim that  $uF$  induces a  $p$ -linear structure on  $E_S$ . Note that  $E_S$  can be identified with  $\text{Ann}_{E_R} I$ . Given any  $z \in E_S$ , that is, an element in  $E_R$  that is annihilated by  $I$ , we have

$$IuF(z) \subseteq I^{[p]}F(z) = F(Iz) = F(0) = 0;$$

hence,  $uF(z)$  is an element of  $\text{Ann}_{E_R} I = E_S$ . Therefore, the restriction of  $uF$  to  $\text{Ann}_{E_R} I$  gives a  $p$ -linear structure on  $E_S$ .

It is an easy exercise, using Matlis duality, to check that  $uF$  gives the trivial  $p$ -linear structure on  $E_S$  if and only if  $u \in I^{[p]}$ . On the other hand, Blicke [Bli, Chapter 3] has shown that every  $p$ -linear structure on  $E_S$  comes from an element in  $(I^{[p]}: I)/I^{[p]}$ .

REMARK 5.5. Suppose now that  $R$  is a regular local ring of positive characteristic and that  $S = R/I$ , for some ideal  $I$  in  $R$ . If we denote by  $\widehat{R}$  and  $\widehat{S}$  the completions of  $R$  and  $S$ , respectively, then  $E_S = E_{\widehat{S}}$ , and the  $p$ -linear structures of this module over  $S$  and over  $\widehat{S}$  can be identified. Using the fact that  $\widehat{R}$  is flat over  $R$ , we deduce from the previous remark that the  $p$ -linear structures on  $E_S$  are in bijection with  $(\widehat{I}^{[p]}: \widehat{I})/\widehat{I}^{[p]}$ , where  $\widehat{I} = I\widehat{R}$ . In particular, this set contains  $(I^{[p]}: I)/I^{[p]}$ .

EXAMPLE 5.6. Let  $R$  be an  $n$ -dimensional regular local ring of characteristic  $p > 0$ , let  $f \in R$  be nonzero and noninvertible, and let  $S = R/(f)$ . We denote by  $\mathfrak{n}$  and  $\mathfrak{m}$  the maximal ideals in  $S$  and  $R$ , respectively. We have  $E_R \simeq H_{\mathfrak{m}}^n(R)$  and  $E_S \simeq \text{Ann}_{E_R}(f) \simeq H_{\mathfrak{n}}^{n-1}(S)$ . (The second isomorphism is a consequence of the exact sequence in the diagram below.) Let  $\Theta$  be the

natural  $p$ -linear structure on  $E_S$  induced by the Frobenius morphism on  $S$ . From the commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & H_n^{n-1}(S) & \longrightarrow & H_m^n(R) & \xrightarrow{f} & H_m^n(R) & \longrightarrow & 0 \\
 & & \downarrow \Theta & & \downarrow f^{p-1}F & & \downarrow F & & \\
 0 & \longrightarrow & H_n^{n-1}(S) & \longrightarrow & H_m^n(R) & \xrightarrow{f} & H_m^n(R) & \longrightarrow & 0
 \end{array}$$

one sees immediately that  $\Theta$  is the restriction of  $f^{p-1}F$  to  $E_S$ . Note also that for every  $i \geq 1$ , we have

$$\{u \in E_R \mid (f^{p-1}F)^i(u) = 0\} \subseteq E_S.$$

Indeed, if  $(f^{p-1}F)^i(u) = 0$ , then  $f^{p^i-1}F^i(u) = 0$ ; hence,  $f^{p^i}F^i(u) = F^i(fu) = 0$ . And since  $F$  is injective on  $E_R$ , we deduce that  $fu = 0$ ; hence,  $u \in E_S$ .

We thus conclude that determining the HSL number of  $(E_S, \Theta)$  is equivalent to determining the one of  $(E_R, f^{p-1}F)$ .

We recall that if  $R$  is a regular  $F$ -finite domain of characteristic  $p > 0$  and if  $f \in R$  is nonzero, then the test ideal  $\tau(f^{m/p^e})$  has a simple description (see [BMS2, Lemma 2.1]):

$$(11) \quad \tau(f^{m/p^e}) = (f^m)^{[1/p^e]}.$$

**PROPOSITION 5.7.** *Let  $R$  be a regular  $F$ -finite local ring of characteristic  $p > 0$ , and let  $f \in R$  be nonzero. If we put  $N_i = \{z \in E_R \mid (f^{p-1}F)^i(z) = 0\}$  for  $i \geq 1$ , then*

$$\text{Ann}_R(N_i) = \tau(f^{(p^i-1)/p^i}).$$

*Proof.* Note first that since taking the test ideal commutes with completion, we may assume that  $R$  is complete. We then use [Kat, Theorem 4.6] (and its proof) with  $I = (f)$  and  $u = f^{p-1}$  to get that  $\text{Ann}_R N_i$  is the smallest ideal  $J_i$  of  $R$  such that

- (i)  $(f) \subseteq J_i$  and
- (ii)  $(f^{p-1})^{(p^i-1)/(p-1)} = f^{p^i-1}$  is contained in  $J_i^{[p^i]}$ .

By definition, this says that  $\text{Ann}_R N_i = (f^{p^i-1})^{[1/p^i]}$ , and the formula in the proposition follows from (11). □

COROLLARY 5.8. *Let  $R$  and  $f$  be as in Proposition 5.7, with  $f$  noninvertible. The HSL number of  $(E_R, f^{p^{-1}}F)$  (or, equivalently, that of  $(E_{R/(f)}, \Theta)$ ) is the smallest positive integer  $\ell$  such that*

$$\tau(f^{1-1/p^\ell}) = \tau(f^{1-1/p^{\ell+i}})$$

for all  $i \geq 1$ .

*Proof.* Since the HSL number of  $(E_R, f^{p^{-1}}F)$  does not change when we pass from  $R$  to its completion, and taking test ideals commutes with completion, we may assume that  $R$  is complete. In this case, the assertion in the corollary follows from the definition of the HSL number for  $(E_R, f^{p^{-1}}F)$  via Matlis duality and Proposition 5.7.  $\square$

Note that the condition in Corollary 5.8 is equivalent to saying that there is no  $F$ -jumping number for  $f$  in the interval  $(1 - (1/p^\ell), 1)$ . We can now reformulate Theorem 3.3 and Corollary 3.4 as follows.

THEOREM 5.9. *Let  $A$  be a finitely generated  $\mathbf{Z}$ -algebra, and let  $X$  be a scheme of finite type over  $\text{Spec } A$ , smooth of relative dimension  $n$ . If  $\mathfrak{a}$  is a locally principal ideal on  $X$ , whose restriction to every fiber is everywhere nonzero, and if  $Z$  is the closed subscheme defined by  $\mathfrak{a}$ , then there is a positive integer  $N$  such that for every closed point  $s \in \text{Spec } A$  and every point  $x$  in the fiber  $Z_s$  of  $Z$  over  $s$ , the HSL number of  $(E_{\mathcal{O}_{Z_s, x}}, \Theta)$  is bounded above by  $N$ .*

*Proof.* The assertion follows from the above interpretation of HSL numbers, and Theorem 4.1 applied to  $\lambda = 1$ , by noting two facts. First, if  $\mathfrak{p}$  is a prime ideal in a regular  $F$ -finite  $R$ , and if  $\mathfrak{b}$  is an everywhere nonzero ideal in  $R$ , then the  $F$ -jumping numbers of  $\mathfrak{b} \cdot R_{\mathfrak{p}}$  are among the  $F$ -jumping numbers of  $\mathfrak{b}$ ; this follows since taking multiplier ideals commutes with localization at  $\mathfrak{p}$ . Second, we may take  $p_0 = 1$ . (This follows by inspecting the proof of Theorem 3.3.)  $\square$

COROLLARY 5.10. *Given  $n$  and  $M$ , there is a positive integer  $N = N(n, M)$  such that for every  $F$ -finite field  $k$  of characteristic  $p > 0$  and every nonzero polynomial  $f \in k[x_1, \dots, x_n]$  with  $\deg(f) \leq M$ , the following holds: for every prime ideal  $\mathfrak{p}$  in  $S = k[x_1, \dots, x_n]/(f)$ , the HSL number of  $(E_{S_{\mathfrak{p}}}, \Theta)$  is bounded above by  $N$ .*

REMARK 5.11. By inspecting the proof of Theorem 3.3, we see that we may take  $N(n, M)$  to be 1 plus the dimension of the vector space of polynomials in  $k[x_1, \dots, x_n]$  of degree at most  $M$ ; that is,  $N(n, m) = \binom{n+M}{n} + 1$ .

**§6. Examples**

In this section we give two examples in order to illustrate how the  $F$ -jumping numbers vary when we reduce modulo various primes. In our computations of test ideals, we will use (1) in Section 2, describing ideals of the form  $\mathfrak{b}^{[1/p^e]}$ . We begin by recalling the following theorem due to Lucas (see [Gra]).

**THEOREM 6.1.** *Let  $p$  be a positive prime integer. Given two positive integers  $m$  and  $n$ , if we write  $m = m_e p^e + m_{e-1} p^{e-1} + \dots + m_0$  and  $n = n_e p^e + n_{e-1} p^{e-1} + \dots + n_0$ , with  $0 \leq m_i, n_i \leq p - 1$ , then*

$$\binom{m}{n} \equiv \prod_{i=0}^e \binom{m_i}{n_i} \pmod{p}.$$

**REMARK 6.2.** It is clear from Lucas’s theorem that  $\binom{m}{n}$  is divisible by  $p$  if and only if there is an  $i$  such that  $n_i > m_i$ .

The following proposition shows that if we want to bound the HSL number of a polynomial, as in Corollary 5.10, we indeed need to bound the degree of the polynomial.

**PROPOSITION 6.3.** *Let  $n$  be a positive integer, and set  $a = 2^{n+1} + 1$  and  $f = x^a + y^a + z^a$ . For all prime numbers  $p$  such that  $p \equiv 2 \pmod{a}$ , the principal ideal  $(f)$ , considered as an ideal in  $\mathbb{F}_p[[x, y, z]]$ , has an  $F$ -jumping number in  $(1 - (1/p^n), 1 - (1/p^{n+1}))$ .*

*Proof.* In order to prove the proposition, it is enough to show the following two assertions.

- (i) The monomial  $x^{a-3}$  lies in  $\tau(f^{1-(1/p^n)})$ .
- (ii) The monomial  $x^{a-3}$  does not lie in  $\tau(f^{1-(1/(p^{n+1}))})$ .

We first prove (i). It follows from (11) that  $\tau(f^{1-(1/p^n)}) = (f^{p^n-1})^{[1/p^n]}$ . Note that  $a$  divides  $p^n - 2^n$ , and we consider the following term in the expansion of  $f^{p^n-1}$ :

$$\begin{aligned} & \binom{p^n - 1}{p^n - 1 - \frac{2}{a}(p^n - 2^n)} \binom{\frac{2}{a}(p^n - 2^n)}{\frac{1}{a}(p^n - 2^n)} (x^a)^{p^n - 1 - \frac{2}{a}(p^n - 2^n)} (y^a)^{\frac{1}{a}(p^n - 2^n)} (z^a)^{\frac{1}{a}(p^n - 2^n)} \\ &= \binom{p^n - 1}{p^n - 1 - \frac{2}{a}(p^n - 2^n)} \binom{\frac{2}{a}(p^n - 2^n)}{\frac{1}{a}(p^n - 2^n)} x^{(a-3)p^n + p^n - 1} y^{p^n - 2^n} z^{p^n - 2^n}. \end{aligned}$$

It follows from the description for  $(f^{p^e-1})^{[1/p^e]}$  given in (1) that if the two binomial coefficients in the above expression are not zero, then (i) holds.



Remark 6.2 implies that  $\binom{p^n-1}{i}$  is not zero in  $\mathbf{F}_p$  for every  $i$  with  $0 \leq i \leq p^n - 1$ . In particular, we have  $\binom{p^n-1}{p^n-1-\binom{2/a}{(1/a)(p^n-2^n)}} \neq 0$ . For the other binomial coefficient, we write

$$\frac{2}{a}(p^n - 2^n) = \sum_{j=0}^{n-1} \frac{2^{j+1}(p-2)}{a} p^{n-1-j}; \quad \frac{1}{a}(p^n - 2^n) = \sum_{j=0}^{n-1} \frac{2^j(p-2)}{a} p^{n-1-j}.$$

Since  $a = 2^{n+1} - 1$ , we see that  $(2^{j+1}(p-2))/a < p$ . Using again Remark 6.2, we deduce that  $\binom{2/a}{(1/a)(p^n-2^n)} \neq 0$ , which completes the proof of (i).

We now prove (ii). It follows from the description of  $\tau(f^{1-(1/(p^{n+1}))}) = (f^{p^{n+1}-1})^{[1/p^{n+1}]}$  given by (1) that if  $x^{a-3}$  lies in this ideal, then we have a monomial  $x^{ar}y^{as}z^{at}$  that appears with nonzero coefficient in the expansion of  $f^{p^{n+1}-1}$  such that  $0 \leq ar - (a-3)p^{n+1} \leq p^{n+1} - 1$ ,  $as \leq p^{n+1} - 1$ , and  $at \leq p^{n+1} - 1$ . In this case we have

$$\begin{aligned} as \leq p^{n+1} - 1 &\Rightarrow s \leq \frac{p^{n+1} - 1}{a} = \frac{p^{n+1} - 2^{n+1}}{a} + \frac{2^{n+1} - 1}{a} \\ &\Rightarrow s \leq \frac{p^{n+1} - 2^{n+1}}{a}, \end{aligned}$$

where the last inequality holds since  $a = 2^{n+1} + 1$  and  $s$  is an integer. We similarly have  $t \leq (p^{n+1} - 2^{n+1})/a$ . Therefore,

$$\begin{aligned} ar &= a(p^{n+1} - 1 - s - t) \geq a(p^{n+1} - 1) - a \cdot \frac{2}{a}(p^{n+1} - 2^{n+1}) \\ &= (a-2)p^{n+1} + (2^{n+2} - a) \geq (a-2)p^{n+1}, \end{aligned}$$

a contradiction. We thus conclude that  $x^{a-3} \notin \tau(f^{1-(1/(p^{n+1}))})$ , proving (ii). □

REMARK 6.4. In [AIM, Problem 1.05], Katzman asks the following question: given  $f \in \mathbf{Z}[x_1, \dots, x_n]$ , if  $\alpha_p$  denotes the HSL number of the injective hull of the residue field of  $\mathbf{F}_p[[x_1, \dots, x_n]]/(f_p)$  with respect to the natural  $p$ -linear structure  $\Theta$ , is  $\limsup_{p \rightarrow \infty} \alpha_p = 1$ ? Note that Proposition 6.3 gives a negative answer to this question. Indeed, since 2 and  $2^{n+1} + 1$  are relatively prime, according to Dirichlet's theorem there are infinitely many prime numbers  $p$  such that  $p \equiv 2 \pmod{2^{n+1} + 1}$ . Therefore, Proposition 6.3 implies that given any  $N$ , there is  $f \in \mathbf{Z}[x_1, x_2, x_3]$  such that  $\alpha_p > N$  for infinitely many primes  $p$ .

EXAMPLE 6.5. Let  $R = \mathbb{F}_p[[x, y, z]]$ , and let  $f = x^5 + y^5 + z^5$ . In this case, we have the following  $F$ -jumping numbers in  $(0, 1)$ .

- (i) When  $p = 2$ , there are three  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = 1/4$ ,  $\lambda_2 = 1/2$ , and  $\lambda_3 = 3/4$ .
- (ii) When  $p = 3$ , there are three  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = 1/3$ ,  $\lambda_2 = 2/3$ , and  $\lambda_3 = 8/9$ .
- (iii) When  $p = 5$ , there are four  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = 1/5$ ,  $\lambda_2 = 2/5$ ,  $\lambda_3 = 3/5$ , and  $\lambda_4 = 4/5$ .
- (iv) When  $p \equiv 1 \pmod{5}$ , there are two  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = 3/5$  and  $\lambda_2 = 4/5$ .
- (v) When  $p \equiv 2 \pmod{5}$  and  $p > 2$ , there are four  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = (3/5) - (1/5p)$ ,  $\lambda_2 = (4/5) - (3/5p)$ ,  $\lambda_3 = 1 - (1/p)$ , and  $\lambda_4 = 1 - (1/p^2)$ .
- (vi) When  $p \equiv 3 \pmod{5}$  and  $p > 3$ , there are four  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = (3/5) - (4/5p)$ ,  $\lambda_2 = (4/5) - (2/5p)$ ,  $\lambda_3 = 1 - (1/p)$ , and  $\lambda_4 = 1 - (1/p^2)$ .
- (vii) When  $p \equiv 4 \pmod{5}$ , there are three  $F$ -jumping numbers in  $(0, 1)$ :  $\lambda_1 = (3/5) - (7/5p)$ ,  $\lambda_2 = (4/5) - (6/5p)$ , and  $\lambda_3 = 1 - (1/p)$ .

Since the computations involved in checking the above example are rather tedious, we omit them. For a sample of such computations proving assertion (v), see the archive version of this paper, available at <http://front.math.ucdavis.edu/1110.5687>.

REMARK 6.6. Let  $R$  and  $f$  be as in Example 6.5. The description of the  $F$ -jumping numbers in this example allows us to compute via Corollary 5.8 the HSL number of  $(E_{R/(f)}, \Theta)$ . We see that this is equal to 2 if  $p \equiv 2$  or  $3 \pmod{5}$ , and it is equal to 1 otherwise.

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